AMERICAN

JOURNAL OF MATHEMATICS

FOUNDED BY THE JOHNS HOPKINS UNIVERSITY

EDITED BY

E. W. CHITTENDEN
UNIVERSITY OF IOWA

A. B. COBLE
THE JOHNS HOPKINS UNIVERSITY

ABRAHAM COHEN
JOHNS HOPKINS UNIVERSITY

G. C. EVANS
RICE INSTITUTE

FRANK MORLEY, CHAIRMAN
THE JOHNS HOPKINS UNIVERSITY

WITH THE COÖPERATION OF

E. T. BELL
F. D. MURNAGHAN
W. A. MANNING

PUBLISHED UNDER THE JOINT AUSPICES OF
THE JOHNS HOPKINS UNIVERSITY
AND
THE AMERICAN MATHEMATICAL SOCIETY

VOLUME L
1928

THE JOHNS HOPKINS PRESS
BALTIMORE, MARYLAND
U. S. A.

P14763

PRINTED BY THE J. H. FURST COMPANY
BALTIMORE, MD.

INDEX

PAGE

Ayres, W. L. Concerning Subsets of a Continuous Curve which can be Connected through the Complement of the Continuous Curve, 521
Baker, Frances E. Compound Statements on Four Classes, . . 195
Brahana, H. R. Certain Perfect Groups Generated by Two Operators of Orders Two and Three, ... 345
Buchanan, H. E. Periodic Orbits of Three Finite Masses about the Equilateral Triangle Points, ... 613
Camp, Chester C. An Expansion Involving p Inseparable Parameters Associated with a Partial Differential Equation, . . . 259
Campbell, Alan D. Nets of Conics in the Real Domain, . . 459
Copeland, Arthur H. Admissible Numbers in the Theory of Probability, ... 535
Dickson, L. E. Additive Number Theory for all Quadratic Functions. 1
Elliott, W. W. Generalized Green's Functions for Compatible Differential Systems, ... 243
Evans, G. C. Note on a Theorem of Bôcher, ... 123
Evans, G. C. Generalized Neumann Problems for the Sphere, . . 127
Farnum, Fay. On Triadic Cremona Nets of Plane Curves, . . 357
Fréchet, Maurice. Démonstration de Quelques Propriétés des Ensembles Abstraits, ... 49
Gergen, J. J. and Widder, D. V. On Taylor's Series Admitting the Circle of Convergence as a Singular Curve, ... 139
Glenn, O. E. A Memoir on the Invariants of Biternary Quantics, . 209
Gray, Marion Cameron. A Boundary Value Problem of Ordinary Self-Adjoint Differential Equations with Singularities, . . 431
Griffiths, Lois W. Generalized Quaternion Algebras and the Theory of Numbers, ... 303
Johnston, F. E. Transitive Substitution Groups Containing Regular Subgroups of Lower Degree, ... 231
Lefschetz, S. A Theorem on Correspondences on Algebraic Curves, 159
Michal, Aristotle D. Affinely Connected Function Space Manifolds, 473
Moore, L. T. and Neelley, J. H. Rational Tacnodal and Oscnodal Quartic Curves Considered as Plane Sections of Quartic Surfaces, 467
Moore, Thomas W. On the Invariant Combinants of Two Binary Quintics, ... 415
Neelley, J. H. See Moore and Neelley.

PATTERSON, BOYD C. The Differential Invariants of Inversive Geometry, ... 55

PERKINS, FRED W. An Intrinsic Treatment of Poisson's Integral, . 38

RICHMOND, D. E. Number Relations between Types of Extremals. Joining a Pair of Points, ... 37

RITT, J. F. On Certain Points in the Theory of Dirichlet Series, . 7

ROOS, C. F. A Mathematical Theory of Depreciation and Replacement, 14

SHARPE, F. R. The Problem of Plane Involutions of Order $t > 2$, . 62

STEVENSON, GUY. Expansions of the Neumann Type in Terms of Products of Bessel Functions, ... 56

SWIFT, ELIJAH. Canonical Forms for Ordinary Homogeneous Linear Differential Equations of the Second Order with Periodic Coefficients, ... 59

THOMAS, TRACY YERKES. A Theorem Concerning the Affine Connection, ... 51

USPENSKY, J. V. On Gierster's Classnumber Relations, ... 9

WHYBURN, GORDON T. Concerning the Structure of a Continuous Curve, ... 16

WIDDER, D. V. See Gergen and Widder.

WRINCH, D. M. On the Asymptotic Evaluation of Functions Defined by Contour Integrals, ... 26

ZARISKI, OSCAR. On a Theorem of Severi, ... 8

ZARISKI, OSCAR. On Hyperelliptic θ-Functions with Rational Characteristics, ... 31

Additive Number Theory for all Quadratic Functions.

By L. E. Dickson.

1. *Introduction.* We consider only those quadratic functions $q(x)$ which take integral values $\geqq 0$ for every integer $x \geqq 0$. The general such $q(x)$ is found in § 2.

One statement of our problem is to find the maximum error committed in affirming that every positive integer p is a sum of s such values of $q(x)$. Or, if $m(p)$ denotes the minimum of the differences between p and all such sums, what is the maximum of $m(p)$ for all p? The answer is evidently $\frac{1}{2}g$ or $\frac{1}{2}(g-1)$, according as g is even or odd, where g is the maximum gap in a table of all sums by s of values of $q(x)$ for integers $\geqq 0$.

Write $e_s = g-1$. Then every p is r plus a sum of s values of $q(x)$, where r is chosen from $0, 1, \cdots, e_s$. For example, let $q(x)$ be $\frac{1}{2}(n-2)(x^2-x)+x$; then $e_4 = n-4$, as first proved by Cauchy. In words, every p is a sum of n polygonal numbers of n sides, all but four of which are chosen from 0 or 1. The case $n=4$ states that every p is a sum of four squares. However perfected, the method of Cauchy is usually applicable only when $s=4$.

For the case $s \geqq 5$ there is developed here a new method of remarkable power and simplicity, which leads readily to the determination of e_s for all $q(x)$. The two effective methods for $s=4$ (§§ 9, 10, and 21) are not only more laborious, but unfortunately require direct verification for very numerous initial values of p (unlike the method for $s>4$). While this would seem to require a table for each $q(x)$, the theory has been so developed that a few such tables suffice.

For each $q(x)$ we find the minimum L of $s+e_s$ for all $s \geqq 5$; this sum is the total number of summands sufficient for every p. We readily detect the cases in which $4+e_4 \geqq L$. It is only in the remaining cases that we discuss the difficult problem[*] to find e_4. Whenever $4+e_4$ is less than L it is exactly $L-1$. By examining also the cases $s=2$ and $s=3$, we find that the minimum of all $s+e_s$ is either L or $L-1$, except (§ 11) when $q(x)=$

---

[*] To be treated in Chicago theses.

$\frac{1}{2}m(x^2 - x) + tx$, where m and t are positive integers such that $m \leqq t$. Then $e_s = t - 1$ for all $s \geqq 3$.

An interesting case is that in which both 0 and 1 are values of $q(x)$. Then every p is a sum of values of $q(x)$. For this case the present problem was solved completely in the *Bulletin of the American Mathematical Society*, November-December, 1927; January-February, 1928; March-April, 1928. These are cited here as *Bulletin*, I, II, III.

Aid was provided by the Carnegie Institution for constructing tables and using them to verify the theorems for the cases below the limits obtained in the proofs.

2. THEOREM 1. *The following and no further quadratic functions* $q(x)$ *have integral values* $\geqq 0$ *for every*[*] *integer* $x \geqq 0$*: First,*

$$f(x) = \tfrac{1}{2}m(x^2 - x) + tx + c, \tag{1}$$

where m, t, c *are integers and* $m > 0$, $t > 0$, $c \geqq 0$. *Second,* $f(x - k)$, *where* k *is an integer* $\geqq 1$, *and* $m \geqq t$.

Since $y = q(x)$ is a parabola whose vertex is its lowest point, there[†] is either a single minimum $q(k)$ of $q(0), q(1), \cdots$, or just two equal minima $q(k-1) = q(k)$. Hence there is a unique integer $k \geqq 0$ such that $q(x) > q(k)$ for every integer $x \geqq 0$ distinct from k and $k - 1$; while $q(k-1) \geqq q(k)$ if $k \geqq 1$.

Write $f(x)$ for $q(x + k)$. Then $f(x)$ is an integer $\geqq 0$ for every integer $x \geqq -k$. Also, $f(x) > f(0)$ for every integer $x \geqq -k$ distinct from 0 and -1; while $f(-1) \geqq f(0)$ if $k \geqq 1$. It will be useful that $x = 0$ gives the minimum of $f(x)$ for all integers $x \geqq -k$.

Write $f(x) = ux^2 + vx + c$. Take $x = 0, 1, 2$. Thus c, $u + v = t$, and $4u + 2v$ are integers, and $u > 0$, $c \geqq 0$. Hence $2u$ is a positive integer m. Thus $f(x)$ is of the form (1). Since $f(1) = t + c > f(0) = c$, $t > 0$. Conversely, this $f(x)$ is evidently an integer $\geqq 0$ for every integer $x \geqq 0$.

In case $k \geqq 1$, $f(-1) = m - t + c \geqq f(0) = c$ implies $m \geqq t$, and conversely. Then $f(-p) > f(0)$ for every $p > 1$. This is evident geometrically, and follows also from

---

[*] It is only an apparent generalization to employ a function $g(x)$ which is an integer $\geqq 0$ for every integer $x \geqq r$, since $q(x) = g(x + r)$ is then an integer $\geqq 0$ for every integer $x \geqq 0$.

[†] Arithmetically, if $q'(\xi) = 0$, then $k = 0$ if $\xi \leqq 0$, $k = \xi$ if ξ is a positive integer, while if ξ lies between $p - 1$ and p, where p is an integer $\geqq 1$, $k = p$ if $q(p-1)$ and $q(p)$ are equal, but, if they are distinct, $k = p - 1$ or p according as $q(p-1)$ or $q(p)$ is the lesser.

$$f(x+1) = f(x) + mx + t, \quad f(-p) = f(-p+1) + mp - t. \tag{2}$$

3. *Symbols* E_s, e_s; *Reduction to* $c = 0$. The least value of $q(x) = f(x-k)$ for integers $x \geqq 0$ was seen to be $f(0) = c$. Hence the least sum of s values of $q(x)$ is sc. Let N be any integer $\geqq sc$. From N we subtract the maximum sum $\leqq N$ of s values of $q(x)$ for integers $x \geqq 0$ and denote the difference by $E_s(N)$. In case $E_s(N)$ has a finite maximum for all integers $N \geqq sc$, we shall denote that maximum by e_s. In other words, every integer $N \geqq sc$ is then a sum of e_s numbers 0 or 1 and s values of $q(x)$ for integers $x \geqq 0$. For example, since every integer $\geqq 0$ is a sum of four squares, $e_4 = 0$ for $q(x) = x^2 + 7$.

Write $h(x)$ for $q(x) - c$, and employ summations over s integers $x \geqq 0$. If $N = r + \Sigma q(x)$, where $N \geqq sc$, $0 \leqq r \leqq e_s$, then $A = N - sc = r + \Sigma h(x)$, $A \geqq 0$, and conversely. Hence $q(x)$ and $h(x)$ have the same e_s. Without loss of generality, we shall therefore take $c = 0$ henceforth. Then

$$q(x) = f(x-k) = \tfrac{1}{2}mx^2 + \tfrac{1}{2}nx + C, \tag{3}$$

$$n = 2t - m(1+2k), \quad C = \tfrac{1}{2}m(k^2+k) - tk. \tag{4}$$

The case $s > 4$ will be seen to depend on facts required in the harder case $s = 4$.

4. *Formulas for the Case* $s = 4$.

We investigate the existence of integers e_4, α, β, γ, δ, all $\geqq 0$, such that

$$A = q(\alpha) + q(\beta) + q(\gamma) + q(\delta) + r, \quad 0 \leqq r \leqq e_4, \tag{5}$$

for every integer $A \geqq 0$. Write

$$a = \Sigma\alpha^2, \quad b = \Sigma\alpha. \tag{6}$$

Then (5) is equivalent to

$$D = \tfrac{1}{2}ma + \tfrac{1}{2}nb + r, \quad D = A - 4C. \tag{7}$$

We shall use two lemmas* due to Cauchy.

LEMMA 1. *If a and b are positive odd integers such that $b^2 < 4a$ and*

$$b^2 + 2b + 4 > 3a, \tag{8}$$

then equations (6) *have solutions α, β, γ, δ in integers $\geqq 0$.*

---

* Cf. Legendre, *Théorie des nombres*, ed. 3, vol. II, Nos. 624-30.

LEMMA 2. *If a and $b=2B$ are positive even integers holds and $a-B^2$ is a sum of three squares, then equations* (6) *$\alpha, \beta, \gamma, \delta$ in integers $\geqq 0$.*

In particular, let $a \equiv 2 \pmod 4$ and $b^2 < 4a$. Then $a -$ and is $\equiv 1$ or $2 \pmod 4$, whence it is a sum of three sq Lemma 1 holds also when a and b are positive even integers s (mod 4).

By (7), (8) is equivalent to

$$mb^2 + 2mb + 4m > 6D - 3nb - 6r,$$

which follows from the inequality obtained by suppressing $6r$. by $4m$ yields

$$(2mb + \tau)^2 > U, \quad U = \tau^2 + 4m(6D - 4m), \tag{9}$$

where $\tau = 2m + 3n$. This inequality holds if

$$b > (U^{1/2} - \tau)/(2m), \quad U \geqq 0. \tag{10}$$

In $m^2(b^2 - 4a) < 0$, replace ma by its value from (7). follows from that with r replaced by e_4 and hence from

$$(mb + 2n)^2 < 4V, \quad V = n^2 + 2mD - 2me_4. \tag{11}$$

This inequality holds if

$$b < (2V^{1/2} - 2n)/m, \quad V \geqq 0, \tag{12}$$

and if $mb + 2n \geqq 0$. When $n \geqq 0$, the latter holds if $b > 0$. it holds by (10) if

$$4n + U^{1/2} - \tau \geqq 0, \tag{13}$$

and hence if

$$3D \geqq 2m - 2n - n^2/m \tag{14}$$

Finally, $b > 0$ if $U^{1/2} \geqq \tau$. If $n < 0$, this follows from $n \geqq 0$, whence $\tau > 0$, it holds if and only if $U \geqq \tau^2$ and hence

$$D \geqq \tfrac{2}{3}m \tag{15}$$

There will be at least d positive integers between the limits for b if

$$4V^{1/2} - U^{1/2} > P, \quad P = 2md - 2m + n. \tag{16}$$

The left member is $\geqq 0$ if

$$16V \geqq U. \tag{17}$$

Then (16) holds automatically if $P < 0$, and always if its square holds and hence if

$$F \equiv (2V + W)^2 - UV > 0, \quad 8W \equiv U - P^2. \tag{18}$$

By the minor conditions we shall mean $U \geqq 0$, (17) and either (14) or (15). We have

$$\begin{aligned} 16V - U &= 8mD - 32me_4 + \phi, \\ \phi &= 12m^2 - 12mn + 7n^2 = 4n^2 + 3(2m - n)^2. \end{aligned} \tag{19}$$

By (9) and (19), $U \geqq 0$ and (17) holds if

$$D \geqq \tfrac{2}{3}m, \quad D \geqq 4e_4. \tag{20}$$

The *reduced* minor conditions are (20) if $n \geqq 0$, but (14) and (20) if $n < 0$.

PART I. THE CASE $k = 0$, §§ 5-12.

We investigate sums of s values of $f(x)$. The case $s \geqq 5$ is treated in §§ 5-7, the case $s = 4$ in §§ 8-10, the case $s = 3$ in § 11, and the case $s = 2$ in § 12.

5. THEOREM 2. *If* $k = 0$, $m \geqq (s - 2)t$, $s \geqq 5$, *then* $e_s = m - (s - 3)t - 1$.

No smaller value of e_s will serve. For, if $B = 2m + 4t - 1$, the maximum sum $\leqq B$ of s values

$$0, \quad t, \quad m + 2t, \quad 3m + 3t, \quad 6m + 4t, \quad 10m + 5t, \quad \cdots \tag{21}$$

of $f(x)$ is $M = m + (s + 1)t$. In fact, no summand exceeds $m + 2t$. If one summand is $m + 2t$, the others are 0 or t, and the largest such sum is $m + 2t + (s - 1)t = M$. The largest sum of s values 0 and t is $st < M$. Hence $E_s(B) = B - M$ has the value stated for e_s.

We are to prove that every integer $N \geqq 0$ is a sum of s values of $f(x)$ and e_s numbers 0 or 1. Write $N = A + R$, where

$$A = \sum_4 f(x), \quad R = r + \sum_{s-4} f(x), \quad 0 \leqq r \leqq e_s. \tag{22}$$

We can find a value $\leqq 2m - 1$ of R which is congruent to any assigned integer modulo m. We add $0, 1, \cdots, t - 1 \leqq e_s$ to each of $0, t, \cdots, (s - 4)t$. But we add $0, 1, \cdots, e_s$ to $m + 2t + (s - 5)t$. We obtain m integers R which are congruent modulo m to the consecutive integers $0, \cdots, m - 1$. The largest such R is $2m - 1$.

We shall treat the first equation (22) by the method[*] of § 4 with $e_4 = 0$. By (4) with $k = 0$, $n = 2t - m$, $C = 0$. Then $D = A$ and (7) becomes

$$A = \tfrac{1}{2}m(a - b) + tb. \tag{23}$$

Let the difference between the limits (10) and (12) on b exceed $d = 2$. Then there is an odd value of b between these limits. We proved that we can choose $R \leqq 2m - 1$ so that $A = N - R$ is congruent modulo m to any assigned integer. As the latter we take tb. Then (23) determines an integer $\tfrac{1}{2}(a - b)$ and hence an odd integer a, which is positive since $4a > b^2$. By (9)-(18).

$$\begin{aligned} &P = 2m + n, \quad W = 3mD - 2m^2 + mn + n^2, \\ &F = m^2(D - 2m - 5n - 4e_4)^2 - 6m\,(n + e_4)\,(2m + n)^2, \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (\text{if } d = 2). \end{aligned} \tag{24}$$

Since $m \geqq 3t$, n is negative. Hence $F > 0$ for every $D = A$. Here (14) becomes $3A \geqq 3m - 4t^2/m$. This and (20) hold if $A \geqq m$.

Hence if $N \geqq 2m - 1 + m$, N is a sum of e_s numbers 0 or 1 and s values of $f(x)$. We next verify this also when $N < 3m - 1$. We employ the sums by s of 0 and t; $m + 2t$ plus their sums by $s - 1$; and $2(m + 2t)$ plus their sums by $s - 2$. Hence sums by s of values of $f(x)$ include.

$$\begin{aligned} &0,\ t, \cdots, st; \quad m + 2t, \cdots, m + (s + 1)t; \\ &\qquad\qquad\qquad\qquad\qquad 2m + 4t, \cdots, 2m + (s + 2)t. \end{aligned}$$

To the last number in each of these three sets we add $0, 1, \cdots, e_s$; to all others we add only $0, 1, \cdots, t - 1$. We obtain the consecutive integers from 0 to $3m + 5t - 1$.

6. THEOREM 3. *If* $k = 0$, $m < (s - 2)t$, $s \geqq 5$, *then* $e_s = t - 1$.

No smaller e_s will serve since 0 is the only value $< t$ of $f(x)$. To $0, t, \cdots, (s - 4)t$, $m + 2t + (s - 5)t$, we add $0, 1, \cdots, t - 1$ and obtain $(s - 2)t > m$ integers R which are congruent modulo m to the consecutive integers from 0 to $(s - 2)t - 1$. The largest such R is $\rho = m + (s - 2)t - 1$.

First, let $m \geqq 2t$. Then $n \leqq 0$. As in § 5, every integer $N \geqq \rho + m$, is a sum of $t - 1$ numbers 0 or 1 and s values of $f(x)$, and the same is true if $N < \rho + m$.

---

[*] After the paper was in type, I found that it is better to take as e_4 the maximum R, add R at the end of (23), and choose R so that $tb + R \equiv A \pmod{m}$. A like remark applies to all theorems with $s > 4$ (§§ 6, 13, 14, 29, 30, 33, 34).

Second, let $t \leqq m < 2t$. Values of R include $0, 1, \cdots, t-1$ and the sum of t and $0, 1, \cdots, m-t \leqq t-1$. These $m+1$ values $\leqq m$ of R include a complete set of residues modulo m. We replace n by its value $2t-m$ in (24) and note that $D = A$. We get

$$F = m^2 w - 48mt^3 - 24m^2t^2 + 12m^3t + 6m^4, \quad w = A + 3m - 10t.$$

Let $A \geqq 7m + 10t$. Then $w \geqq 10m$, $m^2w^2 > 96m^4 \geqq 96mt^3$,

$$F \geqq 24mt^2(2t - m) > 0.$$

Hence every integer $N \geqq 8m + 10t$ is a sum of e_s numbers 0 or 1 and s values of $f(x)$. To verify this also for $N \leqq 9m + 11t$, note that the sums by four of the values (21) of $f(x)$ include

$$\begin{array}{ll} & 0, t, 2t, 3t, 4t; \quad m+3t, 4t, 5t; \quad 2m+4t, 5t, 6t; \\ (25) & 3m+5t, 6t, 7t; \quad 4m+6t, 7t, 8t; \quad 5m+7t, 8t; \quad 6m+7t, 8t, 9t; \\ & 7m+8t, 9t; \quad 8m+8t, 9t, 10t; \quad 9m+9t, 10t, \end{array}$$

whose successive gaps are t or $m-t < t$.

Finally, it remains only to prove the following theorem: *If $m < t$, every integer $N \geqq 0$ is a sum of five values of $f(x)$ and $t-1$ numbers* 0 *or* 1. We seek five integers $x \geqq 0$ such that

$$N = A + R, \quad A = \sum_4 f(x), \quad R = f(x) + r, \quad 0 \leqq r \leqq t-1.$$

To the value 0 of $f(x)$ we add all values of r. To the value t of $f(x)$ we add only the values $0, 1, \cdots, 2m-t-1 < t-1$ of r. We obtain the consecutive integral values $0, 1, \cdots, 2m-1$ of R. Hence we can choose $R < 2m$ so that A is congruent modulo $2m$ to any assigned integer.

If b is odd, we take $A \equiv tb \pmod{2m}$. Then (23) yields an integral value of $\frac{1}{2}(a-b)$ and hence an odd integral value of a. But if b is even, we take

$$b = 2B, \quad A \equiv 2tB + m(1-B) \qquad (\text{mod } 2m).$$

Then $A - 2tB + mB$ is the product of m by an odd integer ϵ. Write $a = 2\epsilon$. Then (23) holds. Since $a \equiv 2 \pmod 4$, Lemma 2 is applicable.

Since we may use either an even or odd value of b, we may employ § 4 with $d = 1$, $e_4 = 0$, $C = 0$, $D = A$. Then $P = n$ is positive. We have

$$2W = 6mA - 3m^2 + 3mn + 2n^2,$$

and find that

$$4F = m^2(2A + 3m - 3n - 8e_4)^2 + 12mn^2(m - n - 2e_4).$$

Replacing n by its value $2t - m$, we get

$$F = m^2w^2 + 6(-4mt^3 + 8m^2t^2 - 5m^3t + m^4), \quad w = A + 3m - 3t.$$

First, let $2m \geqq t$. If $A \geqq 5m + 9t$, then $w \geqq 8m + 6t$,

$$mw \geqq 8m^2 + 6mt \geqq 2t^2 + 3t^2, \quad m^2w^2 \geqq 25t^4 > 24mt^3,$$

since $t > m$. Then $F > 0$. Hence the theorem is proved for integers $N \geqq 7m + 9t$. To verify it for $N \leqq 9m + 11t$, we omit from (25) the initial values $m + 3t, 2m + 4t, \cdots, 9m + 9t$ from each block and see that all gaps are now t or $m < t$.

There remains the case $2m < t$. Then $3m^2t^2 > 6m^3t$. Hence $F > 0$ if $m^2w^2 \geqq 24mt^2 - 30m^2t^2$. Let $5m \geqq t$. Then $120m^2t^2 \geqq 24mt^3$. Thus $F > 0$ if $m^2w^2 \geqq (120 - 30)m^2t^2$ and hence if $w \geqq 10t$, i. e., if $A \geqq -3m + 13t$. This holds if $A \geqq -3m + 13t + 2(5n - t) = 7m + 11t$, and hence if $N \geqq 9m + 11t$. For the smaller N's, the theorem was verified above.

There remains the case $5m < t$. Then $m^2t^2 > 5m^3t$, so that $F > 0$ if $m^2w^2 \geqq 24mt^3 - 42m^2t^2$. To dispose of the subcase $pm \geqq t$, we proceed as follows. Then $24pm^2t^2 \geqq 24mt^3$. Thus $F > 0$ if $m^2w^2 \geqq q^2m^2t^2$, where $q^2 \geqq 24p - 42$, and hence if $A \geqq -3m + (q + 3)t$, or

$$N \geqq -m + (q + 3)t. \tag{26}$$

Since $2(pm - t) \geqq 0$, (26) holds if

$$N \geqq (2p - 1)m + (q + 1)t.$$

Take $p = 18$. The least q is 20. It remains to investigate the integers $N < 35m + 21t$. Those $\leqq 9m + 11t$ were discussed above. It suffices to employ the following sums by four of numbers (21):

$$9m + 10t;\ 14m + 10t,\ 11t,\ 12t;\ 19m + 12t,\ 13t,\ 14t;\ 24m + 14t,\ 15t,\ 16t;$$
$$29m + 16t,\ 17t;\ 34m + 17t,\ 18t;\ 38m + 18t,\ 19t;\ 43m + 19t,\ 20t,$$

whose gaps are t, $4m$ and $5m < t$.

There now remains the case $18m < t$. We now proceed by induction on l from the case $2lm < t$ to the case $2(l + 1)m < t$. Hence let $2lm < t$, $pm \geqq t$, where $p = 2(l + 1)$, and l is an integer $\geqq 9$. We may take $q = 2l + 3$ since $q^2 \geqq 24p - 42$ for $l \geqq 9$. Then

$$f(2l) = l(2l - 1)m + 2lt \geqq (2l + 6)t,$$

since

$$12(l + 1)m \geqq 6t, \quad l(2l - 1) > 12(l + 1),$$

for $l \geqq 9$. Hence $f(2l)$ exceeds the limit (26), whence the theorem holds for all integers $\geqq f(2l)$.

To verify it for integers $< f(2l)$, we apply the following result for $\lambda = 2l$, $2lm < t$.

LEMMA 3. *In a table of all sums by two of values of* $f(x)$, *all gaps are* $\leqq t$ *to* $f(\lambda)$ *if* $(\lambda - 1)m \leqq t$.

For a given g, the sums $gt + s_1m, gt + s_2m, \cdots$, arranged in ascending order of magnitude, will be said to form the *block of rank* g. That of rank 4 is composed of $4t + 2m$, $4t + 3m$, $4t + 6m$. The block of rank 2ξ is composed of

$$(27) \qquad f(\xi + \delta) + f(\xi - \delta) = 2\xi t + (\delta^2 + \xi^2 - \xi)m \quad (\delta = 0, 1, \cdots, \xi).$$

When δ is changed to $\delta + 1$, the increment to this sum is $(2\delta + 1)m$. Hence the largest number of the block is evidently $f(2\xi)$, whence $2\xi \leqq \lambda$. Hence the maximum gap within this block is $(2\xi - 1)m \leqq t$. The numbers of the block of rank $2\xi - 1$ are

$$(28) \quad f(\xi + \delta - 1) + f(\xi - \delta) = (2\xi - 1)t + (\delta^2 - \delta + \xi^2 - 2\xi + 1)m \quad (\delta = 1, \cdots, \xi).$$

When δ is changed to $\delta + 1$, the increment to this sum is $2\delta m$. Hence the largest number of this block is given by $\delta = \xi$ and is $f(2\xi - 1)$. Thus $2\xi - 1 \leqq \lambda$, and the maximum gap within this block is $2(\xi - 1)m \leqq t$.

Subtracting the largest number $f(2\xi - 1)$ in the block of rank $2\xi - 1$ from the least number $2f(\xi)$ in the block of rank 2ξ, we get $t - m(\xi - 1)^2$, which is either $\leqq 0$ or is positive and $\leqq t$. Hence either these blocks overlap or the gap between them is $\leqq t$. Likewise, if we subtract the largest number $f(2\xi - 2)$ of the block of rank $2\xi - 2$ from the least number of that of rank $2\xi - 1$, we get $t - (\xi - 1)(\xi - 2)m \leqq t$ for $\xi \geqq 2$.

7. THEOREM 4. *Let* $k = 0$ *and write* L *for the least* $s + e_s$ *for* $s \geqq 5$. *When* $t = 1$, $L = m + 2$ *if* $m \geqq 3$, $L = 5$ *if* $m = 1$ *or* 2. *When* $t > 1$, *write* $m = ut + v$, $0 \leqq v < t$; *then* $L = u + t + 1$ *if* $v = 0$, $u \geqq 3$, $L = u + t + 2$ *if* $v \geqq 1$, $u \geqq 3$, $L = t + 4$ *if* $u = 0$, 1, *or* 2.

First, let $m \geqq (s - 2)t$, $s \geqq 5$. By § 5,

$$s + e_s = m - s(t - 1) + 3t - 1.$$

If $t = 1$, this is $m + 2$ and is independent of s. But if $t > 1$, the sum is a minimum for the maximum permissible value $u + 2$ of s, since $u \geqq s - 2$. The minimum is $\lambda = v + u + t + 1$ if $u \geqq 3$.

Second, let $m < (s-2)t$, $s \geqq 5$. By § 6, $s + e_s = s + t - 1$ is least for the least permissible s. If $t = 1$, the least s is 5 if $m < 3$, but is $m + 3$ if $m \geqq 3$. Let $t > 1$. Then $u \leqq s - 3$. When $u = 0$, 1, or 2, the least permissible s is 5. When $u \geqq 3$, the least permissible s is $u + 3$, and the least $s + e_s$ is $u + t + 2$. This is $\geqq \lambda$ if $1 \leqq v$.

CASES IN WHICH $k = 0$, $4 + e_4 < L$, §§ 8-10.

8. When $t = 1$, it is known (*Bulletin*, I) that $4 + e_4$ is $m + 2$ if $m \geqq 2$, but is 4 if $m = 1$. Henceforth let $t > 1$.

THEOREM 5. *If $k = 0$, $t > 1$, $m > 2t$, then $4 + e_4 \geqq L$. But if $m = 2t$, then $4 + e_4 = t + 3 = L - 1$.*

No sum of four values (21) of $f(x)$ lies between $m + 5t$ and $2m + 4t$, since the value $3m + 3t$ exceeds the latter. Write $m = ut + v$. Then $u \geqq 2$,

$$4 + E_4(2m + 4t - 1) = 4 + m - t - 1 = ut + v - t + 3 \equiv \mu.$$

First, let $v = 0$, $u \geqq 3$. Then $\mu - L = (u-2)(t-1) > 0$.

Second, let $v \geqq 1$, $u \geqq 3$. Then $\mu - L = (u-2)(t-1) + v - 1 > 0$.

Third, let $u = 2$. Then $\mu - L = v - 1$. There remains only the case $u = 2$, $v = 0$, whence $m = 2t$. Then $f(x) = tx^2$, whence every multiple of t is a sum of four values of $f(x)$. Evidently $e_4 = t - 1$.

9. THEOREM 6. *If $k = 0$, $m \leqq t$, then $e_4 = t - 1$.*

If $m = t$, $f(x)$ is the product of the general triangular number by t. Hence every multiple of t is a sum of four (in fact, three) values of $f(x)$. Hence $e_4 = t - 1$.

For $k = 0$, (4), (7), (9), and (11) give

$$n = 2t - m, \quad C = 0, \quad D = A = m(a-b)/2 + tb + r, \tag{29}$$

$$U = 24mA + 36t^2 - 12tm - 15m^2, \tag{30}$$

$$V = 2mA + 4t^2 - 6tm + m^2 + 2m. \tag{31}$$

For convenience write $d = 2\delta$. By (16) and (18), we get

$$\begin{gathered} F = m^2\omega^2 - Jt^3m - Kt^2m^2 - Ltm^3 - Mm^4 \\ + 24t^2m + (96\delta - 72)tm^2 + (96\delta^2 - 144\delta + 54)m^3, \\ \omega = A - 14\delta t - (14\delta^2 - 21\delta + 4)m + 4, \quad J = 48\delta + 24, \\ K = 240\delta^2 - 120\delta - 72, \quad L = 384\delta^3 - 624\delta^2 + 180\delta + 54, \\ M = 192\delta^4 - 576\delta^3 + 540\delta^2 - 162\delta. \end{gathered} \tag{32}$$

(I) Let $2m \geqq t > m$. Since the values $0, 1, \cdots, t-1$ include a complete set of residues modulo m, the same is true of the values of $tb + r$ for any fixed b. There will exist an odd b between its limits if $d = 2$. We take $tb + r \equiv A \pmod{m}$. Then (29) determines an integer $(a-b)/2$ and hence an odd integer a.

Since $\delta \geqq 1$, the last three terms of F in (32) are positive. Here $\delta = 1$. Then $L = M = -6$. Hence $F > 0$ if

$$m^2\omega^2 \geqq 72t^3m + 48t^2m^2, \quad \omega = A - 14t + 3m + 4.$$

Multiply $2m \geqq t$ by $72t^2m$. Thus $F > 0$ if $m^2\omega^2 \geqq (144 + 48)t^2m^2$ and hence if $\omega \geqq 14t$, and therefore if

$$A \geqq 13(2m - t) + 28t - 3m = 23m + 15t.$$

But the gaps are $\leqq t$ to $23m + 15t$ in a table T_0 of sums by four of values of $f(x)$. Such a table is derived from Table I in *Bulletin*, I, by multiplying all terms free of m by t.

(II) Let $2m < t$. We may employ a single odd b as in (I) or a single even b. For, if $b \equiv 2 \pmod{4}$, $tb + r \equiv A \pmod{2m}$ holds for one of the available values $0, 1, \cdots, t-1$ of r. Then (29) determines an even integer $(a-b)/2$, whence $a \equiv 2 \pmod{4}$, and Lemma 2 is applicable. But, if $b \equiv 0 \pmod{4}$, $tb + r \equiv A + m \pmod{2m}$ holds for one of the available values of r. Then (29) determines an odd integer $(a-b)/2$, whence $a \equiv 2 \pmod{4}$.

Hence $d = 1$, $\delta = \frac{1}{2}$. By (32),

$$F = m^2\omega^2 - 48t^3m + 72t^2m^2 - 36tm^3 + 6m^4 + 6m^3 + 24tm(t - m),$$

where $\omega = A - 7t + 3m + 4$. Since $18t^2m^2 - 36tm^3 = 18tm^2(t - 2m) > 0$, $F > 0$ if $m^2\omega^2 \geqq 48t^3m - 54t^2m^2$.

Consider the case $5m \geqq t$. Then $240t^2m^2 \geqq 48t^3m$ and the preceding condition holds if $m^2\omega^2 \geqq ht^2m^2$, $h = 240 - 54 = 186 < 14^2$ and hence if $\omega \geqq 14t$. Thus $F > 0$ if

$$A \geqq 6(5m - t) + 21t - 3m - 4 = 27m + 15t - 4.$$

For $A \leqq 27m + 16t$, $E_4(A) \leqq t - 1$ by table T_0.

(III) There remains the case $5m < t$. Then $8t^2m^2 > 40tm^3$, and $F > 0$ if $m^2\omega^2 \geqq 48t^3m - 64t^2m^2$. Consider the subcase $pm \geqq t$. Then $48pt^2m^2 \geqq 48t^3m$ and $F > 0$ if $\omega \geqq qt$, where $q^2 \geqq 48p - 64$, and hence if

$$A \geqq a, \quad a = (q + 7)t - 3m - 4. \tag{33}$$

Take $p = 34$. The least q is 40. Then (33) holds if

$$A \geqq 6(34m - t) + a = 201m + 41t - 4.$$

For smaller A's, $E_4(A) \leqq t - 1$ by table T_0.

(IV) For the final case $34m < t$, we proceed by induction on l from the case $2lm < t$ to the case $pm < t$, $p = 2(l+1)$. Hence let $2lm < t$, $pm \geqq t$, $l \geqq 17$. We may take $q = 2l + 7$ since $q^2 \geqq 96l + 32$. Then $f(2l) \geqq (2l + 14)t > a$. In fact,

$$l(2l - 1)m > 14(2l + 2)m \geqq 14t,$$

since $(4l - 29)^2 \geqq 39^2 > 29^2 + 8 \cdot 28$. This proves the theorem for numbers $\geqq f(2l)$. It follows from Lemma 3 for numbers $\leqq f(2l)$.

10. THEOREM 7. *If* $k = 0$, $t < m$, $7t \geqq 4m$, *then* $e_4 = t - 1$. *If* $7t < 4m < 8t$, *then* $e_4(A) = t - 1$ *for* $A \leqq 400m + 58t$.

First, let $t < m$, $2m \leqq 3t$. We assign to r the values $0, 1, \cdots, t-1$, and to b two odd values β, $\beta + 2$. Then the residues modulo m of the values of $tb + r$ include $t\beta + j$ $(j = 0, 1, \cdots, m-1)$ and hence a complete set of residues. In fact, $t(\beta + 2) \equiv t\beta + h \pmod{m}$, where $h = 2t - m$ is positive and $< t$. The values of r and $h + r$ together include $0, 1, \cdots, 3t - m - 1 \geqq m - 1$. The desired two odd values of b will occur between the limits for b if their difference exceeds $d = 4$. Since $\delta = 2$, $F > 0$ by (32) if

$$m^2\omega^2 \geqq 120t^3m + 648t^2m^2 + 990tm^3 + 300m^4,$$

where $\omega = A - 28t - 18m + 4$. Multiply $m > t$ by $120t^2m$, $3t \geqq 2m$ by $495\ tm^2$, and $9t^2 \geqq 4m^2$ by $75m^2$. Hence $F > 0$ if

$$m^2\omega^2 \geqq gt^2m^2, \quad g = 120 + 648 + 1485 + 675 = 2928 < 55^2,$$

and hence if $\omega \geqq 55t$. This holds if

$$A \geqq 57(m - t) + 83t + 18m = 75m + 26t.$$

Here $n > 0$ and the reduced minor conditions (20) hold if $A \geqq 4t$. Using only $m < 2t$, we see that all gaps are $\leqq t$ down to $75m + 25t$ in table T_0.

Second, let $2m > 3t$, $7t \geqq 4m$. Give to b the odd values β, $\beta + 2$, $\beta + 4$, $\beta + 6$. Then the values of $tb + r$ are congruent modulo m to the sum of $t\beta$ and the values of r, $h + r$, $2h + r$, $3h + r$, which include the consecutive numbers from 0 to $3h+t-1 \geqq m-1$. Thus $d=8$, $\delta=4$. By (32), $F>0$ if

$$m^2\omega^2 \geqq 216t^3m + 3288t^2m^2 + 15366tm^3 + 20280m^4,$$

where $\omega = A - 56t - 144m + 4$. Multiply $2m > 3t$ by $72t^2m$, $7t \geqq 4m$ by $3842tm^2$, and $49t^2 \geqq 16m^2$ by $1268m^2$. Hence $F>0$ if $\omega \geqq 305t$, since

$$305^2 > 92458 = 144 + 3288 + 26894 + 62132.$$

Thus $F>0$ if

$$A \geqq 361t + 144m + 103(2m-3t) = 350m + 52t.$$

When $m < 2t$, we saw that all gaps are $\leqq t$ in T_0 to $75m+25t$; this is true also to $115m+27t$. From thence to $354m+55t$, we employ only $m<3t$ and find that all gaps are $\leqq t$ in T_0 and the following extension of it.

TABLE T_0.

All Sums of Four Polygonal Numbers.

$200m+25$—27, 29, 30, 32—35, 37—42; $201m+27$, 28, 31—37, 39—42; $202m+28$—42; $203m+28$—33, 35—41; $204m+30$, 32—42; $205m+26$—28, 31—37, 39—42; $206m+28$—31, 33—35, 37—42; $207m+28$—33, 35—41; $208m+29$—42; $209m+31$—37, 39—42; $210m+21$—24, 30—39, 41—43; $211m+23$—25, 27—33, 35—41, 43; $212m+25$, 26, 29, 30, 32—43; $213m+24$—27, 31—37, 39—43; $214m+26$, 27, 30—43; $215m+28$, 31—33, 35—41, 43; $216m+25$—31, 33—43; $217m+27$—29, 31—37, 39—43; $218m+28$—30, 33—43; $219m+28$—33, 35—41, 43; $220m+26$—28, 30, 32—44; $221m+28$, 29, 31—37, 39—44; $222m+29$—31, 33—35, 37—44; $223m+29$—31, 35—41, 43, 44; $224m+31$—44; $225m+27$—29, 31—37, 39—44; $226m+29$—39, 41—44; $227m+31$—33, 35—41, 43, 44; $228m+30$, 31, 33—44; $229m+32$—37, 39—44; $230m+31$, 32, 34—44; $231m+22$—25, 28—33, 35—41, 43—45; $232m+24$—26, 30, 31, 33—45; $233m+26$, 27, 32, 34—37, 39—45; $234m+25$—28, 31, 32, 34—43, 45; $235m+27$, 28, 30—33, 36—41, 43—45; $236m+29$, 32—39, 41—45; $237m+26$—29, 31—37, 39—45; $238m+28$—31, 33—35, 37—45; $239m+30$—32, 35—41; 43—45; $240m+29$—45; $241m+27$—29, 31—37, 39—45; $242m+29$, 30, 34—39, 41—46; $243m+30$—32, 35—41, 43—46; $244m+30$—46; $245m+31$—37, 39—45; $246m+28$—46; $247m+30$—33, 35—41, 43—46; $248m+32$—35, 37—46; $249m+31$—34, 36, 37, 39—45; $250m+33$—43, 45, 46; $251m+32$, 33, 35—41, 43—46; $252m+29$—46; $253m+23$—26, 31, 32, 34—37, 39—45, 47; $254m+25$—27, 33, 35, 37—47; $255m+27$, 28, 31—33, 35—41, 43—47; $256m+26$—29, 32—47; $257m+28$, 29, 34—37, 39—45, 47; $258m+30$, 33—39, 41—47; $259m+27$—32, 35—41, 43—47; $260m+29$—33, 37—47; $261m+31$, 34—37, 39—45, 47; $262m+30$—47; $263m+28$—30, 32, 35, 36. 38—41, 43—47; $264m+30$, 31, 37—48; $265m+31$—37, 39—45, 47, 48; $266m+31$—43, 45—48; $267m+31$—33, 35—41, 43—48; $268m+29$—31, 33—48; $269m+31$—37, 39—45, 47, 48; $270m+33$—35, 37—48; $271m+32$, 33, 35—41, 43—48; $272m+34$—36, 38—48; $273m+33$—37, 39—45, 47, 48; $274m+30$—39, 41—48; $275m+32$, 33, 35—41, 43—48; $276m+24$—27, 32—43, 45—49; $277m+26$—28, 33—37, 39—45, 47—49; $278m+28$, 29, 34—47, 49; $279m+27$—30, 35—41, 43—49; $280m+29$, 30, 34—49; $281m+31$—37, 39—45, 47—49; $282m+28$—31, 33, 34, 36—43, 45—49; $283m+30$—32, 35—41, 43—49; $284m+32$, 34—38, 40—49; $285m+31$—33, 36, 37, 39—45, 47—49; $286m+29$—31, 33—35, 37—49; $287m+31$, 32, 35—41, 43—49; $288m+32$—50; $289m+32$—37, 39—45, 47—50; $290m+34$, 35, 37—39, 41—50; $291m+30$—32, 35—41, 43—49; $292m+32$—50; $293m+34$, 35, 37, 39—45, 47—50; $294m+33$, 34, 36, 38—47, 49, 50; $295m+35$—41, 43—49; $296m+34$, 35, 37—50; $297m+31$—36, 39—45, 47—50; $298m+33$—43, 45—50; $299m+35$—38, 40, 41, 43—49; $300m+25$—28, 34, 35, 37—39, 41—51; $301m+27$—29, 35—37, 39—45, 47—51; $302m+29$, 30, 37—51; $303m+28$—31, 35—41, 43—49, 51; $304m+30$—34, 37—51; $305m+32$, 34, 35, 39—45, 47—51; $306m+29$—32, 36—39, 41—51; $307m+31$—33, 35—49, 51; $308m+33$—51; $309m+32$—37, 39—45, 47—51; $310m+30$—32, 34, 36—47, 49—51; $311m+32$, 33, 37—41, 43—49, 51; $312m+33$—35, 37—52; $313m+33$—36, 39—45, 47—52; $314m+35$, 37—43, 45—52; $315m+31$—33, 36—41, 43—49, 51, 52; $316m+33$—36, 38—52; $317m+35$, 36, 40—45, 47—52; $318m+34$, 35, 37—52; $319m+35$—41, 43—49, 51, 52; $320m+35$—38, 40—47, 49—52; $321m+32$—37, 39—45, 47—52; $322m+34$—39, 41—52; $323m+36$, 38—41, 43—49, 51, 52; $324m+35$—52; $325m+26$—29, 36, 37, 39—45, 47—53; $326m+28$—30, 37—39, 41—47, 49—51, 53; $327m+30$, 31, 36—41, 43—49, 51—53; $328m+29$—35, 38—53; $329m+31$, 32, 35, 36, 40—45, 47—53; $330m+33$, 36—53; $331m+30$—33, 35—40, 43—53; $332m+32$—34, 37—53; $333m+34$, 39—53 $334m+33$—35, 37—53; $335m+31$—33, 35, 39—41, 43—49, 51—53; $336m+33$—53; $337m+34$—37, 39—45, 47—54; $338m+34$—36, 38, 39, 41, 42, 44—54; $339m+36$—38, 40, 41, 43—54; $340m+32$—34, 37, 39, 41—43, 45—54; $341m+34$—37, 40—45, 47—53; $342m+36$—54; $343m+35$—41, 43—49, 51—54; $344m+37$—54, $345m+35$—37, 39—54; $346m+33$—43, 45—54; $347m+35$, 36, 38—49, 51—54; $348m+37$—54; $349m+36$, 37, 39—45, 47—54; $350m+37$, 38, 41—55; $351m+27$—30, 38—41, 43—55; $352m+29$—31, 37—55; $353m+31$, 32, 34—36, 39, 40, 42—45, 47—55; $354m+30$—33, 36—39, 41—55.

Third, let $7t < 4m < 8t$. Is $e_4 = t - 1$? Let l be the least integer for which $t(2l-1) \geqq ml$. By using the odd values $\beta, \beta+2, \cdots, \beta+2l-2$ of b, we get $d = 2l$, and the method of § 21 gives the same d. The resulting limit for A is beyond our table. An incomplete extension of the latter to $400m + 58t$ was made quickly by adding 0, 1, and* occasionally $m + 2$ to the tabulated sums by three of values of $f(x)$. When $m < 3t$, the gaps are $\leqq t$ in this extension.

---

[*] Also $3m+3$ to $383m+40$ and $397m+49$, 50; and $6m+4$ to $386m+44$, $394m+50$.

All the results in §§ 8-10 may be combined into

THEOREM 8. *Let $k = 0$ and L be the least $s + e_s$ for $s \geqq 5$. If $m > 2t$, then $4 + e_4 \geqq L$. But $4 + e_4 = t + 3 = L - 1$ if $m = 2t$ or $4m \leqq 7t$ and doubtless also if $7t < 4m < 8t$.*

11. THEOREM 9. *If $k = 0$, $m > t$, $3 + e_3$ is not less than the minimum $s + e_s$ for $s \geqq 4$. But if $m = t$, $e_3 = t - 1$, while if $m < t$, $e_3(A) \leqq t - 1$ for $A \leqq 400m + 58t$.*

The sums by three of the values of $f(x)$ are

$$\begin{gathered} 0,\ t,\ 2t,\ 3t;\ m + 2t,\ m + 3t,\ m + 4t;\ 2m + 4t,\ 2m + 5t; \\ 3m + 3t, \cdots, 3m + 6t;\ 4m + 5t,\ 4m + 6t;\ 5m + 7t; \\ 6m + 4t, \cdots, 6m + 7t;\ 7m + 6t, \cdots;\ 21m + 7t, \cdots, 21m + 12t; \\ 22m + 9t,\ 22m + 10t,\ 22m + 12t,\ 22m + 13t; \\ 23m + 11t,\ 23m + 13t;\ 24m + 10t, \cdots. \end{gathered}$$

If $m \geqq 3t$, there is a gap $m + t$ from $4m + 6t$ to $5m + 7t$, which is less than all later entries. Thus $3 + E_3(5m + 7t - 1) = m + t + 2 \equiv \alpha$. If $t = 1$, then $m \geqq 3$, $\alpha = L + 1$, for $L = m + 2$ of Theorem 4. If $t > 1$, write $m = ut + v$, $0 \leqq v < t$, whence $u \geqq 3$. Since $m > u$, $\alpha > L$.

Next, let $2t < m < 3t$, whence $u = 2$, $t > 1$. There is a gap $2m - 2t$ from $4m + 6t$ to $6m + 4t$, and $3 + E_3 = 2m - 2t + 2 > L = t + 4$.

Next, let $t < m \leqq 2t$. There is a gap m from $21m + 12t$ to $v = 22m + 12t$. Consider $\beta = 3 + E_3(v - 1) = m + 2$. If $t = 1$, then $m = 2$, and $\beta = 4 = 4 + e_4$. Let $t > 1$. If $m > t + 1$, then $\beta \geqq L = t + 4$. There remains the case $m = t + 1$. Then $4m \leqq 7t$ and $\beta = 4 + e_4$ by Theorem 7.

When $m = t$, $f(x)$ is the product of t by the triangular number $\frac{1}{2}(x^2 + x)$. Hence every multiple of t is a sum of three values of $f(x)$, so that $e_3 = t - 1$.

The methods of this paper do not apply directly to the remaining case $m < t$. An extension of the above table of sums by three was used to verify that $e_3(A) \leqq t - 1$ for $A \leqq 400m + 50t$.

12. *Comparison of $2 + e_2$ with $s + e_s$.*

First, let $m \geqq t$. There is a gap from $39m + 12t$ to $a = 42m + 13t$ in the table of sums of two values of $f(x)$, and $\sigma = 2 + E_2(a - 1) = 3m + t + 1$. In all cases, $\sigma \geqq L$ of Theorem 4. Also, $\sigma > t + 3$ in the last part of Theorem 8. If $m = t$, $\sigma > 3 + e_3 = t + 2$ of Theorem 9.

Second, let $2m \geqq t > m$. There is a gap from $3m + 4t$ to $4m + 5t$, whence $\tau = 2 + E_2 = m + t + 1$. If $m \geqq 2$, $\tau \geqq t + 3$ of Theorem 8. In the remaining case $m = 1$, $t = 2$, there is a gap 4 from $27m + 11t = 29m + 10t = 49$ to $31m + 11t = 53$, and $2 + E_2(52) = 5 = t + 3$.

Third, let $3m \geqq t > 2m$. There is a gap from $8t + 21m$ to $9t + 22m$, and $2 + E_2 = t + m + 1 \geqq t + 3$ unless $m = 1$, whence $t = 3$. In the latter case, there is a gap 5 from $70 = 11t + 37m = 12t + 34m$ to $75 = 10t + 45m = 12t + 39m = 13t + 36m$, and $2 + E_2(74) = 6 = t + 3$.

Fourth, let $4m \geqq t > 3m$. There is a gap from $9t + 18m$ to $9t + 22m$, and $2 + E_2 = 4m + 1 \geqq t + 3$ unless $4m = t$ or $t + 1$. When $t = 4m$, the (first) gap $6m$ is from $394m$ to $400m$, and $2 + E_2 = 6m + 1 \geqq t + 3$. Finally, let $4m = t + 1$. Since $4m \geqq t$ and $7m \leqq 2t$, there is a gap $5m$ from $16t + 60m$ to $16t + 65m$. Here $2 + E_2 = 5m + 1 \geqq t + 3$.

Fifth, let $5m \geqq t > 4m$. There is a gap $5m$ from $10t + 24m$ to $10t + 29m$, and $5m + 1 \geqq t + 3$ unless $5m = t$ or $t + 1$. When $t = 5m$, the (first) gap $8m$ is from $962m$ to $970m$. Finally, let $5m = t + 1$. Since $5m \geqq t$, $9m \leqq 2t$, there is a gap $6m$ from $17t + 70m$ to $17t + 76m$, and $6m + 1 \geqq t + 3$.

Sixth, let $6m \geqq t > 5m$. Write $t = 5m + g$, $0 < g \leqq m$. If $m \leqq 2g$, there is a gap $6m$ from $\alpha = 15t + 55m$ to $\beta = 15t + 61m$, and $6m+1 \geqq t+3$ unless $6m = t$ or $t + 1$. If $m > 2g$, $\gamma = 13t + 66m$ is the only entry between α and β, so that there is a gap $2t - 5m = 5m + 2g$ from γ to β. It is $\geqq t + 2$ unless $g = 1$. Hence there remain only three cases:

$$(34) \qquad t = 6m; \quad t = 6m - 1,\ m > 1; \quad t = 5m + 1,\ m > 2.$$

When $t = 6m$, the first gap $> 6m$ is from $952m = 41t + 706m$ to $960m = 49t + 666m$. In the last two cases (34), we have found a gap $t + 1$, whence $2 + E_2 = t + 2$. That this is the value of $3 + e_3$ is plausible[*] by the last part of Theorem 9.

Seventh, let $7m > t > 6m$. Write $t = 6m + g$. The only possible entries between $\alpha = 32t + 249m$ and $\beta = 32t + 256m$ are $\gamma = 29t + 268m$, $\delta = 25t + 300m$, $\epsilon = 28t + 282m$. If $2m \geqq 7g$ there is a gap $6m + 3g$

---

[*] If we do not admit this, we need gaps $\geqq t + 2$. There is a gap $6m - 3g$ from $17t + 70m$ to $14t + 91m$; in the third case (34), it is $\geqq t + 2$ if $m \geqq 6$. For $m = 3$, $t = 16$, the first gap 18 is from $1664 = 32t + 384m$ to $1682 = 29t + 406m = 38t + 358m$. For $m = 4$, $t = 21$, the first gap $\geqq 23$ is from $3516 = 40t + 669m$ to $3540 = 48t + 633m$. For $m = 5$, $t = 26$, the gaps $> t$ below 7000 are $28 = 4413 - 4385$, $29 = 4847 - 4818$, $41 = 4899 - 4858$, $36 = 5397 - 5361$, $33 = 5726 - 5693$, $30 = 5799 - 5769$, $30 = 6503 - 6473$, $31 = 6582 - 6551$.

$\geqq t+2$ from γ to β. If $2m < 7g$, $m \geqq 3g$, there is a gap $8m-4g \geqq t+1$ from γ to δ. If $m < 3g$, there is a gap $z = 9m - 7g$ from α to δ, and $z > t$ if $3m > 8g$. Next, the only possible entries between $a = 36t + 315m$ and $b = 36t + 322m$ are $c = 32t + 340m$ and $d = 30t + 354m$. If $m \leqq 2g$, there is a gap $7m > t$ from a to b. If $m > 2g$, there is a gap $w = 4m + 6g$ from d to b, and $w > t$ if $5g > 2m$. We have now found a gap $> t$ except when both $3m \leqq 8g$ and $5g \leqq 2m$. These imply $2g \leqq m \leqq 6g$, which are the conditions for a gap $6m + 3g > t$ from $39t + 601m$ to $42t + 589m$.

THEOREM 10. *At least when $t < 7m$, $2 + e_2$ is not less than the minimum $s + e_s$ for $s \geqq 3$.*

PART II. THE CASE $k > 0$, $m \geqq 2t$, §§ 13-27.

13. THEOREM 11. *If $k > 0$, $s \geqq 5$, $m \geqq (s+2)t$, then* $e_s = m - (s+1)t - 1$.

No smaller value of e_s will serve since no number between st and $m-t$ is a sum of s values of $q(x) = f(x-k)$ for integers $x \geqq 0$. To show that $0 = f(0)$ and $t = f(1)$ are the only such values $< m - t = f(-1)$, we recall from § 2 that

$$(35) \qquad f(-p) > f(-1), \quad f(p) > f(1) \quad \text{if } p > 1.$$

If N is any integer $\geqq 0$, it remains to prove the existence of s integers $x \geqq 0$ such that

$$(36) \qquad N = r + \sum_{s} q(x), \quad 0 \leqq r \leqq e_s.$$

We shall first prove this for $k = 1$. Then the summands are chosen from $f(-1)$ and $f(z)$, $z \geqq 0$. Since these are values of $f(y)$ when $y \geqq -k$, (36) will hold also for $k > 1$. Write $N = A + R$, where

$$(37) \qquad A = \sum_{4} q(x), \quad R = r + \sum_{s-4} q(x), \quad 0 \leqq r \leqq e_s.$$

We can find a value of R congruent to any assigned integer modulo m such that $R \leqq \rho = 11m - t - 1$.

If $s = 5$, we employ $m - t$ and the first six values

$$(38) \qquad 0,\ t,\ \kappa = m + 2t,\ \lambda = 3m + 3t,\ \mu = 6m + 4t,\ \nu = 10m + 5t$$

of $f(x)$. To $10m + 5t$ we add $0, 1, \cdots, e_5 = m - 6t - 1$. To the remaining six we add only $0, 1, \cdots, t - 1 \leqq e_5$. We get m integers $R \leqq \rho$ which are congruent modulo m to the consecutive integers from $-t$ to $m - t - 1$ inclusive.

If $s \geqq 6$, we add $0, 1, \cdots, t-1 \leqq e_s$ to the $s-3$ sums by $s-4$ of 0 and t, and also to $(s-5)t+\kappa$, $(s-6)t+2\kappa$, $2(m-t)$, and $m-t$. But we add $0, 1, \cdots, e_s$ to $\phi=(s-6)t+\kappa+\lambda$. We obtain m integers R which are congruent modulo m to the consecutive integers from $-2t$ to $m-2t-1$. The maximum such R is $\phi+e_s=5m-2t-1<\rho$.

We shall treat the first equation (37) by the method of § 4 with $e_4=0$. By (4) with $k=1$, and (7),

$$n=2t-3m, \quad C=m-t, \tag{39}$$

$$D=A-4C=m(a-3b)/2+tb. \tag{40}$$

Let $d=2$. Then there is an odd value of b between the limits (10) and (12) on b. We proved that we can choose $R \leqq \rho$ so that $A=N-R$ is congruent modulo m to any assigned integer. As the latter we take $4C+tb$. Then (40) determines an integer $(a-3b)/2$ and hence an odd integer a, which is positive since $4a>b^2$. By (16), $P=2t-m<0$. Hence it remains only to satisfy the minor conditions in § 4. In (9), (19), and (14), we replace n by its value (39) and D by $A-4m+4t$, and get

$$U=24mA-63m^2+12mt+36t^2>0 \quad \text{if } A \geqq 3m, \tag{41}$$

$$16V-U=8mA+79m^2-76mt+28t^2-32me_4>0 \quad \text{if } A \geqq 0, \tag{42}$$

$$3A \geqq 11m-4t-4t^2/m. \tag{43}$$

The last holds if $A \geqq 4m-3t$. Hence (36) holds for every $N \geqq 15m-4t-1$.

We next verify (36) when $N \leqq 15m+t-1$. We employ the abbreviations (38) and $\delta=m-t$. Consider a sum Σ of p of $\delta, \kappa, \lambda, \mu, \nu$, where $p \leqq 5$. To Σ add the sums by $s-p$ of 0 and t. Hence $\Sigma, \Sigma+t, \Sigma+2t, \cdots, \sigma=\Sigma+(s-p)t$ are sums by s of values of $q(x)$. To each we add $0, 1, \cdots, t-1 \leqq e_s$. We get the consecutive integers from Σ to $M=\Sigma+(s+1-p)t-1$ inclusive. We readily verify the following table:

Σ	$M+1$	Σ	$M+1$
0	$(s+1)t^*$	$\delta+\lambda$	$4m+(s+1)t$
δ	$m+(s-1)t$	$\kappa+\lambda$	$4m+(s+4)t^*$
κ	$m+(s+2)t^*$	$2\delta+\lambda$	$5m+(s-1)t$
2δ	$2m+(s-3)t$	$\delta+\kappa+\lambda$	$5m+(s+2)t$
$\delta+\kappa$	$2m+st$	$2\kappa+\lambda$	$5m+(s+5)t^*$
2κ	$2m+(s+3)t^*$	$2\delta+\kappa+\lambda$	$6m+st$
$2\delta+\kappa$	$3m+(s-2)t$	μ	$6m+(s+4)t$
λ	$3m+(s+3)t$	2λ	$6m+(s+5)t^*$
3κ	$3m+(s+4)t^*$	$\delta+\mu$	$7m+(s+2)t$

Σ	$M+1$
$\kappa+\mu$	$7m+(s+5)t^*$
$2\delta+\mu$	$8m+st$
$\delta+\kappa+\mu$	$8m+(s+3)t$
$2\kappa+\mu$	$8m+(s+6)t^*$
$2\delta+\kappa+\mu$	$9m+(s+1)t$
$2\delta+\kappa+2\lambda$	$9m+(s+2)t$
$\delta+2\kappa+\mu$	$9m+(s+4)t$
3λ	$9m+(s+7)t^*$
ν	$10m+(s+5)t$
$\kappa+\lambda+\mu$	$10m+(s+7)t^*$
$\delta+\nu$	$11m+(s+3)t$
$\kappa+\nu$	$11m+(s+6)t$
$2\kappa+\lambda+\mu$	$11m+(s+8)t^*$
$\delta+\kappa+\nu$	$12m+(s+4)t$
2μ	$12m+(s+7)t^*$
$2\delta+\kappa+\nu$	$13m+(s+2)t$
$\delta+2\mu$	$13m+(s+5)t$
$\kappa+2\mu$	$13m+(s+8)t^*$
$2\delta+2\mu$	$14m+(s+3)t$

Except for the cases marked *, $M+1$ is $\geqq$ the Σ in the next line, since $s \geqq 5$. For the cases marked *, $m-(s+2)t+M+1$ is $\geqq$ the Σ in the next line. This addition to $M+1$ is permissible since we may add e_s instead of $t-1$ to σ. Making the same addition to the final entry, we see that (36) holds for every $N \leqq 15m+t-1$. This completes the proof of Theorem 11.

14. THEOREM 12. *If $k>0$, $s \geqq 5$, $2t \leqq m < (s+2)t$, then $e_s = t-1$.*

No smaller value of e_s will serve since (35) and $f(-1)=m-t \geqq t$ show that no value of $q(x)$ lies between 0 and t.

We can find a small value of R in (37) which is congruent modulo m to any assigned integer.

If $s=5$, we add $0, 1, \cdots, t-1$ to $m-t$ and the six numbers (38). We get $7t$ integers R congruent modulo m to the consecutive integers from $-t$ to $6t-1$ inclusive. They include a complete set of residues modulo $m<7t$. The maximum such R is $10m+6t-1$. If $s \geqq 6$, we proceed as in § 13, except that we add only $0, 1, \cdots, t-1$ to ϕ. We get $(s+2)t>m$ integers R congruent modulo m to the consecutive integers from $-2t$ to $st-1$ inclusive. The maximum such R is $4m+st-1$.

It suffices to prove (36) for $k=1$. We study (37_1) by the method of § 4 with $e_4=0$, $d=2$. We have (39)—(43). Also F is given by (24). Since n is negative, $F>0$ for every D. Since $m \geqq 2t$, (43) holds if $A \geqq 4m-2t$. Adding to this the above maximum of R, we conclude that (36) holds if $N \geqq$ the larger of $14m+4t-1$ and $8m+(s-2)t-1$.

For the remaining values of N we proceed as at the end of § 13. Each entry $M+1$ marked by a * in the table exceeds the Σ in the next line since $(s+2)t>m$.

15. THEOREM 13. *Let $k>0$, $m \geqq 2t$, and write L for the least $s+e_s$ for $s \geqq 5$. When $t=1$, $L=m-2$ if $m \geqq 7$, $L=5$ if $2 \leqq m < 7$. When*

$t > 1$, write $m = ut + v$, $0 \leqq v < t$, whence $u \geqq 2$. Then $L = u + t - 3$ if $u \geqq 7$, $v = 0$; $L = u + t - 2$ if $u \geqq 7$, $v \geqq 1$; $L = t + 4$ if $2 \leqq u \leqq 6$.

First, let $m \geqq (s+2)t$, $s \geqq 5$. By Theorem 11 of § 13, the least $s + e_s$ is $m - 2$ if $t = 1$, $m \geqq 7$; but is $u + v + t - 3$ *if* $t > 1$, $u \geqq 7$, since $u \geqq s + 2 \geqq 7$ and the maximum s is $u - 2$.

Second, let $2t \leqq m < (s+2)t$, $s \geqq 5$. By Theorem 12, $e_s = t - 1$. If $t = 1$, the least $s + e_s = s$ is 5 if $2 \leqq m < 7$, but is $m - 1$ if $m \geqq 7$. If $t > 1$, then $s \geqq u - 1$ and the least $s + e_s$ is $u - 1 + t - 1$ if $u \geqq 7$, but is $5 + t - 1$ if $u \leqq 6$.

Note that $m < 7t$ if and only if $u \leqq 6$. Then $4 + e_4 \geqq L$ if and only if $e_4 \geqq t$ and hence if and only if there is a gap $> t$ in table T_k of sums by four of values of $f(x-k)$ for $x \geqq 0$.

16. *Case* $s = 4$, $t = 1$. First, let $m \geqq 7$. Then 0 and 1 are the only values $< m - 1$ of $f(x-k)$. Hence to obtain the sum $m - 2$, we must employ $m - 2$ values 1. Thus $E_4(m-2) = m - 6$, and $m - 2$ is the minimum $s + e_s$ for $s \geqq 4$. Second, let $m < 7$. We employ the following results proved in *Bulletin*, II. If $k = 1$, $4 + e_4$ is 6 if $m = 6$, 5 if $m = 4$ or 5, 4 if $m = 3$. If $k \geqq 2$, $4 + e_4$ is 5 if $m = 6$, 4 if $m = 3$, 4, or 5. When $m = 2$, $f(x) = x^2$ and $e_4 = 0$. This proves

THEOREM 14. *If $k > 0$, $t = 1$, the minimum of $s + e_s$ for $s \geqq 4$ is $m - 2$ when $m \geqq 7$, 5 when $m = 6$, 4 when $m = 2$ or 3, but when $m = 4$ or 5, it is 5 if $k = 1$, 4 if $k \geqq 2$.*

17. *Case* $s = 4$, $t > 1$, $k = 1$, $m \geqq 2t$. A table T_1 of all sums by four of values of $f(x-1)$ for integers $x \geqq 0$ is derived from Table II of *Bulletin*, II, by multiplying the terms free of m by t. We employ the notations of Theorem 13. If $u \geqq 6$, there is a gap $m - 5t$ from $4t$ to $m - t$, whence

$$g = 4 + E_4(m - t - 1) = ut + v - 5t + 3.$$

If $u \geqq 7$, $v = 0$, $g - L = (u-6)(t-1) > 0$. If $u \geqq 7$, $v \geqq 1$, $g - L = (u-6)(t-1) + v - 1 > 0$. If $u = 6$, $g - L = v - 1 \geqq 0$ unless $v = 0$. But if $m = 6t$, there is a gap $2t > t$ from $5m + 2t = 4m + 8t$ to $5m + 4t < 6m$. If $u = 5$, there is a gap $2t - v > t$ from $4m - 4t$ to $3m + 3t$. If $u = 4$, there is a gap $t + v$ from $2m + 6t$ to $3m + 3t$. But if $u = 4$, $v = 0$, there is a gap $2t$ from $5m + 5t$ to $5m + 7t$. If $u = 3$, there is a gap $t + v$ from $5m + 5t$ to $6m + 3t$; if also $v = 0$, we apply the following with $h = 3$.

If $m = ht$, $f(x) = tp(x)$, where $p(x)$ is a polygonal number of order $h + 2$. For $h = 2$ or 3, every integer $\geqq 0$ is a sum of four values of $p(x-1)$ by *Bulletin*, II. Then every multiple of t is a sum of four values of $f(x-1)$. Hence $e_4 = t - 1$.

This covers the case $u = 2$, $v = 0$. There remains only the case $2t < m < 3t$.

18. THEOREM 15. *If* $k = 1$, $t > 1$, $2m \geqq 5t$, $m < 3t$, *then* $e_4 = t - 1$.

No smaller e_4 will serve since t is the least value > 0 of $f(x-1)$. Here n and C have the values (39). By (7),

$$D = A - 4C = mq + bt + r, \quad q = \tfrac{1}{2}(a - 3b). \tag{44}$$

Assign to b the odd values β, $\beta + 6$, $\beta + 4$, $\beta + 2$, and to r the values $0, 1, \cdots, t-1$. The residues modulo m of $bt + r$ are βt plus

$$0, \cdots, t-1;\ 6t - 2m, \cdots, 7t - 2m - 1;\ 4t - m, \cdots, 5t - m - 1;$$
$$2t, \cdots, 3t - 1.$$

Here $0 < 6t - 2m \leqq t$, $0 < 4t - m < 7t - 2m$, $2t < 5t - m$. Hence these four sets include the consecutive integers from 0 to $3t - 1 > m - 1$. Hence $d = 8$. We have U in (41) and

$$V = 2mA + 4t^2 - 6tm + m^2 + 2m, \quad P = 2t + 11m,$$
$$F = m^2\omega^2 - 216t^3m - 2664t^2m^2 - 9702tm^3 - 8712m^4 + 24t^2m + 264tm^2 + 726m^3,$$

where $\omega = A - 52t - 96m + 4$. Multiply $2m \geqq 5t$ by $44t^2m$, $3t > m$ by $9702tm^2$, and $9t^2 > m^2$ by $8712m^2$. Hence $F > 0$ if $m^2\omega^2 \geqq ht^2m^2$,

$$h = 88 + 2664 + 29106 + 78408 = 110266 < 333.^2$$

Hence $F > 0$ if $\omega \geqq 333t$ and therefore if

$$A \geqq 69(2m - 5t) + 385t + 96m = 234m + 40t.$$

Then the reduced minor conditions in § 4 are satisfied. For, (14) holds if $3D \geqq 8t - m - 4t^2/m$ and hence if $D \geqq 2t$, $A \geqq 4m - 2t$. Also, (20) hold if $A \geqq 4m - 4$.

The gaps in table T_1 are all $\leqq t$ down to $d = 27m + 9t$, and from $b = 25m + 15t$ to the limit $54m + 19t$, when $m < 3t$. Now $6t - 2m$ is positive and $\leqq t$; adding it to d we get b.

We employ table T_0, cited in § 9, of sums by four of values of $f(x)$. To it we annex the sums

014743

$$m-t+2(21m+7t)+45m+10t=88m+23t,$$
$$m-t+m+2t+10m+5t+105m+15t\ =117m+21t=\sigma$$

of four values of $f(x-1)$. Using only $m<3t$, we see that all gaps are now $\leqq t$ from $54m+19t$ to either $112m+32t$ or σ and (§ 10) from $\tau=115m+27t$ to $234m+45t$. But τ is the sum of σ and $6t-2m\leqq t$. This completes the proof of Theorem 15, which covers the case $2v\geqq t$ in the notation of Theorem 16.

19. THEOREM 16. *If* $k=1$, $m=2t+v$, $0<2v<t$, *then* $e_4\geqq t$ *except when* $8v\leqq t$ *or* $4v=t$, *and then* e_4 *is probably* $t-1$, *with certainty that* $e_4(A)=t-1$ *if* $A\leqq 400m+58t$.

In § 18 we cited the entries b, d, of table T_1. They may be written as $b=65t+25v$, $d=63t+27v$. All entries involving gm, where $g\geqq 29$, exceed b. Those involving lm, where $l\leqq 23$, are $<d$. Also, $24m+15t<d$. From the entry $e=24m+16t$ and the entries involving $25m$, $26m$, $27m$, $28m$, we eliminate m and see that $e=64t+24v$ and $f=64t+28v$ are the only possible entries between d and b.

If $3v\geqq t$, then $e\leqq d$, $f\geqq b$, and there is a gap $2t-2v>t$ from d to b. Hence let $3v<t$. Then $e-d=b-f=t-3v$, and there is a gap $4v$ from e to f.

There remains only the case $4v\leqq t$. As just noted, the gaps from d to e, e to f, and f to b are all $\leqq t$. Then by § 18, the gaps in T_1 are $\leqq t$ to $\alpha=112m+32t=256t+112v$, and from $\beta=113m+31t=257t+113v$ to τ and thence to $234m+45t$ and (§ 10) on to $400m+58t$. We employ the following extension of table T_1:

$113m+17, 20, 21, 23—29, 31, 32$; $114m+19, 22—27, 29—32$; $115m+20—22,$
$24, 25, 27—32$; $116m+19, 20, 22, 23, 25—32$; $117m+18, 21, 23—29, 31, 32$;
$118m+22—32$; $119m+22—25, 27—32$; $120m+16—19, 21—33$;
$121m+15—21, 23—29, 31—33$; $122m+14, 15, 17—23, 25—33$;
$123m+13, \cdots,$

in which the terms free of m are to be multiplied by t.

The only possible entries between α and β are

$$\gamma=117m+21t=255t+117v,\quad \delta=120m+16t=256t+120v.$$

In fact,

$$113m+29t=255t+113v<\alpha,\quad 114m+27t=255t+114v<\alpha,$$
$$114m+29t=257t+114v>\beta,\quad 115m+27t=257t+115v>\beta,$$
$$115m+25t=255t+115v<\alpha,\quad 116m+23t=255t+116v\leqq\alpha,$$

$$\begin{array}{ll} 116m+25t=257t+116v>\beta, & 117m+18t=252t+117v<\alpha, \\ 117m+23t=257t+117v>\beta, & 118m+22t=258t+118v>\beta, \\ 120m+17t=257t+120v>\beta, & 121m+15t=257t+121v>\beta, \\ 122m+14t=258t+122v>\beta, & 123m+13t=259t+123v>\beta. \end{array}$$

Each entry $gm+ht$ with $g \geqq 124$ exceeds β since $h>13$.

First, let $5v \leqq t$. We drop $\gamma \leqq \alpha$. If $7v \geqq t$, we drop also $\delta \geqq \beta$ and have a gap $t+v>t$ from α to β. Hence let $7v<t$. Then there is a gap $8v$ from α to δ, and a gap $t-7v$ from δ to β. There remains only the case $8v \leqq t$. If we attempt to prove that $e_4=t-1$, we see that $d=20$ by either of our methods and that there remain numbers far beyond our tables.

Second, let $5v>t$, $4v \leqq t$. We drop $\delta>\beta$. There is a gap $5v-t$ from α to γ and a gap $w=2t-4v$ from γ to β. Since $w>t$ if $t>4v$, there remains only the case $t=4v$. If we then attempt to prove that $e_4=t-1$, we may take $v=1$ and see that $d=12$ by either of our methods. We find that $F>0$ if $A \geqq 6098$. The limit at the end of Theorem 16 is now 3832.

Combining Theorems 15 and 16 with the results in § 17, we get

THEOREM 17. *If* $k=1$, $t>1$, $m \geqq 2t$, *the minimum of* $s+e_s$ *for* $s \geqq 4$ *is* L *of Theorem* 13 *if* $m>3t$ *or if* $m=2t+v$, $0<2v<t$, *and neither* $8v \leqq t$ *nor* $4v=t$. *But it is* $4+e_4=t+3=L-1$ *if* $m=2t$ *or* $m=3t$ *or* $m=2t+v$, $0<v<t \leqq 2v$, *and also up to* $400m+58t$ *if either* $8v \leqq t$ *or* $4v=t$.

20. If $s=4$, $t>1$, $k \geqq 2$, $m>6t$, there is a gap from $4t$ to $m-t$ and the first part of § 17 shows that $4+e_4 \geqq L$.

21. *The Case* $s=4$, $k=2$, $t>1$, $m \geqq 2t$. If $5t \leqq m \leqq 6t$, there is a gap from $9m+5t$ to $10m+t$ in T_2 and the gap exceeds t unless $m=5t$. In that case, $e_4=t-1$ by the end of § 17 and *Bulletin*, II, Theorem 4.

Let $m=4t+v$, $0 \leqq v<t$. If $3v>2t$, there is a gap $2t-v>t$ from $28m+2t$ to $27m+8t$. If $3v \leqq 2t$, there is a gap $2v$ from $25m+16t$ to $27m+8t$. There remains only the case $2v \leqq t$. We shall prove that $e_4=t-1$. The use of only odd values of b leads to $d=18$ and hence to a limit for A beyond our tables. That we may take $d=9$ is shown by the following method to be applied frequently in the sequel.

For any k, (4) and (7) give

$$D=mH+tb+r,\quad H=\tfrac{1}{2}[a-(1+2k)b]. \tag{45}$$

LEMMA 4. *If* $b \equiv 2 \pmod{4}$, *the choice* $D \equiv tb + r \pmod{2m}$ *yields an integer* $a \equiv 2 \pmod{4}$ *which satisfies* (45). *The same is true of the choice* $D \equiv tb + r + m \pmod{2m}$ *if* $b \equiv 0 \pmod{4}$. *If* b *is odd, either of these two choices yields an odd integer* a *satisfying* (45).

In the first case, H is even, whence $a \equiv b \equiv 2 \pmod{4}$. In the second case, $m \equiv mH \pmod{2m}$ and H is odd, whence $a - b \equiv 2$, $a \equiv 2 \pmod{4}$. If b is odd, either choice makes $D \equiv tb + r \pmod{m}$, whence H is an integer and a is odd. In all these cases we may apply Lemma 1 or 2.

We seek the smallest set* of consecutive values $\beta, \beta + 1, \cdots$ of b such that, when r takes its values $0, 1, \cdots, t-1$, the values of D computed by Lemma 4 shall yield a complete set of residues modulo $2m$. Then every integer D can be given the form (45) by use of permissible values of a and b for which Lemmas 1 and 2 are applicable.

TABLE T_2.

All Sums by Four of Values of $f(x-2)$.

(terms free of m are to be multiplied by t)

0—4; m—1, 0—5; 2m—2, —1, 0—6; 3m—3—1, 0, 1, 3—7; 4m—4—1, 0—8;
5m—4, —3, —1, 0—5, 7, 8; 6m—5—2, 0—9; 7m—5, —4, —2, —1, 0, 1, 3—9;
8m—6, —3, —1, 0, 2—10; 9m—6, —5, —1, 0—5, 7—10; 10m—7, —4, —2,
1—11; 11m+0, 3—9, 11; 12m—8, —3, 0—12; 13m—1, 2—5, 7—12;
14m+2, 3, 5—12; 15m—2, 1, 3, 4, 6—9, 11—13; 16m+1, 2, 5—13;
17m+0, 3—5, 7—13; 18m+3—7, 9—14; 19m—1, 3, 4, 6—9, 11—14;
20m+2, 5—14; 21m+2, 3, 7—13, 15; 22m+1, 4—15; 23m+5, 6, 8, 9,
11—15; 24m+0, 4—16; 25m+4, 5, 7—13, 15, 16; 26m+3, 6, 8—11, 13—16;
27m+3, 4, 6, 8, 9, 11—16; 28m+2; 5, 7—17; 29m+7—13, 15—17;
30m+1, 6, 7, 9—17; 31m+5—8, 11—17; 32m+5, 6, 8—18; 33m+4, 7,
9—13, 15—18; 34m+4, 5, 9—18; 35m+3, 6, 8, 9, 11—17; 36m+8—19;
37m+2, 7—13, 15—19; 38m+7—15, 17—19; 39m+6—9, 11—17, 19;
40m+6—20; 41m+5, 8, 10—13, 15—20; 42m+5, 6, 9—11, 13—20;
43m+4, 7, 9, 12—17, 19, 20; 44m+9, 11—20; 45m+3, 8, 10—13, 15—21;
46m+9—21; 47m+8, 9, 11—17, 19—21; 48m+7—10, 12—21; 49m+7,
8, 10—13, 15—21; 50m+6, 9, 11, 12, 14—22; 51m+6, 7, 11—17, 19—22;
52m+5, 8, 10, 13—22; 53m+12, 13, 15—21; 54m+4, 9, 12—14, 17—19,
21, 22; 55m+11—17, 19—23; 56m+10—23; 57m+9—13, 15—21, 23;
58m+8—11, 13—23; 59m+8, 9, 11—17, 19—23; 60m+7, 10, 12—24;
61m+7, 8, 11—13, 15—21, 23, 24; 62m+6, 9, 11, 12, 14—24; 63m+12—17,
19—24; 64m+5, 10, 13—24; 65m+12, 15—21, 23, 24; 66m+12—25;
67m+11—17, 19—25; 68m+10, 11, 13—25; 69m+9—13, 15—21, 23—25;
70m+9, 10, 12—15, 17—19, 21—25; 71m+8, 11—14, 16, 17, 19—25;
72m+8, 9, 13—26; 73m+7, 10, 12, 15—21, 23—26; 74m+14, 16—23,

---

* This was true for a single b in the simple case of (II), § 9.

25, 26; $75m+6$, 11, 14—17, 19—25; $76m+13$, 15—26; $77m+15$—21, 23—26; $78m+13$—27; $79m+12$—17, 19—25, 27; $80m+11$, 12, 14, 15, 17—27; $81m+10$—13, 16—21, 23—27; $82m+10$, 11, 13—27; $83m+9$, 12, 14—17, 19—25, 27; $84m+9$, 10, 14—28; $85m+8$, 11, 13, 15—21, 23—28; $86m+15$—18, 21—28; $87m+7$, 12, 14—17, 19—25, 27, 28; $88m+14$, 17—28; $89m+15$, 17, 18, 20, 21, 23—28; $90m+13$, 16—23, 25—28; $91m+14$—17, 19—25, 27—29; $92m+13$—29; $93m+12$, 13, 15, 16, 18—21, 23—29; $94m+11$—14, 17—29; $95m+11$, 12, 14—17, 19—25, 27—29; $96m+10$, 13, 15—18, 21—29; $97m+10$, 11, 15, 16, 18—21, 23—29; $98m+9$, 12, 14, 17—27, 29, 30; $99m+15$, 16, 19—25, 27—30; $100m+8$, 13, 16—30; $101m+15$, 18—21, 23—29; $102m+18$, 19, 21—30; $103m+14$, 17, 19—25, 27—30; $104m+17$—19, 21—30; $105m+15$—21, 23—29, 31; $106m+14$—23, 25—31; $107m+13$—17, 19—25, 27—31; $108m+12$—15, 17—22, 24—31; $109m+12$, 13, 15—21, 23—29, 31; $110m+11$, 14, 16—31; $111m+11$, 12, 16, 17, 19—25, 27—31; $112m+10$, 13, 15—32; $113m+17$, 20, 21, 23—29, 31, 32; $114m+9$, 14, 17—27, 29—32; $115m+16$, 19—25, 27—32; $116m+18$—23, 25—32; $117m+15$, 18, 20, 21, 23—29, 31, 32; $118m+17$—19, 22—32; $119m+17$, 20, 22—25, 27—32; $120m+16$—19, 21—33; $121m+15$—21, 23—29, 31—33; $122m+14$, 15, 17—23, 25—33; $123m+13$—16, 19—25, 27—33; $124m+13$, 14, 16—19, 21—33; $125m+12$, 15, 17, 18, 20, 21, 23—29, 31—33; $126m+12$, 13, 17, 18, 20—31, 33; $127m+11$, 14, 16, 19—25, 27—33; $128m+18$, 20—34; 129m+10, 15, 18—21, 23—29, 31—34; $130m+17$, 19—27, 29—34; $131m+20$, 21, 23—25, 27—33; $132m+16$, 18, 19, 21—34; $133m+19$, 20, 23—29, 31—34; $134m+18$, 21—34; $135m+21$—25, 27—33; $136m+17$—35; $137m+16$—21, 23—29, 31—35; $138m+15$, 16, 18—22, 25—35; $139m+14$—17, 19—25, 27—33, 35; $140m+14$, 15, 17—24, 26, 27, 29—35; $141m+13$, 16, 18, 19, 21, 23—29, 31—35; $142m+13$, 14, 18—31, 33—35; $143m+12$, 15, 17, 20—25, 27—33, 35; $144m+19$, 21—36.

Write $[j]$ for the set $j, j+1, \cdots, j+t-1$.

(I) $\beta \equiv 1 \pmod 4$. For $b=\beta, \beta+2, \beta+4, \beta+6, \beta+8$, the values of D computed by Lemma 4 are congruent modulo $2m$ to the sum of $t\beta$ and

$$(46) \qquad [0], [m] \quad [2t], [2t+m] \quad [4t], [4t+m] \quad [6t], [6t+m] \quad [8t], [8t+m].$$

For $b=\beta+1, \beta+3, \beta+5, \beta+7$, we get

$$(47) \qquad [t] \quad [3t+m] \quad [5t] \quad [7t+m].$$

In the present problem, $4t \leqq m \leqq 5t$. We subtract the modulus $2m$ from $7t+m$ and $8t+m$. The sets

$$[0] \ [t] \ [2t] \ [7t-m] \ [8t-m] \ [4t] \ [5t] \ [6t] \ [2t+m] \ [3t+m] \ [4t+m]$$

overlap and are arranged in ascending order, since the first number of any set is $\leqq 1$ plus the last number of the preceding set. The largest number of the final set is $5t+m-1 \geqq 2m-1$. Hence the sets include a complete set of residues modulo $2m$.

(II) $\beta \equiv 3 \pmod 4$. We again have (46). But (47) is replaced by

$$[t+m] \quad [3t] \quad [5t+m] \quad [7t]. \tag{48}$$

The following are overlapping sets in ascending order:

$$[0]\ [5t-m]\ [6t-m]\ [2t]\ [3t]\ [4t]\ [m]\ [t+m]\ [2t+m]\ [7t]\ [8t].$$

The largest number of the final set is $9t-1 \geqq 2m-1$ since $2v \leqq t$.

(III) $\beta \equiv 0 \pmod 4$. For $b = \beta+1,\ \beta+3,\ \beta+5,\ \beta+7$, we get

$$[t], [t+m] \quad [3t], [3t+m] \quad [5t], [5t+m] \quad [7t], [7t+m]. \tag{49}$$

For $b = \beta,\ \beta+2,\ \beta+4,\ \beta+6,\ \beta+8$, we get

$$[m] \quad [2t] \quad [4t+m] \quad [6t] \quad [8t+m]. \tag{50}$$

The set $[4t+m]$ terminates with $5t+m-1 \equiv 5t-m-1 \geqq 0 \pmod{2m}$ and hence contains the residues $0, 1, \cdots, 5t-m-1$. We employ these and the following overlapping sets in ascending order:

$$\begin{aligned}&[5t-m]\ [t]\ [2t]\ [3t]\ [8t-m]\ [m]\\ &\qquad\qquad\qquad [t+m]\ [6t]\ [7t]\ [3t+m]\ [4t+m].\end{aligned}$$

(IV) $\beta \equiv 2 \pmod 4$. We again have (49). But (50) is replaced by

$$[0] \quad [2t+m] \quad [4t] \quad [6t+m] \quad [8t]. \tag{51}$$

Overlapping sets in ascending order are

$$\begin{aligned}&[0]\ [t]\ [6t-m]\ [7t-m]\ [3t]\ [4t]\\ &\qquad\qquad\qquad [5t]\ [t+m]\ [2t+m]\ [3t+m].\ [8t].\end{aligned}$$

Since in all four cases we obtain a complete set of residues modulo $2m$, we have $d=9$. Then

$$\begin{aligned}&n = 2t-5m, \qquad U = 24mD + 36t^2 - 156tm + 153m^2,\\ &\qquad C = 3m-2t, \qquad V = 2mD + 4t^2 - 22tm + 25m^2 + 2m,\\ &P = 2t+11m, \qquad W = 3mD + 4t^2 - 25tm + 4m^2,\\ &F = m^2\omega^2 - 240t^3m - 2856t^2m^2 - 9636tm^3 - 6534m^4\\ &\qquad\qquad\qquad\qquad + 24t^2m + 264tm^2 + 726m^3,\end{aligned}$$

where $\omega = D - 63t - 75m + 4$. Multiply $m \geqq 4t$ by $60t^2m$, $9t \geqq 2m$ by $4818tm^2$, and $81t^2 \geqq 4m^2$ by $1633\frac{1}{2}m^2$. Hence $F > 0$ if $m^2\omega^2 \geqq pt^2m^2$, $p = 60 + 2856 + 43362 + 132314 = 178592 < 423^2$, and hence if $\omega \geqq 423t$. This holds if

$$A \geqq 109(m - 4t) + 12m - 8t + 423t + 63t + 75m = 196m + 42t.$$

The gaps are $\leqq t$ in T_2 to $122m + 33t$ and thence to $196m + 41t$ in T_0 after annexation of

$$\begin{aligned}164m + 28t &= m - t + 3m + 3t + 55m + 11t + 105m + 15t,\\ 164m + 29t &= m - t + 6m + 4t + 66m + 12t + 91m + 14t,\\ 194m + 33t &= m - t + 28m + 8t + 45m + 10t + 120m + 16t.\end{aligned}$$

Next, let $3t \leqq m < 4t$. To prove that $e_4 = t - 1$, give to b the values $\beta, \beta + 1, \cdots, \beta + 7$. We suppress the final pair in (46), and the final entry in both (50) and (51). If $\beta \equiv 1 \pmod 4$, the sets

$$[0] \quad [t] \quad [2t] \quad [6t - m] \quad [7t - m] \quad [4t] \quad [5t] \quad [6t] \quad [3t + m]$$

overlap and are in ascending order. The largest number in the last set is $4t + m - 1 > 2m - 1$. If $\beta \equiv 3 \pmod 4$, the sets

$$[0] \quad [4t - m] \quad [5t - m] \quad [2t] \quad [3t] \quad [4t] \quad [t + m] \quad [2t + m] \quad [6t] \quad [7t]$$

overlap and terminate with $8t - 1 > 2m - 1$. If $\beta \equiv 0 \pmod 4$, the sets

$$[4t - m] \quad [t] \quad [2t] \quad [3t] \quad [m] \quad [t + m] \quad [5t] \quad [6t] \quad [7t]$$

overlap and terminate with $8t - 1 \equiv 8t - 2m - 1 \geqq 4t - m \pmod{2m}$, and hence include a complete set of residues modulo $2m$. If $\beta \equiv 2 \pmod 4$, the sets

$$[0] \quad [t] \quad [5t - m] \quad [6t - m] \quad [3t] \quad [4t] \quad [5t] \quad [2t + m] \quad [3t + m]$$

overlap and terminate with $4t + m - 1 > 2m - 1$. In every case, $d = 8$. Using the above U and V, we get

$$F = m^2\omega^2 - 216t^3m - 2040t^2m^2 - 5238tm^3 - 1944m^4 + 486m^3 + 24t^2m + 216tm^2,$$

where $\omega = D - 56t - 40m + 4$. Multiply $m \geqq 3t$ by $72t^2m$, $4t > m$ by $5238tm^2$, and $16t^2 > m^2$ by $1944m^2$. Hence $F > 0$ if

$$m^2\omega^2 \geqq gt^2m^2, \quad g = 72 + 2040 + 20952 + 31104 = 54168 < 233^2,$$

and therefore if

$$A \geqq 56t + 40m + 233t + 12m - 8t + 82(m - 3t) = 134m + 35t.$$

The gaps are $\leqq t$ to $77m + 26t$ in T_2 and thence to $134m + 34t$ in T_0 after annexation of $m - t + 15m + 6t + 2(36m + 9t)$.

Finally let $m = 2t + v$. In view of §§ 17, 18, it remains to consider only the case $0 < 2v < t$. In T_2, the gaps are $\leqq t$ from $27m + 9t$ to $28m + 17t$. Hence by § 18, the gaps are $\leqq t$ down to $\alpha = 112m + 32t$ and from $\beta = 113m + 31t$ to $400m + 58t$. We employ the entries

$$\epsilon = 117m + 20t = 254t + 117v, \quad \rho = 118m + 18t = 254t + 118v,$$
$$\sigma = 118m + 19t = 255t + 118v, \quad \tau = 119m + 17t = 255t + 119v.$$

(i) Let $5v > t$. Then the gaps from α to β are all $< t$. For, if $5v \geqq 2t$, then $\epsilon - \alpha = 5v - 2t < t$ since $5v < 6v < 3t$, $\rho - \epsilon = v$, $\gamma - \rho = t - v$, $\beta - \gamma = 2t - 4v < t$ since $8v > 5v \geqq 2t$, and these differences are all $\geqq 0$. But if $5v < 2t$, then $\gamma - \alpha = 5v - t$, $\sigma - \gamma = v$, $\beta - \sigma = 2t - 5v$ are all positive and $< t$.

(ii) Let $5v \leqq t$. Then δ, σ, and τ are the only possible entries between α and β in T_2.

First, let $t \leqq 7v$. We drop $\delta \geqq \beta$. If $6v < t$, we drop also $\sigma < \alpha$ and have

$$\tau - \alpha = 7v - t \geqq 0, \quad \beta - \tau = 2t - 6v > t.$$

But if $6v \geqq t$, then $\sigma - \alpha = 6v - t$, $\tau - \sigma = v$, $\beta - \tau = 2t - 6v$ are all $\geqq 0$ and $\leqq t$, whence the gaps from α to β are $\leqq t$.

Second, let $7v < t$. We drop $\sigma < \alpha$, $\tau < \alpha$. Then $\beta - \delta = t - 7v$ and there is a gap $8v$ from α to δ.

In summary, there is a gap $> t$ if and only if $6v < t < 8v$, while the gaps are all $\leqq t$ to $400m + 58t$ if $t \leqq 6v$ or $t \geqq 8v$.

If $v = 1$, then $t = 3, \cdots, 6$ or $\geqq 8$. Is $e_4 = t - 1$?

For $t = 3$, $m = 7$, $e_4 = 2$, either of our methods gives $d = 10$. Then

$$F = m^2D^2 - 6250mD - 2658761 > 0 \text{ if } MD \geqq 6650,$$

and hence if $A \geqq 1010$. But the gaps are $\leqq t$ to $130m + 34t = 1012$.

For $t = 4$, $m = 9$, $e_4 = 3$, either method gives $d = 12$. Then

$$F = m^2D^2 - 41940mD - 9573714 > 0 \text{ if } mD \geqq 42168,$$

and hence if $A \geqq 4762$. This exceeds the limit $400m + 58t = 3832$ of our tables.

For $t = 5$, $m = 11$, or $t = 6$, $m = 13$, or $t = 8$, $m = 17$, the use of odd b's only gives $d = 14$, 16, or 20, respectively, and leads to values of A beyond our tables.

When $v = 2$, let first t be even, $t = 2T$. Then $m = 2M$, $M = 2T + 1$. Then $f(x) = 2F(x)$, where $F(x)$ has the parameters T and M. In certain cases we just proved that its e_4 is $T - 1$, so that every positive integer is a sum of r and four values of $F(x-2)$ where $r = 0, \cdots, T-1$. Hence every even integer is a sum of $2r$ and four values of $f(x-2)$. Thus every odd is a sum of $2r + 1$ and four such values. Hence every integer is a sum of R and four values, where $R \leqq 2T - 1$. Thus $e_4 = t - 1$ for $f(x)$.

For $v = 2$, $t = 5$, $m = 12$, $e_4 = 4$, either method gives $d = 10$. Then

$$F = m^2D^2 - 42000mD - 23263488 > 0 \text{ if } mD \geqq 42547,$$

and hence if $A \geqq 3650 = 300m + 10t$, and the proof is complete.

But for $v = 2$, $t = 7$ or 9, $d = 12$ or 14 and A is too large. For $v = 3$, $t = 7$, $d = 10$.

THEOREM 18. *If $k = 2$, $t > 1$, $m \geqq 2t$, the minimum of $s + e_s$ for $s \geqq 4$ is L of Theorem 13 when $2m > 9t$, $m \neq 5t$, and when $m = 2t + v$, $6v < t < 8v$; but is $4 + e_4 = t + 3 = L - 1$ when $m = 2t$, $m = 5t$, $5t \leqq 2m \leqq 9t$, and for certain (and probably all) cases of $m = 2t + v$ for which either $t \leqq 6v$ or $t \geqq 8v$.*

22. *The Case $s = 4$, $k = 3$, $t > 1$, $m \geqq 2t$.* If $m > 6t$, see § 20. We need the following part of table T_3:

$$38m + 2,\ 5,\ 7-15,\ 17-19;\ 39m - 2,\ +5-9,\ 11-17,\ 19;$$
$$40m + 1 - 3,\ 6 - 20;$$

all entries $gm + ht$ with $g \leqq 37$ have $h \leqq 19$; all with $g \geqq 41$ have $h \geqq 1$, except for $46m - 1$. Hence if $5t < m \leqq 6t$, there is a gap $m - 4t > t$ from $38m + 15t$ to $39m + 11t$ in T_3. If $m = 5t$, $e_4 = t - 1$ by § 21.

Let $m = 4t + v$, $0 \leqq v < t$. We shall prove that $e_4 = t - 1$. Since this is true by § 21 if $2v \leqq t$, let $2v > t$, whence $2m > 9t$. If $\beta \equiv 1 \pmod 4$, we employ § 21 unchanged. If $\beta \equiv 3 \pmod 4$ we employ after $[8t]$ the set $[4t + m]$ there ignored; it terminates with $5t + m = 1 > 2m - 1$. If $\beta \equiv 0 \pmod 4$, we must use the new value $b = \beta + 9$, which gives the set $[9t - m]$; we insert it before $[m]$. If $\beta \equiv 2 \pmod 4$, we must use also

$b = \beta + 9$; we place the new set $[9t]$ at the end; it terminates with $10t - 1 > 2m - 1$. In each case we may take $d = 10$. Then

$$F = m^2\omega^2 - 264t^3m - 3024t^2m^2 - 9306tm^3 - 3630m^4 + 24t^2m \\ + 264tm^2 + 726m^3,$$

where $\omega = D - 70t - 51m + 4$. Multiply $2m > 9t$ by $30t^2m$, $5t > m$ by $9306tm^2$, and $25t^2 > m^2$ by $3630m^2$. Hence $F > 0$ if $m^2\omega^2 \geqq lt^2m^2$, $l = 60 + 3024 + 46530 + 90750 = 140364 < 375^2$. But $A = D + 4C$, $C = 6m - 3t$. Hence $F > 0$ if

$$A \geqq 75m + 433t + 44(2m - 9t) = 163m + 37t.$$

To T_2 we annex $27m + 7t = 6m - 3t + 6m + 4t + 15m + 6t$, which is a sum of three and hence of four values of $f(x - 3)$. Hence by § 21 the gaps in T_3 are $\leqq t$ to $163m + 37t$.

Next, let $m = 2t + v$. The only possible numbers between α and β in T_3, but not in T_2, are $120m + 14t$ and $123m + 6t$. They are both $< \alpha$ in the case $6v < t < 8v$ in which there was a gap $> t$ in T_2.

THEOREM 19. *If $k = 3, t > 1$, $m \geqq 2t$, the minimum of $s + e_s$ for $s \geqq 4$ is L of Theorem* 13 *when $m > 5t$, and when $m = 2t + v$, $6v < t < 8v$; but is $4 + e_4 = t + 3 = L - 1$ when $m = 2t$, $5t \leqq 2m \leqq 10t$, and for certain (and probably all) cases of $m = 2t + v$ for which either $t \leqq 6v$ or $t \geqq 8v$.*

23. THEOREM 20. *In $k = 4$, $t > 1$, $m \geqq 2t$, the minimum of $s + e_s$ for $s \geqq 4$ is L of Theorem* 13 *when $2m > 11t$, and when $m = 2t + v$, $6v < t < 8v$. But it is $4 + e_4 = t + 3 = L - 1$ for certain (if not all) cases of $m = 2t + v$, $t \leqq 6v$ or $t \geqq 8v$, for $m = 2t$, for $5t \leqq 2m \leqq 10t$, and probably for $10t < 2m \leqq 11t$ since then $E_4(A) = t - 1$ if $A \leqq 354m + 55t$ and if $A \geqq 689m + 75t$.*

If $m > 6t$, see § 20. If $5t \leqq m \leqq 6t$, there is a gap $2m - 10t$ from $B = 69m + 21t$ to $C = 71m + 11t$ in T_4:

$69m + 4, 7, 9-13, 15-21, 23-25$; $70m + 3, 8-10, 12-15, 17-19, 21-25$;
$71m - 1, +3-8, 11-17, 19-25$; $72m + 3, 6, 8-26$; $73m + 2, 7-13, \cdots$.

Any entry $gm + ht$ with $g \leqq 68$ has $h \leqq 25$ and is $< B$; any with $g \geqq 74$ has $h \geqq -2$ and is $> C$. Also, $70m + 15t$, $71m + 8t$, and $72m + 3t$ are $\leqq B$; $69m + 23t$, $70m + 17t$, $72m + 6t$, and $73m + 2t$ are $\geqq C$. This proves the theorem if $2m > 11t$.

Let $m = 2t + v$. In the case $6v < t < 8v$ in which there is a gap $> t$

in T_2 (and T_3), table T_4 contains no entry, not in T_2, which is between α and β of § 21, whence there is a gap $> t$ in T_4.

It remains only to consider the case $10t < 2m \leqq 11t$. Is $e_4 = t - 1$? We obtain $d = 16$ by using only odd values β, $\beta + 2 \cdots, \beta + 14$ of b. Write $[j]$ for the set $j, j + 1, \cdots, j + t - 1$. Take $b = \beta, \beta + 6, \beta + 12, \beta + 2, \beta + 8, \beta + 14, \beta + 4, \beta + 10$. We obtain the overlapping sets

$$[0] \ [6t - m] \ [12t - 2m] \ [2t] \ [8t - m] \ [14t - 2m] \ [4t] \ [10t - m].$$

The final number of the last set is $11t - m - 1 \geqq m - 1$. The same $d = 16$ is obtained by the method of § 21 and also by the use of modulus $8m$ instead of $2m$. We get

$$F = m^2\omega^2 - 408t^3m + 472t^2m^2 - 65142tm^3 - 105840m^4 + 24t^2m + 504tm^2 + 2646m^3,$$

where $\omega = D - 112t - 300m + 4$. Then $F > 0$ if $\omega \geqq 1887t$. But $A = D + 4C, C = 10m - 4t, \lambda = 4C + 112t + 300m = 340m + 96t$. Hence $F > 0$ if

$$A \geqq 1887t + \lambda + 381(m - 5t) = 721m + 78t.$$

We may lower this limit by subdivision of the interval from $5t$ to $5.5t$ for m. Making five equal divisions we find that the successive values of h in $\omega \geqq ht$ are 1757, 1789, 1822, 1855, 1887. To $h + \lambda$ add $356(m - lt) \geqq 0$ for $l = 5, 5.1, 5.2, 5.3, 5.4$, respectively. The respective limits for A are $696m + pt$, where $q = 73, 70, 67, 65, 61$. The essential point is the decrease, so that the maximum A is to be chosen.

Let the number of equal divisions increase indefinitely. Hence at the first step we have m approximately $5t$, and $F > 0$ if $\omega > 1724t$. We add $\lambda + 349(m - 5t)$ and obtain the best limit[*] $689m + 75t$ for A.

We test values of A below this limit by a table t_4 of certain sums by four of values of $f(x - 4)$, which includes table T_2 to $144m$, table T_0 for larger numbers, the above entries $69m + \cdots$, and the following, in which $p = m - t$, $q = 6m - 3t$, $r = 10m - 4t$, and (x, y) denotes $xm + yt$:

---

[*] Approximately 4 times the value $171m + 19t$ of $f(x)$.

$(10,0)=p+q+1+(3,3),\qquad (16,4)=q+1+1+(10,5),$

$(20,4)=p+q+(3,3)+(10,5),\qquad (27,7)=r+(1,2)+(6,4)+(10,$

$(30,8)=q+1+(3,3)+(21,7),\qquad (35,4)=p+q+(28,8),$

$(40,5)=p+r+(1,2)+(28,8),\qquad (52,12)=q+1+(10,5)+(36,9),$

$(76,14)=q+(1,2)+(3,3)+(66,12),\quad (104,20)=q+2(10,5)+(78,13),$

$(148,28)=p+2(21,7)+(105,15),\qquad (159,27)=p+(1,2)+(66,12)+(9$

$(164,28)=p+(3,3)+(55,11)+(105,15),$

$(164,29)=p+(6,4)+(66,12)+(91,14),$

$(168,28)=p+(10,5)+(21,7)+(136,17),$

$(176,28)=p+(3,3)+(36,9)+(136,17),$

$(191,27)=r+(6,4)+(55,11)+(120,16),$

$(194,33)=p+(28,8)+(45,10)+(120,16),$

$(200,31)=p+(3,3)+(91,14)+(105,15),$

$(202,25)=3m-2t+(28,8)+(171,19),$

$(202,26)=p+(1,2)+(10,5)+(190,20),$

$(202,27)=q+(10,5)+(15,6)+((171,19),$

$(207,27)=p+(1,2)+(15,6)+(190,20),$

$(228,29)=3m-2+(105,15)+(120,16),$

$(248,36)=p+(21,7)+(55,11)+(171,19),$

$(260,36)=p+(10,5)+(78,13)+(171,19),$

$(278,33)=p+(1,2)+(45,10)+(231,22),$

where each (a, b) in the right members is a value of $f(x)$. In this t_4 the gaps are $\leqq t$ to $354m + 55t$. Write $m = 5t + v$. Then $2v \leqq t$. We employ the gaps $2v$ beginning with $(17,13)$ [and ending with $(19,3)$], $(33,13)$, $(38,15)$, $(53,21)$, $(74,23)$, $(90,23)$, $(95,25)$, $(106,23)$, $(121,29)$, $(136,35)$, $(139,33)$, $(142,31)$, $(157,37)$, $(166,38)$, $(172,39)$, $(173,37)$, $(189,37)$, $(194,39)$, $(198,41)$, $(200,35)$, $(205,37)$, $(210,39)$, $(216,43)$, $(226,39)$, $(241,45)$, $(257,45)$, $(262,47)$, $(273,45)$, $(276,43)$, $(278,47)$, $(289,45)$, $(294,47)$, $(304,51)$, $(310,47)$, $(325,53)$. Finally, we replace $(19,4)$ by $18m + 9t + v$, and similarly replace $(27,9)$, $(35,4)$, $(97,16)$, $(108,15)$, $(108,22)$, $(115,25)$, $(123,25)$, $(130,27)$, $(169,29)$, $(174,31)$, $207,33)$, $(212,30)$, $(228,31)$, $(237,37)$, $(283,41)$, $(298,43)$, $(321,45)$. Since there are no annexations * after $(278,33)$, and a rarer use of the two devices, with none after $(325,53)$, it is highly probable that the gaps are $\leqq t$ to the above limit $(689,75)$ beyond which the gaps were proved $\leqq t$.

---

* None are necessary after $(1542, 156)$. Taking $\delta = 8$ in (32), we find that $F > 0$ if $\omega \geqq 4094t$. But $\omega = A - 112t - 732m + 4$. We add $810(m - 5t)$ to the limit for A.

24. *Case* $k = 5$. First, let $m = 2t + v$, $6v < t < 8v$. Then T_5 contains

$$G = 121m + 13t = 15m - 5t + t + m + 2t + 105m + 15t.$$

We may now pass from α to β by gaps $\leqq t$ since $G - \alpha = 9v - t$ and $\beta - G = 2t - 8v$ are both > 0 and $< t$. By § 21, the gaps are $\leqq t$ to $400m + 58t$. For $e_4 = t - 1$, either of our methods gives $d = 18$.

Next, let $m = 5t + v$, $2v > t$. From T_5 we need

$54m$—5, —3, —2, 0, 1—15, 17—19, 21, 22; $55m$—19, —10, —8, —3, 0, 1, 3—9, 11—17, 19—23; $56m$—6, —3, 1—3, 5, 6, 8—23; $57m$—6, 0, 1—5, 7—13, 15—21, 23; $58m$—4, —2, —1, 0, 3—11, 13—23; $59m$—4, —3, —1, 0, 3—9, 11—17, 19—23; $60m$—20, —9, —1, 2, 4—7, 9—24;

while all earlier entries are $\leqq \alpha = 53m + 21t$, and in every later entry $gm + h$, $g \geqq 61$, $h \geqq -8$, whence it exceeds $\gamma = 56m + 5t$. Now $54m + 15t$, $55m + 9t$, $56m + 3t$, $57m - 6t$ are $\leqq \alpha$, $54m + 17t$, $55m + 11t$, $57m$, $58m - 4t$, $60m - 9t$ are $\geqq \gamma$. Hence the only possible entry between α and γ is $\delta = 60m - 20t$.

If $7v \leqq 6t$, we drop $\delta \leqq \alpha$ and have a gap $3m - 16t = 3v - t > 0$ from α to γ. If $7v > 6t$, $0 < \delta - \alpha = 7v - 6t \leqq t$, but $\gamma - \delta = 25t - 4m = 5t - 4v \geqq t$ since $t \geqq v$. Hence between α and γ there occurs some gap $> t$ if and only if

$$7v \leqq 6t < 9v \quad \text{or} \quad 7v > 6t,\ t > v. \tag{52}$$

But we can pass from α to γ by gaps $\leqq t$ if either $3v \leqq 2t$ or $v = t$.

If $3v \leqq 2t < 4v$, the gaps are $\leqq t$ at least to $354m + 55t$. The proof is similar to, but longer than, that at the end of § 23. For example, to (304, 51) we add $3v - t \leqq t$ and obtain (307, 35). Likewise from (325, 53), we get (328, 37), which is the sum of $m - t$, (6, 4), (45, 10), (276, 24). We annex the sum (312, 36) of $m - t$, (3, 3), (55, 11), (253, 23). After these changes, the proof in § 23 shows that the gaps are $\leqq t$ from (296, 50) to (354, 55). The least v is 2 and then $t = 3$, $m = 17$, and either method gives $d = 22$ and hence a very large limit for A.

If $v = t$, then $e_4 = 2t - 1$. Since $m = 6t$, the maximum gap is the product of t by the maximum gap 2 when $t = 1$, the first one being then from 979 to 981 (*Bulletin*, II). This fact indicates the delicacy of the case $k \geqq 5$.

THEOREM 21. *If* $k = 5$, $t > 1$, $m \geqq 2t$, *the minimum of* $s + e_s$ *for* $s \geqq 4$ *is* L *of Theorem* 13 *when* $m \geqq 6t$ *and when* $m = 5t + v$, *subject to conditions* (52). *In all remaining cases, it is doubtless* $4 + e_4 = t + 3 = L - 1$, *but complete proof has not been given in special cases.*

25. THEOREM 22. *If $k > 0$, $m \geqq 2t$, $t > 1$, $3 + e_3$ is equal to or greater than the minimum $s + e_s$ for $s \geqq 4$, except possibly when both $k \geqq 5$ and $m \leqq 4t$, a case not examined.*

We employ the existence of a gap $A = m - 4t$ from $3t$ to $m - t$ if $m \geqq 4t$, gap $B = m - 3t$ from $m + 4t$ to $2m + t$ if $4t \leqq m \leqq 5t$, gap $C = 2t$ from $2m - t$ to $2m + t$ if $m \geqq 5t$, gap $D = 3t$ from $3m$ to $3m + 3t$ if $m \geqq 6t$ or if $k = 1$, $m \geqq 5t$, gap $E = 3t$ from $5m + 4t$ to $5m + 7t$ if $k = 1$, $m \geqq 3t$.

(I) $t > 1$. First, let $m \geqq 7t$. By A, $g \equiv 3 + E_3(m - t - 1) = m - 4t + 2$. We use the notations of Theorem 13. If $u \geqq 7$, $v = 0$, then $g - L = (u - 5)(t - 1) > 0$. If $u \geqq 7$, $v \geqq 1$, $g - L = (u-5)(t-1) + v - 1 > 0$.

Henceforth let $m = ut + v$, $2 \leqq u \leqq 6$, $0 \leqq v < t$. Theorems 17-21 state that the minimum M of $s + e_s$ for $s \geqq 4$ is either $t + 4$ or $t + 3$. Then $3 + e_3 \geqq M$ if $e_3 > t$ or $> t - 1$ and hence if there is a gap $> t + 1$ or $> t$, respectively. If $u = 5$ or 6, $C > t + 1$.

Let $k = 1$. If $u > 2$, $E > t + 1$. Let $u = 2$. There is a gap $b=4t--m$ from $3m$ to $2m + 4t$ if $m \leqq 4t$. Either $b = 2t - v > t + 1$ or $t - v = 1$. In the latter case, $2v \geqq t$ and $M = t + 3$ by Theorem 17, whence $b > t$.

Henceforth let $k > 1$, $u \leqq 4$. First, let $u = 4$. Then $B > t + 1$ unless $m = 4t$ or $4t + 1$ and then $M = t + 3$ by Theorems 18-21, whence $B > t$ if $m = 4t + 1$. When $m = 4t$, there is a gap $2t$ beginning with $41t$ if $k = 2$, $79t$ if $k = 3$ or 4, $98t$ if $k = 5$, $215t$ if $k = 6$, $292t$ if $k = 7$, $409t$ if $k = 8$ or 9, etc.

There remain the cases $u = 2$, $u = 3$, with $k \geqq 2$. By Theorems 18-21, we have $M = t + 3$ and seek a gap $> t$, except that $M = t + 4$ if $u = 2$, $k = 2, 3, 4$, $m = 2t + v$, $6v < t < 8v$, and then we seek a gap $> t + 1$.

Let $u = 3$. If $2v \geqq t$, there is a gap $> t$ from $4m - 2t$ to $3m + 3t$. Hence let $2v < t$. If $k = 2$ or 3, there is a gap $t + v$ from $\alpha = 16m + 6t$ to $\beta = 17m + 4t$, while if $v = 0$ there is a gap $2t$ from $8m + 8t = 10m + 2t = 13m - 7t$ to $10m + 4t = 12m - 2t$. If $k \geqq 2$, there is a gap $3v$ from $2m + 5t$ to $5m - 4t$. Hence let $3v \leqq t$. Let also $k = 4$. The only possible entries between α and β are $\gamma = 20m - 7t < \beta$ and $\delta = 21m - 9t \geqq \alpha$. First, let $t < 4v$. Then $\delta > \beta$, $\gamma > \alpha$. While $\gamma - \alpha < t$, we have a gap $\beta - \gamma = 2t - 3v > t$ unless $t = 3v$, and then there is a gap $4v > t$ from $22m + 10t = 250v$ to $23m + 8t = 26m - 2t = 254v$. Finally, if $k = 4$, $t \geqq 4v$, there is a gap $t + v$ from $19m + 12t$ to $20m + 10t$. If also $v = 0$, whence $m = 3t$, there is a gap $2t$ from $112t$ to $114t$.

Let $u = 2$. If $k = 2$, there is a gap $4t - m$ from $7m$ to $6m + 4t$. It exceeds t and also $t+1$ if $t > 6v \neq 0$. For $k \geqq 2$, there is a gap $8t - 3v$ from $5m - 4t$ to $2m + 4t$. It exceeds t if $t > 3v$ and exceeds $t+1$ if $t > 6v \neq 0$. Henceforth let $t \leqq 3v$.

Let $u = 2$, $k = 3$. The only possible entry between $\rho = 12m + 3t$ and $\sigma = 10m + 9t$ is $\tau = 15m - 3t$. If $5v \geqq 2t$, then $\tau \geqq \sigma$ and the gap from ρ to σ is $2t - 2v$, which exceeds t when $t > 2v$. But if $5v < 2t$, the gap from ρ to τ is $3v \geqq t$. When $3v = t$, then $m = 7v$ and there is a gap $4v > t$ from $14m + 2t$ to $12m + 8t$. Finally, let $t \leqq 2v$. The only possible entry between $\alpha = 16m + 11t$ and $\beta = 17m + 10t$ is $\gamma = 19m + 3t$. If $3v \leqq 2t$, then $\gamma \leqq \alpha$ and there is a gap $\beta - \alpha = t + v > t$. But if $3v > 2t$, the gap from γ to β is $3t - 2v > t$.

Let $u = 2$, $k = 4$. If $2t > 3v$, there is a gap $3t - 3v > t$ from $22m + 2t$ to $19m + 11t$. Hence let $2t \leqq 3v$. If also $4v \leqq 3t$, there is a gap $3v - t \geqq t$ from $23m + 11t = \delta$ to $26m + 4t = \epsilon$. But if $4v > 3t$, $27m$ lies between δ and ϵ, while $\epsilon - 27m = 2t - v > t$. There remains only the case $3v = 2t$. There is always a gap $2v$ from $40m + 14t$ to $42m + 10t$. Here $2v > t$.

26. THEOREM 23. *If* $t = 1$, $k > 0$, $m \geqq 2$, $3 + e_3$ *is equal to or greater than the minimum* $s + e_s$ *for* $s \geqq 4$ *except possibly when* $m = 4$, $k > 9$, *and when* $m = 3$, $k > 14$.

We employ gaps noted at the beginning of § 25. If $m \geqq 7$, $3 + A - 1 = m - 2$ is the minimum M in Theorem 14. If $m = 6$, $3 + D - 1 = 5 = M$. The same holds also if $m = 5$, $k = 1$. But if $m = 5$, $k \geqq 2$, $3 + C - 1 = 4 = M$. If $m = 3$ or 4, $k = 1$, $3 + E - 1 = 5 \geqq M$. If $m = 3$ or 4, $k \geqq 2$, it suffices to exhibit a gap 2, whence $3 + e_3 \geqq 4 = M$. For $m = 4$, this was done up to $k = 9$ in § 25, case $u = 4$. For $m = 3$, there is a gap 2 beginning with 32 if $k = 2$ or 3, 112 if $k = 4$, 134 if $k = 5$ or 6, 449 if $k = 7$ or 8, 1562 if $9 \leqq k \leqq 13$.

27. THEOREM 24. *If*[*] $k = 1, 2, 3, 4$, $m \geqq 2t$, $2 + e_2$ *is equal to or greater than the minimum* M *of* $s + e_s$ *for* $s \geqq 4$.

Consider a table R_k of sums by two of values of $f(x - k)$. In both R_1 and R_2 there is a gap from $\alpha = 39m + 12t$ to $\beta = 42m + 13t$ and $2 + E_2(\beta - 1) = 3m + t + 1 \geqq M$ by Theorems 13, 14, 17, 18. In both R_3 and R_4 there is a gap from α to $\gamma = 42m + 6t$, and $g = 2 + E_2(\gamma - 1) = 3m - 6t + 1$. If $m \geqq 3t$, then $g \geqq M$. If $t = 1$, $m = 2$, then $g = 1$; but there is a gap from 5 to 8 in a table of sums of two squares, whence

---

[*] Also if $k = 5$, except possibly when $t > 1$, $m < 3t$.

$2+E_2(7)=4=M$. In R_5, the new entry $43m+3t$ is $\geqq \gamma$ if $m \geqq 3t$ and then $g \geqq M$.

For $k=1, 2, 3, 4$, and $m \leqq 6t$, there is a gap from $115m+20t$ to $\delta = 120m+16t$ and $2+E_2(\delta - 1) = 5m-4t+1 \geqq t+4$.

28. *Preliminary on $k>0$, $m<2t$.* The first remark in § 10 yields

THEOREM 25. *If $k>0$, $m=t$, $s \geqq 3$, then $e_s = t-1$.*

Since $k>0$, we have $m \geqq t$ by § 2. Hence it remains to treat only the case $m>t$. It will greatly facilitate the discussion to write $m=t+q$ and employ the parameters t and q instead of t and m. We have $0<q<t$. While the values $\geqq 2$ of k may be treated together (in Part IV), a separate discussion is necessary for $k=1$.

PART III. THE CASE $k=1$, $m=t+q$, $0<q<t$, §§ 29-32.

29. THEOREM 26. *If $t \leqq 3q$, $e_5 = q-1$. If $t \leqq (s-1)q$, $s \geqq 6$, $e_s = q-1$.*

No smaller e_s will serve since (35) implies that no value of $f(x-1)$ lies between 0 and $q=f(-1)$.

We can find a small value of R in (37) which is congruent modulo m to any assigned integer.

If $s=5$, we employ the values $t \equiv -q$, $m+2t \equiv -2q \pmod{m}$, 0, q of $f(x-1)$. Adding $0, 1, \cdots, q-1$ to each, we get $4q \geqq m$ integers R congruent modulo m to the consecutive integers from $-2q$ to $2q-1$ inclusive. The maximum such R is $m+2t+q-1$.

If $s \geqq 6$, we add $0, 1, \cdots, q-1$ to t, $2t$, $t+m+2t$, 0, $q, \cdots,$ $(s-4)q$ and get $sq \geqq m$ integers R congruent modulo m to the consecutive integers from $-3q$ to $(s-3)q-1$. The maximum such R is the greater of $m+3t+q-1$ and $(s-3)q-1$.

We choose R so that $A=N-R$ is congruent to $4C+tb$, whence (44) determines an integer $(a-3b)/2$. Since $n<0$, F in (24) is positive for every D. Take $A \geqq 4m-2t$. Then (43) holds since $(m-t)^2+3t^2>0$. It remains only to examine integers less than the sum of $4m-2t$ and the preceding maximum R. If $s \geqq 5$, they are less than the greater of $6t+6q-1$ and $2t+(s+1)q-1$.

We obtain overlapping sets of consecutive integers if we add $0, 1, \cdots,$

$q-1$ to the following seven groups of sums of s of $0,\ q,\ t,\ \kappa=m+2t=3t+q$:

$$\begin{array}{ll} jt,\ jt+q,\cdots,\quad jt+(s-2)q & (j=0,1,2); \\ 3t,\ \kappa,\ \kappa+q,\cdots,\quad \kappa+(s-1)q; & \\ \kappa+t,\ \kappa+t+q,\cdots,\quad \kappa+t+(s-2)q; & \\ \kappa+2t,\ \kappa+2t+q,\cdots,\quad \kappa+2t+(s-3)q; & \\ 6t\ (\text{if } s>5),\ \kappa+3t,\ 2\kappa,\ 2\kappa+q,\ 2\kappa+2q,\ 2\kappa+3q. & \end{array}$$

30. THEOREM 27. *If* $k=1,\ m=t+q,\ 0<q<t$, *then* $e_5=t-2q-1$ *for* $t\geqq 3q$, $e_s=t-(s-2)q-1$ *for* $t\geqq(s-1)q,\ s\geqq 6$.

No smaller value of e_s will serve. For, if $s\geqq 6$, no sum of s values of $f(x-1)$ lies between $\rho=2t+(s-2)q$ and $3t$. In fact, $0,\ q,\ t$ are the only values of f less than $3t$. But if we use a single t, the maximum sum is $t+(s-1)q<\rho$, and $sq<\rho$. Next, if $s=5$, no sum of s values lies between $5t+3q$ and $6t+q$. The values not too large are $0,\ q,\ t,\ \kappa=3t+q$. The sum 2κ is too large. The case of a single summand κ is excluded, since no sum by four of $0,\ q,\ t$ lies between $2t+2q$ and $3t$. Finally, if κ is not used, the maximum sum $5t$ is too small.

We can find a small value of R in (37) which is congruent modulo m to any assigned integer.

If $s=5$, we add $0,1,\cdots,e_5$ to q, and add only $0,1,\cdots,q-1\leqq e_5$ to $0,\ t\equiv -q,\ m+2t\equiv -2q \pmod{m}$. We get $m=t+q$ integers R congruent modulo m to the consecutive integers from $-2q$ to $t-q-1$ inclusive. The maximum such R is $m+2t+q-1$.

If $s\geqq 6$, we add $0,1,\cdots,e_s$ to $(s-4)q$, but add only $0,1,\cdots,q-1\leqq e_s$ to $t,\ 2t,\ t+m+2t,\ 0,\ q,\cdots,(s-5)q$. We get $m=t+q$ integers R congruent modulo m to the consecutive integers from $-3q$ to $t-2q-1$ inclusive. The maximum such R is $\rho=m+3t+q-1$.

We proceed as in § 29. It remains to examine integers $<4m-2t+\rho=\sigma$. We employ the seven groups of sums of s values of $f(x-1)$ given at the end of § 29. To the final entry in each group except the last, we add $0,1,\cdots,e_s$; to all remaining entries we add only $0,1,\cdots,q-1$. We obtain seven overlapping sets of consecutive integers which include $0,1,\cdots,\sigma=6t+6q-1$.

31. THEOREM 28. *For* $k=1,\ m=t+q,\ 0<q<t$, *write* L *for the least* $s+e_s$ *when* $s\geqq 5$. *When* $q=1,\ L=t+1$ *if* $t\geqq 5$ (*given by* $s=6,\ 7,\cdots$, *or* $t+1$), $L=t+2$ *if* $t=3$ *or* 4 $(s=5)$, $L=5$ *if* $t=2$ $(s=5)$. *When* $q>1$, *write* $t=pq+\rho,\ 0\leqq\rho<q$. *Then* $L=q+q+1$ if $p=4$ *or if* $p\geqq 5,\ \rho>0$ $(s=p+2)$; $L=p+q$ *if* $p\geqq 5,\ \rho=0$ $(s=p+1)$;

$L = q + 5$ if $p = 3$, $\rho > 0$ ($s = 6$); $L = q + 4$ if $p = 3$, $\rho = 0$, or if $p = 1$ or 2 ($s = 5$).

The proof by means of Theorems 26 and 27 is similar to that in §§ 7, 15. We next seek the cases in which $4 + e_4 < L$.

THEOREM 29. *With the notations of Theorem 28, we have $4 + e_4 \geqq L$ if $t > 2q$, but $e_4 = q - 1$, $4 + e_4 = L - 1$, if $2t \leqq 4q \leqq 3t$ and doubtless if merely $t \leqq 2q$, with certainty for integers $\leqq 473t + 427q$.*

In a Table T_1 of sums by four of values of $f(x-1)$, there is a gap from $5t + 2q$ to $6t + q$, and $\lambda \equiv 4 + E_4(6t + q - 1) = t - q + 3$.

First, let $t = 2q$, whence $m = 3q$, $f(x) = q \cdot e(x)$, where $e(x) = p_5(-x)$ is an extended pentagonal number. By *Bulletin*, III, every integer $\geqq 0$ is a sum of four numbers chosen from 1 and values of $e(x)$. Hence every multiple of q is a sum of four numbers chosen from $q = f(-1)$ and values of $f(x)$. Hence $e_4 = q - 1$.

Second, let $t > 2q$. If $q = 1$, $\lambda = t + 2 \geqq L$. Next, let $q > 1$, whence $p \geqq 2$ in Theorem 28. Then

$$\lambda - (p + q + 1) = (p - 2)(q - 1) + \rho \geqq 0.$$

Hence if $\rho = 0$, $\lambda - (p + q) > 0$. If $p = 3$, $\rho > 0$, $\lambda - L = \rho + q - 2 > 0$. If $p = 3$, $\rho = 0$, $\lambda - L = q - 1 > 0$. Finally, if $p = 2$, then $\rho > 0$, $\lambda - L = \rho - 1 \geqq 0$. In all these cases, $\lambda \geqq L$.

Third, let $t < 2q$. Then $t = q + \rho$, $0 < \rho < q$. By (39), $n = -4q - \rho$, $C = q$. Then

$$U = 24mD - 48q\rho - 15\rho^2,$$
$$V = 2mD + 12q^2 + 6q\rho + \rho^2 + 4q + 2\rho.$$

(I) Let $q \leqq 2\rho$. We shall prove that $e_4 = q - 1$. Write $[j]$ for the set $j, j + 1, \cdots, j + q - 1$. For $b = \beta, \beta + 2, \beta + 4, \beta + 6$, and $r = 0, 1, \cdots, q - 1$, the values of $bt + r$ are congruent modulo m to βt plus the numbers in the overlapping sets $[0]$, $[\rho]$, $[2\rho]$, $[3\rho]$, since $2t \equiv \rho$. The largest number of the last set is $3\rho + q - 1 \geqq m - 1$. Since the four sets include a complete set of residues modulo m, we have $d = 8$. Then

$$P = 24q + 13\rho, \qquad W = 3mD - 72q^2 - 84q\rho - 23\rho^2,$$

$$F = m^2D^2 - 960mq^2D - 1056mq\rho D - 288m\rho^2D + 16mqD + 8m\rho D$$
$$+ 2304q^4 + 7488q^3\rho + 7668q^2\rho^2 - 768q^3 - 1344q^2\rho + 3162q\rho^3$$
$$- 756q\rho^2 + 456\rho^4 - 138\rho^3 + 64q^2 + 64q\rho + 16\rho^2.$$

The terms involving D suggest the substitution

$$mD = 960q^2 + 1056q\rho + 288\rho^2 - 16q - 8\rho - p.$$

Multiply this by mD and make the same substitution in pmD. Write $p = xq^2 + yq\rho + z\rho^2$. In F the coefficients of the terms of degree 4 in q and ρ are now all positive and small if $x = 2$, $y = 5$, $z = 1$. Then

$$F = 169\rho^4 + 676q\rho^3 + 881q^2\rho^2 + 596q^3\rho + 388q^4 - 736q^3 - 1248q^2\rho \\ - 700q\rho^2 - 130\rho^3.$$

If $\rho = 1$, then $q = 2$, $F > 0$. If $\rho \geqq 2$, the term involving $q^i\rho^j$ is numerically less than that involving $q^i\rho^{j+1}$ for $i, j = 0, 1, 2, 3$, whence $F > 0$. Division of mD by $m = 2q + \rho$ yields the remainder $-2q\rho$ and quotient $479q + 287\rho - 8$. Since $A = D + 4q$, $F > 0$ if

$$A \geqq 483q + 287\rho + 20(q - p) = 503q + 267\rho.$$

To this point in table T_1 the gaps were verified to be $\leqq q$.

(II) Let $3\rho \geqq q > 2\rho$. Is $e_4 = q - 1$? The use of the further value $b = \beta + 8$ and hence the set $[4\rho]$ gives $d = 10$ as in (I). Also the method of § 21 yields $d = 10$. We find that

$$F = m^2D^2 - 1744mq^2D - 1868mq\rho D - 498m\rho^2D + 16mqD + 8m\rho D \\ + 10816q^4 + 27616q^3\rho + 24856q^2\rho^2 + 1311\rho^4 + 9498q\rho^3 \\ - 1664q^3 - 2720q^2\rho - 1460q\rho^2 - 258\rho^3 + w^2, \quad w = 8q + 4\rho.$$

Proceeding as in (I), we get $x = 6$, $y = 9$, $z = 2$, and find that $F > 0$ if $D \geqq 869q + 496\rho$. But $A = D + 4q$. Thus $F > 0$ if

$$A \geqq 873q + 496\rho + 15(q - 2\rho) = 888q + 466\rho.$$

To this point (and on to $900q + 473\rho$), it was verified that all gaps in T_1 are $\leqq q$ when $\rho \leqq q$.

(III) Let $q > 3\rho$. Since the preceding verification becomes nearly automatic for large numbers, it is highly probable that $e_4 = q - 1$. Proof would be long since $d \geqq 12$ by either method.

32. THEOREM 30. *If* $k = 1$, $m = t + q$, $0 < q < t$, *both* $3 + e_3$ *and* $2 + e_2$ *are* $\geqq L$ *of Theorem* 28.

In a table of sums by three of values of $f(x - 1)$, there is a gap from $5t + q$ to $\alpha = 6t + 2q$, and $3 + E_3(\alpha - 1) = t + q + 2 \geqq L$ in all cases. In a table of sums by two, there is a gap from $4t + q$ to α, and $2 + E_2(\alpha - 1) = 2t + q + 1$, which exceeds $t + q + 2 \geqq L$.

COROLLARY. *If* $t > 2q$, *the minimum of all* $s + e_s$ *for* $s \geqq 1$ *is* L *of Theorem* 28 *and is given by a value* $\geqq 5$ *of* s. *But if* $t \leqq 2q$, *the minimum is* $4 + e_4 = q + 3$.

PART IV. THE CASE $k \geqq 2$, $m = t + q$, $0 < q < t$, §§ 33-41.

33. THEOREM 31. *If* $s \geqq 5$, $t \leqq (s+1)q$, *then* $e_s = q - 1$.

No smaller e_s will serve since no value of $f(x-k)$ lies between 0 and q. For, if $p > 2$, $f(-p)$ exceeds $\epsilon = f(-2) = 3m - 2t = t + 3q$.

There exists a small value of R in (37) which is congruent modulo m to any assigned integer.

If $s = 5$, we employ the following values of $f(x-k)$:

$$\epsilon \equiv 2q,\ q,\ 0,\ t \equiv -q,\ m + 2t \equiv -2q, \qquad (53)$$
$$3m + 3t \equiv -3q,\ 6m + 4t \equiv -4q \pmod{m}.$$

Adding $0, 1, \cdots, q-1$ to each, we get $7q \geqq m$ integers R which are congruent modulo m to the consecutive integers from $-4q$ to $3q - 1$ inclusive. The maximum such R is $6m + 4t + q - 1$.

If $s \geqq 6$, we employ

$$2\epsilon + (s-6)q \equiv (s-2)q,\ \epsilon + (s-5)q \equiv (s-3)q,\ (s-4)q, \cdots,$$
$$q,\ 0,\ t \equiv -q,\ 2t \equiv -2q,\ t + (m + 2t) \equiv -3q \pmod{m}. \qquad (54)$$

Adding $0, 1, \cdots, q-1$ to each, we get $(s+2)q \geqq m$ integers R which are congruent modulo m to the consecutive integers from $-3q$ to $(s-1)q - 1$ inclusive. The maximum such R is the greater of $2t + (s+1)q - 1$ and $m + 3t + q - 1$.

First, let $k = 2$. Then

$$n = 2t - 5m, \quad C = 3m - 2t, \quad A = D + 4C. \qquad (55)$$

We employ a single odd b and apply § 4 with $d = 2$, $e_4 = 0$. Then P is negative, and only the minor conditions are to be satisfied. We have

$$16V - U = 8mA + 151m^2 - 100mt + 28t^2 - 32me_4 > 0,$$

for every $A \geqq 0$. Here (14) becomes

$$3A \geqq 23m - 8t - 4t^2/m,$$

which holds if $A \geqq 8m - 3t$. Then

$$U = 24mA + 36t^2 + 36mt - 135m^2 > 0.$$

It remains only to examine the integers less than the sum of $8m - 3t$ and the preceding maximum R. For $s \geqq 5$, these are less than the greater of $15t + 15q - 1$ and $7t + (s+9)q - 1$. The left column of the following table contains only sums of s values of $f(x-2)$. Consecutive entries in any line differ by q. To each entry we add $0, 1, \cdots, q-1$. We therefore obtain a set of consecutive integers terminating with $M-1$, where M is listed in the right column in a relation $M \geqq Q$ found by use of $(s+1)q \geqq t$. Since Q is the first entry of the next line of the first column, the sets of consecutive integers overlap, and hence together include all integers from 0 to $15t + 15q - 1$.

$0, q, \cdots, sq$	$(s+1)q \geqq t$
$t,\ t+q,\ t+2q,\ \epsilon + jg \quad (j = 0, \cdots, s-3)$	$t + (s+1)q \geqq 2$
$2t, 2t+q, 2t+2q, t+\epsilon+jg$	$2t + (s+1)q \geqq 3$
$3t, 3t+q, 3t+2q, 2t+\epsilon+jq$	$3t + (s+1)q \geqq 4$
$4t, \kappa+t+lq\ (l = 0, 1, 2), \epsilon+\kappa+jq$	$4t + (s+2)q \geqq 5$
$\kappa+2t+lq, t+\epsilon+\kappa+jq$	$5t + (s+2)q \geqq 6$
$\kappa+3t, 2\kappa, \lambda+pq\ (p = 0, \cdots, s-1)$	$6t + (s+3)q \geqq 7$
$2\kappa+t, \lambda+t+lq, \epsilon+\lambda+uq\ (u = 0, \cdots, s-2)$	$7t + (s+5)q \geqq 8$
$\lambda+2t+q, \lambda+2t+2q, \epsilon+\lambda+t+jq$	$8t + (s+4)q \geqq 9$
$\left\{ \begin{array}{l} 3\kappa+lq, \epsilon+\lambda+2t, \epsilon+\lambda+2t+q, \\ \quad 2\epsilon+2\kappa+t, 2\epsilon+\lambda+t+wq\ (w = 0, \cdots, s-4) \end{array} \right.$	$9t + (s+6)q \geqq 10t$
$\kappa+\lambda+t+q, \mu+pq$	$10t + (s+6)q \geqq 11t$
$\kappa+\lambda+2t+q, \mu+t+uq$	$11t + (s+5)q \geqq 12t$
$4\kappa, 4\kappa+q, 2\lambda+uq$	$12t + (s+5)q \geqq 13t$
$4\kappa+t, 2\kappa+\lambda+t, 2\kappa+\lambda+t+q, \kappa+\mu+uq$	$13t + (s+6)q \geqq 14t$
$\left\{ \begin{array}{l} 2\kappa+\lambda+2t, 2\lambda+2t+zg\ (z = 0, 1), \\ \quad \kappa+\mu+t+q+zq, \kappa+\epsilon+\mu+jq \end{array} \right.$	$14t + (s+8)q \geqq 15t$
$\kappa+2\lambda+lq, \nu+pq$	$15t + (s+10)q \geqq 16t$

The case $k > 2$ of Theorem 31 is a corollary to the case $k = 2$.

34. THEOREM 32. *If* $k \geqq 2$, $m = t + q$, $0 < q < t$, $t \geqq (s+1)q$, $s \geqq 5$, *then* $e_s = t - sq - 1$.

No smaller value of e_s will serve. For, 0 and q are the only values $< t$ of $f(x-k)$, whence no sum of s values lies between sq and t.

If $s = 5$, add $0, 1, \cdots, e_5$ to the first number ϵ in (53), but add only $0, 1, \cdots, q-1 \leqq e_5$ to the others. We get $t + q = m$ integers R which are congruent modulo m to the consecutive integers from $-4q$ to $t - 3q - 1$ inclusive. The maximum such R is $6m + 4t + q - 1$.

If $s \geqq 6$, we proceed similarly with (54) instead of (53), and get $t + q = m$ integers R which are congruent modulo m to the consecutive integers from $-3q$ to $-2q - 1$ inclusive. The maximum such R is $m + 3t + q - 1$.

As in § 33, it remains only to examine the integers less than the sum $15t+15q-1$ of $8m-3t$ and the larger of the preceding maxima. We employ the left column of the table in § 33. We now add $0, 1, \cdots, e_s$ to the last entry of each line and obtain $Q-1$, where Q is the second member of the inequality in the right column.

35. THEOREM 33. *When* $k \geqq 2, m=t+q, 0<q<t$, *write* L *for the least* $s+e_s$ *for* $s \geqq 5$. *When* $q=1, L=t-1$ *if* $t \geqq 6$ (*given by* $s=5, 6, \cdots$, *or* $t-1$); $L=5$ *if* $t=2, 3, 4$, *or* 5 $(s=5)$. *When* $q>1$, *write* $t=pq+\rho, 0 \leqq \rho < q$. *Then* $L=p+q-1$ *if* $p \geqq 6, \rho \geqq 1$ $(s=p)$; $L=p+q-2$ *if* $p \geqq 6, \rho=0$ $(s=p-1)$; $L=q+4$ *if* $1 \leqq p \leqq 5$ $(s=5)$.

This follows from Theorems 31 and 32.

36. THEOREM 34. *If* $q=1, 4+e_4<L$ *only when* $k \geqq 2, t=2$ *or* 3, *and when* $k \geqq 3, t=4$, *and then* $4+e_4=4=L-1$.

Here $t=m-1$, $f(x)=e(x)=\frac{1}{2}m(x^2+x)-x$ is an extended polygonal number. By *Bulletin*, III, when $k=2$, $e_4=t-5$ if $t \geqq 7$, $e_4=2$ if $t=6$, $e_4=1$ if $t=4$ or 5, $e_4=0$ if $t=2$ or 3; when $k \geqq 3$, $e_4=t-5$ if $t \geqq 6$, $e_4=1$ if $t=5$, $e_4=0$ if $t=2, 3, 4$.

37. THEOREM 35. *When* $m=t+q, 1<q<t$, *then* $4+e_4 \geqq L$ *if* $p \geqq 5, k \geqq 2$; *and if* $k=2, p=4$, *or* $p=3, \rho>0$. *But if* $t=3q, k \geqq 2$, $e_4=q-1$.

In a table T_k of sums by four of values of $f(x-k)$, there is a gap from $4q$ to t if $p \geqq 4$. First, let $p \geqq 6$. Then

$$\lambda=4+E_4(t-1)=pq+\rho-4q+3$$

exceeds L by $(p-5)(q-1)+\delta>0$, where $\delta=\rho-1$ if $\rho \geqq 1$, but $\delta=0$ if $\rho=0$. Second, let $p=5$. Then $\lambda-L=\rho-1$. But if $p=5, \rho=0$, then $t=5q$, $f(x)=q \cdot e(x)$, for $e(x)$ as in § 36 with $m=6$, whence its e_4 is 1. In other words, a table of sums by four of values of $e(x-k)$ has the maximum gap 2 if $k \geqq 2$. Hence T_k has the maximum gap $2q$. Thus, for $f(x-k)$, $e_4=2q-1$, $4+e_4>L$.

In the following table T_2, the terms free of t are to be multiplied by q:

0—4; $t+0$—6; $2t+0$—8; $3t+0$—4, 6, 7, 9, 10; $4t+0$—6, 9, 12;
$5t+1$, 2, 4, 5, 7, 8; $6t+1$—7, 10; $7t+2$—8; $8t+2$—10; $9t+3$—6, 9, 12;
$10t+3$—9; $11t+4$, 6—11; $12t+4$—10, 12, 13; $13t+5$—10, 12, 15;
$14t+6$—12; $15t+6$—8, 10—13; $16t+7$—15.

All entries $xt + yq$ with $x \geqq 17$ have $y \geqq 8$ and exceed $z = 15t + 10q$ since $t > q$.

If $p = 4$, $k = 2$, there is a gap from $5t + 2q$ to $y = 5t + 4q$, and $4 + E_4(y - 1) = 2q + 3 > L$. If $p = 3$, $k = 2$, there is a gap from $14t + 12q$ to z, and $4 + E_4(z - 1) = q + \rho + 3 \geqq L$ if $\rho > 0$. But if $\rho = 0$, then $t = 3q$ and $f(x) = q \cdot e(x)$, with $m = 4$ in the $e(x)$ of § 36. Since every integer $\geqq 0$ is a sum of four values of $e(x - k)$, every multiple of q is a sum of four values of $f(x - k)$, whence $e_4 = q - 1$.

38. THEOREM 36. *Let* $m = t + q$, $t = pq + \rho > q$, $p = 1$ *or* 2, $0 \leqq \rho < q$, $k \geqq 2$. *Then* $e_4 = q - 1$.

No smaller e_4 will serve since there is a gap from 0 to q. If $p = 1$ or if $p = 2$, $\rho = 0$, then $t \leqq 2q$ and the theorem is a corollary to Theorem 29.

Let $p = 2$, $\rho > 0$, $k = 2$. We have (55). Thus

$$n = -11q - 3\rho, \quad U = 24mD + 585q^2 + 282q\rho + 33\rho^2,$$
$$C = 5q + \rho, \quad V = 2mD + 115q^2 + 64q\rho + 9\rho^2 + 6q + 2\rho.$$

We obtain the smaller d by the method of § 21. We employ only $b = \beta, \cdots, \beta + 7$ and hence suppress the final pair in (46) and the final set in (50) and (51). When $\beta \equiv 1 \pmod 4$, there are two cases. If $2\rho \leqq q$, the sets

$$], [6t - 4m] = [2\rho], [2t - m] = [\rho + q], [t] = [\rho + 2q], [m] = [\rho + 3q],$$
$$[3t - m] = [2\rho + 3q], [2t] = [2\rho + 4q], [5t - 2m] = [3\rho + 4q],$$
$$[4t - m] = [3\rho + 5q]$$

overlap and terminate with $3\rho + 6q - 1 > 2m - 1$. But if $2\rho > q$, we insert $[7t - 5m] = [2\rho - q]$ after $[0]$. If $\beta \equiv 3 \pmod 4$, the sets

$$[0], [3t - 2m] = [\rho], [2t - m] = [\rho + q], [5t - 3m] = [2\rho + q],$$
$$[4t - 2m] = [2\rho + 2q], [7t - 4m] = [3\rho + 2q],$$
$$[6t - 3m] = [3\rho + 3q], [2t] = [2\rho + 4q], [t + m] = [2\rho + 5q]$$

overlap and terminate with $2\rho + 6q = 2m - 1$. If $\beta \equiv 0 \pmod 4$, the set $[4t - m] = [3\rho + 5q]$ terminates with $3\rho + 6q - 1 \equiv \rho - 1 \pmod{2m}$ and includes $0, 1, \cdots, \rho - 1$. We employ these and

$$[3t - 2m] = [\rho], [6t - 4m] = [2\rho], [5t - 3m] = [2\rho + q],$$
$$[t] = [\rho + 2q], [m] = [\rho + 3q], [3t - m] = [2\rho + 3q],$$
$$[2t] = [2\rho + 4q], [t + m] = [2\rho + 5q],$$

which overlap and terminate with $2m-1$. If $\beta \equiv 2 \pmod 4$, the sets

$$\begin{gathered}[0],\ [3t-2m]=[\rho],\ [2t-m]=[\rho+q],\ [5t-3m]=[2\rho+q],\\ [4t-2m]=[2\rho+2q],\ [3t-m]=[2\rho+3q],\\ [6t-3m]=[3\rho+3q],\ [5t-2m]=[3\rho+4q],\ [t+m]=[2\rho+5q]\end{gathered}$$

overlap and terminate with $2m-1$. Hence $d=8$ in every case. Then

$$\begin{gathered}P=31q+11\rho,\quad W=3mD-47q^2-50q\rho-11\rho^2,\\ F=m^2D^2-1368mq^2D-1008mq\rho D-184m\rho^2D+24mqD+8m\rho D\\ -33786q^4-41322q^3\rho-18462q^2\rho^2-3558q\rho^3-248\rho^4\\ +882q^3+474q^2\rho+30q\rho^2-10\rho^3+w^2,\quad w=12q+4\rho.\end{gathered}$$

Proceeding as in (I) of § 31, we get $x=25$, $y=11$, $z=2$,

$$\begin{aligned}F=1039q^4-524q^3\rho+183q^2\rho^2+526q\rho^3+124\rho^4\\ +282q^3+10q^2\rho-106q\rho^2-26\rho^3+w^2>0.\end{aligned}$$

Hence $F>0$ if $mD \geqq 1393q^2+1019q\rho+186\rho^2-24q-8\rho$, and hence if $D \geqq 465q+186\rho$. Adding $4C=20q+4\rho$, we get $F>0$ if

$$A \geqq 485q+190\rho+15(q-\rho)=500q+175\rho.$$

It was verified that all gaps are $\leqq q$ in T_2 to this point.

39. *Case* $p=3$. First, let $k=3$. We use the following part of table T_3 of sums by four of values of $f(x-3)$:

$48t+31$—45; $49t+32$—38, 40—43, 45, 46; $50t+32$—40, 42, 43, 45, 48;
$51t+33$—36, 38—44, 48, 49; $52t+34$—43, 46, 48, 51;

in which terms free of t are to be multiplied by q. All entries $gt+hq$ with $g<48$ have $h \leqq 43$ except $45t+46q$, and all such are $<\alpha=50t+40q$. To show that $\gamma=49t+45q$ is the only possible entry between α and $\delta=52t+34q$, note that $48t+45q<\alpha$, $49t+43q \leqq \alpha$, $49t+46q>\delta$, $50t+42q>\delta$, $51t+36q<\alpha$, $51t+38q>\delta$. All entries $gt+hq$ with $g \geqq 53$ have $h \geqq 36$ and exceed δ.

If $2q<3\rho$, then $\gamma<\delta$ and there is a gap $2q-\rho>q$ from α to γ. But if $2q \geqq 3\rho$, the gap is from α to δ and is 2ρ.

Hence there remains only the case $2\rho \leqq q$. We shall prove that $e_4=q-1$. The use of only odd values of b yields too large a value of d. We obtain the small value $d=9$ by using the same sets as in § 21, but expressed in terms of q and ρ.

$\beta \equiv 1 \pmod{4}$: $[0]$, $a = [8t - 6m] = [2\rho]$, $b = [7t - 5m] = [2\rho + q]$,
$c = [2t - m] = [\rho + 2q]$, $[t]$, $[m]$, $f = [8t - 5m] = [3\rho + 4q]$,
$g = [3t - m] = [2\rho + 5q]$, $h = [2t] = [2\rho + 6q]$,
$i = [6t - 3m] = [3\rho + 6q]$, $j = [5t - 2m] = [3\rho + 7q]$.

$\beta \equiv 3 \pmod{4}$: $[0]$, a, $k = [3t - 2m] = [\rho + q]$, c, $l = [6t - 4m]$
$= [2\rho + 2q]$, $p = [5t - 3m] = [2\rho + 3q]$, $r = [4t - 2m] = [2\rho + 4q)$,
f, $s = [7t - 4m] = [3\rho + 5q]$, h, $u = [t + m] = [2\rho + 7q]$.

$\beta \equiv 0 \pmod{4}$: j, $v = [4t - 3m] = [\rho]$, k, b, l, p, $[m]$, f, g, h, u.

$\beta \equiv 2 \pmod{4}$: $[0]$, a, k, c, $[t]$, p, r, g, s, i, u.

In each case the sets overlap, terminate with a number $\geqq 2m - 1$, and hence include a complete set of residues modulo $2m$.

In view of its application below, we shall prove that $E_4(A) = q - 1$ when $2\rho > q$ for every A not between the above α and δ. If $\beta \equiv 1 \pmod{4}$, insert the sets $v = [\rho]$ and r after $[0]$ and $[m]$ respectively. If $\beta \equiv 3 \pmod{4}$, insert only v. If $\beta \equiv 0 \pmod{4}$, insert only r. In these three cases, the use of nine consecutive values of b leads to a complete set of residues modulo $2m$. But this fails if $\beta \equiv 2 \pmod{4}$, since the only available sets not used above are j and b, neither of which contains any one of the now missing numbers between $q - 1$ and 2ρ. All of the latter occur in the new set $[9t - 7m] = [2\rho - q]$ obtained from the new value $\beta + 9$ of b. Hence we may take $d = 10$ in all cases.*

By taking $d = 10$, we may treat together the above two cases $2\rho \leqq q$ and $2\rho > q$. We shall find a limit on A beyond which $e_4(A) = q - 1$. For $k = p = 3$, $e_4 = q - 1$, we have

$$n = -22q - 5\rho, \quad U = 24mD + 3108q^2 + 1380q\rho + 153\rho^2,$$

$$C = 15q + 3\rho, \quad V = 2mD + 476q^2 + 218q\rho + 25\rho^2 + 8q + 2\rho,$$

$$P = 50q + 13\rho, \quad W = 3mD + 76q^2 + 10q\rho - 2\rho^2,$$

$$\begin{aligned} F = {} & m^2D^2 - 3248mq^2D - 1748mq\rho D - 234m\rho^2D + 32mqD + 8m\rho D \\ & - 422624q^4 - 417448q^3\rho - 153764q^2\rho^2 - 25038q\rho^3 - 1521\rho^4 \\ & + 8032q^3 + 5240q^2\rho + 1120q\rho^2 + 78\rho^3 + w^2, \quad w = 16q + 4\rho. \end{aligned}$$

Write $mD = 3248q^2 + 1748q\rho + 234\rho^2 - 2w + p$. Multiply this by mD and make the same replacement in pmD. Take $p = xq^2 + yq\rho + z\rho^2$. In F the coefficients of the terms of degree 4 in q and ρ are small and positive if $x = 126$, $y = 57$, $z = 7$. Then

---

* No improvement results by working modulo $8m$.

$$F = 2500q^4 + 2300q^3\rho + 3105q^2\rho^2 + 1334q\rho^3 + 166\rho^4$$
$$+ 4000q^3 + 2408q^2\rho + 440q\rho^2 + 22\rho^3 + w^2.$$

Hence $F > 0$ if $mD \geqq 3374q^2 + 1805q\rho + 241\rho^2$. Division by $m = 4q + \rho$ yields the remainder $-2q^2 - 3q\rho$ and quotient $844q + 241\rho$. But $A = D + 4C$. Hence $F > 0$ if

$$A \geqq 904q + 253\rho + 15(q - \rho) = 919q + 238\rho.$$

To this entry of T_3 the gaps are $\leqq q$ except between α and δ. This proves the above statements.

For $k = 4$, we annex to T_3 the sum

$$50t + 41q = t + 6t + 10q + 15t + 10q + 28t + 21q$$

of four values of $f(x - 4)$. Now the gaps are $\leqq q$ between α and δ. Hence the above statements imply $e_4 = q - 1$.

THEOREM 37. *Let* $m = t + q$, $t = 3q + \rho$, $0 < \rho < q$. *If* $k = 3$ *and* $2\rho > q$, *then* $4 + e_4 \geqq L$. *If* $k \geqq 3$ *and* $2\rho \leqq q$, *then* $e_4 = q - 1$, $4 + e_4 = L - 1$. *The latter holds always if* $k \geqq 4$.

40. *Case* $p = 4$. First, let $k = 3$. From T_3 we need

$12t$+4—16, 19, 24; $13t$+5—10, 12—14, 15, 18; $14t$+6—13, 15, 16;
$15t$+6—8, 10—13, 15, 16, 18, 21; · · · ; $23t$+12—16, 18, 19, 21, 22;
$24t$+12—24, 28; $25t$+13, 14, 16—22, 24, 25; $26t$+14—17, 19—22, 24, 27.

and the following facts. The only entries $xt + yq$ with $x < 12$, $y > 18$ are $9t + 19q$ and $10t + 21q$, both $< \alpha = 13t + 18q$. All entries with $x > 15$ have $y \geqq 7$ and exceed $\beta = 15t + 10q$. All with $x \leqq 22$ have $y \leqq 25$ and are $< \delta = 25t + 22q$. All with $x \geqq 27$ have $y \geqq 15$ and are $\geqq \epsilon = 26t + 19q$. Hence the only possible entry between α and β is $\gamma = 12t + 24q$. If $2q < 3\rho$, then $\gamma < \beta$ and there is a gap $2q - \rho > q$ from α to γ. Hence let $2q \geqq 3\rho$. Then there is a gap 2ρ from α to β. Hence let $2\rho \leqq q$. Then there is a gap $q + \rho$ from δ to ϵ. But if $\rho = 0$, then $t = 4q$, $m = 5q$, and, as at the end of § 37 with $m = 5$ instead of $m = 4$, we have $e_4 = q - 1$.

Next, let $k = 4$. From T_4 we need

$68t$+47, 49—59, 61, 62, 65, 68; $69t$+48—50, 52—58, 60—65;
$70t$+49—53, 55—57, 59—62, 64—66, 71;

and the facts that every entry $xt + yq$ with $x \leqq 67$ has $y \leqq 66$ and is $\leqq \phi = 69t + 58q$, while every one with $x \geqq 71$ has $y \geqq 50$ and is $\geqq \psi =$

$71t + 50q$. Hence there is a gap[*] 2ρ from ϕ to ψ. But if $2\rho \leqq q$, it was verified that all gaps are $\leqq q$ to $200t + 185q$. Then an attempt to prove that $e_4 = q - 1$ leads to $d = 16$ by use of $b = \beta + j$, $j = 0, 10, 4, 14, 8, 2$. $12, 6$. The method of § 21 gives the same d. But the limit for A would be very large.

Finally, if $k = 5$ and $2\rho > q$, there is a gap 2ρ from $85t + 74q$ to $87t + 66q$.

THEOREM 38. *Let* $m = t + q$, $t = 4q + \rho$, $0 \leqq \rho < q$. *If* $k = 3$. $e_4 \geqq q$ *or* $e_4 = q - 1$ *according as* $\rho > 0$ *or* $\rho = 0$. *If* $k = 4$ *or* 5, $e_4 \geqq q$ *when* $2\rho > q$, *but when* $2\rho \leqq q$, $E_4(A) = q - 1$ *at least for* $A \leqq 200t + 185q$.

Theorems 35-38 may be combined into

THEOREM 39. *Let* $k \geqq 2$, $m = t + q$, $1 < q < t$. *Then* $e_4 = q - 1$ *if and only if* $k = 2$, $t \leqq 3q$; $k = 3$, $2t \leqq 7q$ *or* $t = 4q$; $k = 4$ *or* 5, $2t \leqq 9q$; $k \geqq 6$, $t \leqq 4q$, *but with no conclusion when* $4q < t < 5q$. *In all further cases (including* $t \geqq 5q$ *if* $k \geqq 6$), $4 + e_4 \geqq L$ *of Theorem* 33.

41. THEOREM 40. *If* $k = 2$, 3 *or* 4, $m = q + t$, $0 < q < t$, *then* $2 + e_2$ *exceeds* L *of Theorem* 33.

If $k = 2$, there is a gap $\tau = 2t + 3q$ from $13t + 7q$ to $15t + 10q$. Then $2 + E_2 = \tau + 1$. If $q = 1$, $\tau + 1 > L$. Let $q > 1$. Since

$$\tau + 1 - (p + q - 1) = (p + 2)(q - 1) + 4 + 2\rho > 0,$$

the theorem follows if $p \geqq 6$. Also $\tau + 1 > q + 4$.

If $k = 3$, there is a gap $3t + 3q > \tau$ from $73t + 57q$ to $76t + 60q$. If $k = 4$, there is a gap τ from $133t + 112q$ to $135t + 115q$.

THEOREM 41. *If* $k = 2$ *or* 3, $m = q + t$, *then* $3 + e_3$ *is equal to or greater than the least* $s + e_s$ *for* $s \geqq 4$.

First, let $k = 2$. There is a gap from $20t + 13q$ to $21t + 12q$ and $\sigma = 3 + E_3 = t - q + 2$. If $q = 1$, $t \geqq 4$, $\sigma \geqq L$. If $q = 1$, $t = 3$, then $\sigma = 4 = 4 + e_4$ by Theorem 34. If $q = 1$, $t = 2$, there is a gap from $17t + 16q = 50$ to $19t + 14q = 52$ and $3 + E_3(51) = 4 = 4 + e_4$. Henceforth, let $q > 1$. Since

$$\sigma - (p + q - 1) = (p - 2)(q - 1) + \rho + 1,$$
$$\sigma - (q + 4) = q(p - 2) + \rho - 2,$$

---

[*] If $3\rho > 2q$, the earliest gap $> q$ is somewhere between $40t + 43q$ and $42t + 35q$. This holds also if $k = 5$.

$\sigma \geqq L$ unless $p = 1$ or 2. Let $t = q + \rho$, $0 < \rho < q$. Then $C = 28t + 21q$ is the only possible entry between $A = 28t + 18q$ and $B = 30t + 18q$. If $q \geqq 2\rho$, then $C \geqq B$ and there is a gap $2t$ from A to B, while $3 + E_3 = 2t + 2 \geqq q + 4$. But if $q < 2\rho$, the gap is from A to C, and $3 + E_3 = 3q + 2 \geqq q + 4$. Finally, let $t = 2q + \rho$. Then $\gamma = 19t + 11q$ is the only possible entry between $\alpha = 17t + 16q$ and $\beta = 19t + 14q$. According as $q \geqq 2\rho$ or $q < 2\rho$, there is a gap $2q + 2\rho$ from α to β or $3q$ from γ to β, and these gaps are the same as before.

Second, let $k = 3$. For every k there is a gap from $3q$ to t if $t > 3q$. Write $\epsilon = 3 + E_3 = t - 3q + 2$. If $q = 1$, $t \geqq 6$, then $\epsilon = L$. If $q > 1$, $p \geqq 5$, then $\epsilon \geqq L$ since $\epsilon - (p + q - 1) = (p - 4)(q - 1) + \rho - 1 \geqq 1$ for $p \geqq 6$.

When $3q \leqq t \leqq 5q$, there is a gap $6q - t$ from $52t + 42q$ to $51t + 48q$. Write ϕ for $3 + E_3 = 6q - t + 2$. If $q = 1$, $t = 3$ or 4, $e_4 = 0$ by Theorem 34 and $\phi = 8 - t \geqq 4 + e_4$, while if $t = 5$ there is a gap 3 from $3t + 3q = 18$ to $3t + 6q = 21$ and $3 + 2 = L$. Next, let $q > 1$. Since $p < 5$, $t < 5q$. Hence $\phi \geqq q + 4$ unless $5q = t + 1$, and then there is a gap $2q$ from $2t + q$ to $2t + 3q$, while $2 + 2q \geqq q + 4$.

Let $t = 2q + \rho$, $0 \leqq \rho < q$. Then $\gamma = 19t + 11q$ is the only possible entry between $\alpha = 17t + 16q$ and $\delta = 18t + 16q$. If $q \geqq 2\rho$, then $\gamma \leqq \alpha$ and there is a gap from α to δ. Then $3 + E_3 = t + 2 \geqq q + 4$ unless $\rho = 0$. But for $t = 2q$, $e_4 = q - 1$ and $t + 2 \geqq 4 + e_4$. But if $q < 2\rho$, there is a gap $5q - t$ from γ to δ and $3 + E_3 = 3q - \rho + 2 \geqq q + 4$.

There remains only the case $t = q + \rho$, $0 < \rho < q$. Then $e_4 = q - 1$ and $K = 51t + 48q$ is the only possible entry between $G = 55t + 39q$ and $H = 56t + 40q$. If $3q \geqq 5\rho$, then $K \geqq H$ and the gap is from G to H, while $3 + E_3 = 2 + 2q + \rho \geqq 4 + e_4$. But if $3q < 5\rho$, there is a gap from G to K and $3 + E_3 = 2 + 5q - 4\rho \geqq 4 + e_4$.

We do not investigate the case $k \geqq 4$, not having a table of sums by three expressed in terms of q and ρ.

Démonstration de Quelques Propriétés des Ensembles Abstraits.

PAR MAURICE FRÉCHET.

Introduction. Dans mon livre sous presse: *Les Espaces Abstraits et leur Théorie considérée comme Introduction à l'Analyse Générale** je n'ai donné qu'un petit nombre des démonstrations, me contentant souvent de renvoyer aux différents mémoires où les démonstrations utilisées avaient été publiées. Cependant, parmi les propriétés citées dans mon livre, quelques unes étaient inédites, d'autres avaient été énoncées sans démonstration par certains auteurs, enfin plusieurs de ces propriétés n'avaient été démontrées que dans des cas particuliers. On trouvera ici les démonstrations de plusieurs de ces propriétés. Pour chacune d'elles, j'indiquerai entre parenthèses la page correspondante de mon livre (approximativement d'après la pagination du manuscrit).

La terminologie sera celle qui a été adoptée dans mon livre.† Les références bibliographiques seront indiquées par les chiffres en caractères gras, qui correspondent aux numéros figurant dans la liste bibliographique placée à la fin de mon livre.

Le mémoire traite des propriétés des ensembles appartenant:

1. à un espace (V), page 49; 2. à un espace accessible (H), page 59;
3. à un espace régulier, page 64; 4. à un espace (L), page 64;
5. à un espace (S), page 65; 6. à un espace (D), page 66.

1. *Propriétés des ensembles appartenant à un espace (V).*

I. *Tout ensemble E compact en soi au sens très strict possède la pripriété de Borel* (page **270**). Il s'agit de montrer que si une famille dénombrable F d'ensembles $I_1, I_2, \cdots, I_n, \cdots$ couvre E, il en est de même d'une famille finie f convenablement extraite de F.

---

* In 8° (25 x 16) x + 300 pages environ, chez Gauthier-Villars, Paris, 1928. Ouvrage faisant partie de la " Collection de monographies sur la Théorie des fonctions " publiée sous la direction d'Emile Borel.

† Ou dans mes derniers mémoires (sensiblement différente par conséquent de celle que j'avais d'abord employée dans ma Thèse).

Nous pourrons employer le mode de raisonnement utilisé par M. Borel pour le cas linéaire.

Ou bien l'ensemble I_1 couvre à lui seul E, et on peut constituer la famille f au moyen de I_1 seul; ou bien il existe au moins un point a_1 de E non intérieur à I_1. Le point a_1 est intérieur à l'un au moins des ensembles de F; soit $J_2 = I_{n_2}$, le premier d'entre eux, (J_2 est nécessairement distinct de I_1). Ou bien, on peut constituer f de $I_1, I_2, \cdots, I_{n_2}$, ou bien il existe au moins un point a_2 de E qui n'est intérieur à aucun des ensembles $I_1, \cdots, I_{n_2}$. Et ainsi de suite; supposons extraite de F une suite d'ensembles $I_{n_2}, \cdots, I_{n_p}$ et une suite de points de E distincts $a_1, \cdots, a_p$, telles que pour $k \leq p$, I_n soit le premier des ensembles de la famille F auquel a_{k-1} est intérieur et que a_k ne soit intérieur à aucun des ensembles $I_1, I_2, \cdots, I_{n_k}$. Ceci a lieu au moins pour $p = 2$. Alors a_p est intérieur à l'un au moins des ensembles de F; soit $I_{n_{p+1}}$ le premier; on a nécessairement $n_1 < n_2 < \cdots < n_p < n_{p+1}$. Ou bien on peut constituer f de $I_1, I_2, \cdots, I_{n_{p+1}}$. Ou bien, il existe au moins un point de E, a_{p+1}, qui n'est intérieur à aucun de ces derniers ensembles et qui est par suite distinct de $a_1, \cdots, a_p$.

Finalement, ou bien on peut constituer la famille finie f comme il était annoncé, ou bien la formation récurrente de la suite des points distincts a_1. $a_2, \cdots$ et des ensembles distincts $I_1, \cdots, I_{n_k}$ pourrait se poursuivre indéfiniment. Mais alors, le sous-ensemble $a_1 + a_2 + \cdots$ de l'ensemble E étant infini, il existerait un point x de E dont tout voisinage contient à son intérieur une infinité de points a_n distincts. Comme x est intérieur à l'un au moins des ensembles de F, soit I_r, il existerait un voisinage de x compris dans I_r et, par suite, I_r comprendrait à son intérieur une infinité de points a_s distincts. Or a_s n'est intérieur à aucun ensemble I_r tel que $r \leq n_s$; I_r ne peut comprendre à son intérieur que les points a_s tels que $n_s < r$, c'est-à-dire seulement un nombre fini d'entre eux.

II. *Tout ensemble E possédant la propriété de Borel est compact en soi* (page **270**). Autrement dit, pour tout sous ensemble infini e de E, il existe un point de E dont tout voisinage comprend une infinité de points de e.

Pour le démontrer, tirons de e une suite infinie, S, de points distincts $a_1, a_2, \cdots, a_n, \cdots$. S'il existait un point de E dont tout voisinage comprenne une infinité de points de S, les voisinages de ce point comprendraient aussi une infinité de points de e et la proposition serait établie.

Sinon, pour tout x de E, il existerait un voisinage de x ne comprenant qu'un nombre fini de points de S. Soit e_n l'ensemble des points x de E dont chacun desquels possède au moins un voisinage $V_x^{(n)}$ ne contenant aucun des points $a_n, a_{n+1}, \cdots$. Certains des e_n sont peut-être vides; mais tout x

appartient à l'un au moins des e_n. Pour tout e_n non vide, on peut former un ensemble I_n, somme des voisinages $V_x^{(n)}$ correspondant aux points x de e_n. L'ensemble E est couvert par la famille dénombrable des ensembles I_n. En vertu de l'hypothèse, il est donc aussi couvert par un nombre fini des I_n. Soit r le plus grand des indices de ceux-ci. Le point a_r appartenant à E appartiendrait à un ensemble I_n tel que $n \leq r$, il appartiendrait donc à un voisinage $V_x^{(n)}$ ne comprenant aucun des points $a_n, a_{n+1}, \cdots, a_r, \cdots$. D'où la contradiction annoncée.

Exemple de M. Sierpinski. Nous venons d'obtenir deux conditions, l'une suffisante, l'autre nécessaire pour qu'un ensemble ait la propriété de Borel. Aucune de ces deux conditions n'est à la fois nécessaire et suffisante. Pour montrer que la première condition n'est pas nécessaire, M. Sierpinski a bien voulu me communiquer l'exemple suivant. (Il sera curieux d'observer plus loin que cet exemple peut aussi servir à démontrer que la seconde condition n'est pas suffisante).

M. Sierpinski considère un espace (V) dont les éléments sont les nombres ordinaux de première et de seconde classe, et ne sont pourvus chacun que d'un seul voisinage. Le voisinage d'un entier a est constitué par a et par l'ensemble des nombres ordinaux de seconde classe. On prend comme voisinage d'un nombre α de seconde classe, l'ensemble des nombres ordinaux $\leq \alpha$. Il est alors clair que l'intérieur du voisinage V_a d'un entier a quelconque est réduit au point a, alors que l'intérieur du voisinage V_a d'un nombre de seconde classe consiste dans l'ensemble des nombres de seconde classe appartenant à V_α.

L'espace considéré jouit de la propriété de Borel. Soit F une famille dénombrable d'ensembles $U_1, \cdots, U_n, \cdots$ qui couvre l'espace. L'un au moins des U, soit U_m, contient à son intérieur une infinité non dénombrable de points de cet espace puisque cet espace n'est pas dénombrable. Quel que soit le nombre ordinal α, il existe donc un nombre ordinal $\beta > \alpha$ et intérieur à U_m. Par suite le voisinage de β appartient à U_m. Donc U_m comprend tous les nombres $\leq \beta$ et en particulier les nombres $\leq \alpha$. Comme α est un point arbitraire de l'espace, on voit que tout l'espace appartient à U_m. Par suite tout l'espace est bien aussi intérieur à U_m, l'espace est couvert par le seul ensemble U_m extrait de F.

Puisque cet espace possède la propriété de Borel, il est compact en soi. Mais il ne l'est pas au sens très strict. En effet, extrayons de l'espace l'ensemble infini, e, constitué de tous les entiers. Il ne peut exister aucun point x de l'espace dont le voisinage contienne à son intérieur une infinité d'entiers. Car si x est un entier, l'intérieur de son voisinage se réduit au seul point x.

si x est un nombre de seconde classe, aucun entier b n'est intérieur au voisinage de x.

On peut encore utiliser ce même espace pour donner un exemple d'un ensemble E compact en soi sans posséder la propriété de Borel. Il suffit de prendre pour E l'ensemble constitué par tous les entiers et par ω, le premier nombre ordinal de seconde classe. Cet ensemble est couvert par la famille dénombrable constitué par les voisinages de ses points. Comme aucun de ces voisinages ne contient à son intérieur plus d'un point, il est impossible d'en extraire une famille finie couvrant E. E n'a pas la propriété de Borel. Par contre, si l'on considère un sous-ensemble infini, e, de E, le voisinage du nombre ω de E contient certainement une infinité de points de e. Donc E est compact en soi.

III. *La condition nécessaire et suffisante pour qu'un ensemble E soit compact* (*en soi*) *est que, quelle que soit la suite monotone de sous-ensembles* (*non vides*) *de E, à rangs entiers*

$$E_1 \supset E_2 \supset \cdots \supset E_n \supset \cdots$$

il existe au moins un point (*de E*) *appartenant aux E_n ou appartenant à leurs dérivés* (page **271**).

1°. Supposons E compact. On ne considère que des sous-ensembles E_n non vides: soit a_n un point de E_n.

Si l'un au moins des a_n est répété une infinité de fois c'est un point commun aux E_n.

Sinon la suite des a_n contient une suite S de points distincts. Alors il existe un point a (de E) dont tout voisinage contient une infinité de points distincts et appartenant à S. Par suite, aussi grand que soit n, tout voisinage de a contiendra au moins un point distinct de a et appartenant à E_n: le point a est commun aux dérivés E_n' des E_n.

2°. Montrons que la condition est suffisante. Soit e un sous-ensemble infini (s'il en existe) de E. Extrayons de e une suite infinie S de points distincts $a_1, a_2, \cdots$, et soit E_n l'ensemble des points $a_n, a_{n+1}, \cdots$. Aucun des E_n n'est vide et chacun contient le suivant. Il y a donc un point a (de E) commun aux E_n ou à leurs dérivés. Comme les E_n n'ont aucun point commun, on voit que dans tout voisinage de a, il y aura au moins un point, distinct de a, appartenant à E_n, quel que soit n. Il y aura donc dans tout voisinage de a, une infinité de points distincts appartenant à e.

IV. *Soit, dans un espace (V), une famille F d'ensembles fermés dont l'un au moins, E, est parfaitement compact. La condition nécessaire et suffisante pour qu'il existe au moins un point commun à tous les ensembles de F est que tout nombre fini d'ensembles de F possèdent un point au moins en commun* (page **271**).

La condition est évidemment nécessaire. Si elle est remplie, nous allons prouver qu'il est impossible que chaque point a de E soit disjoint de l'un au moins E_a des ensembles de F. En effet, dans ce cas, puisque E_a est fermé il existerait au moins un voisinage V_a de a qui serait disjoint de E_a. La famille des ensembles V_a couvrant E, qui est parfaitement compact en soi, il existerait d'après la généralisation du théorème de Borel-Lebesgue par M. Chittenden, un nombre fini des V_a : $V_{a_1}, \cdots, V_{a_n}$, qui suffisent pour couvrir E. Les ensembles de F, $E_{a_1}, \cdots, E_{a_n}$, et aussi E, étant en nombre fini, auraient par hypothèse, au moins un point commun α. Ce point α appartenant à E appartient à l'un des ensembles V_{a_i}, par exemple à V_{a_1}. Or les ensembles E_{a_1} et V_{a_1} sont disjoints, ils ne sauraient avoir le point α en commun.[*]

V. *Soit dans un espace (V) une famille F d'ensembles, dont l'un au moins est parfaitement compact et fermé. Si pour toute famille finie extraite de F, il y a au moins un point commun aux ensembles de cette famille finie, alors il y a sûrement un point qui est commun à tous les ensembles de F ou qui est commun à tous leurs dérivés* (page **272**).

Cette proposition généralise en partie le théorème III. Sa démonstration sera obtenue plus loin à propos d'un énoncé distinct de celui-ci (XXIV).

VI. *Si deux ensembles mutuellement connexes E, F sont chacun connexes, leur somme est connexe* (page **267**).

Soit $E+F=G+H$, il faut démontrer que si G, ni H ne sont vides, on a $GH'+G'H\neq 0$.

Or $E=E\cdot G+E\cdot H=G_1+H_1$; $F=F\cdot G+F\cdot H=G_2+H_2$.

Si G_1, ni H_1 ne sont vides alors G_1 et H_1 doivent être mutuellement connexes, (puisque E est connexe), donc G et H, qui les contiennent respectivement, sont aussi mutuellement connexes. On peut refaire le même raisonnement quand G_2, ni H_2 ne sont vides.

---

[*] Cette proposition a été énoncée pour la première fois dans le cas des espaces euclidiens par M. F. Riesz. Elle a été retrouvée et sa démonstration publiée pour la première fois par M. Sierpinski (CIII). On verra en comparant sa démonstration avec la nôtre, qu'en passant du cas euclidien au cas infiniment plus général des espaces (V), bien loin d'avoir eu à compliquer sa démonstration, nous avons pu la simplifier.

Reste le cas où l'un des ensembles G_1, H_1 est vide et où l'un des ensembles G_2, H_2 est vide. Si par exemple $G_1 = 0$, $G_2 = G \neq 0$; on a donc le cas où $H_2 = 0$ et alors $H_1 = H \neq 0$. Dès lors $E = H_1 = H$ et $F = G_2 = G$, G et H sont encore mutuellement connexes, comme E et F.

VII. *Si des ensembles deux à deux mutuellement connexes sont chacun connexe, leur somme est connexe* (page **267**). En effet, dans ce cas, d'après le théorème précédent, deux points quelconques de la somme appartiennent à un sous-ensemble connexe de cette somme. Cela suffit (page **267**) pour prouver que cette somme est connexe.

VIII. *Si un ensemble F est fermé, chacun de ses composants est un continu ou est réduit à un seul point* (page **269**).

Tout composant f d'un ensemble F étant connexe ou réduit à un point, il suffit de prouver que si F est fermé, ses composants sont aussi fermés. Si f n'était pas fermé, il existerait un point a de f' n'appartenant pas à f. Mais a appartiendrait à F', donc à F qui est fermé. L'ensemble $f + a$ est connexe, d'après l'avant-dernière proposition. Ce serait donc un sous-ensemble de F qui serait connexe et contiendrait f, c'est-à-dire contiendrait le plus grand sous-ensemble de F qui est connexe et contient un point de f.

IX. Par définition de l'intérieur I et de l'extérieur E d'un ensemble G, aucun point de I ne peut appartenir au dérivé du complémentaire de G, donc ne peut appartenir au dérivé de E. De même, aucun point de E ne peut appartenir à I'. Donc I et E ne peuvent être mutuellement connexes.

On va en conclure qu'*un ensemble ouvert et connexe I est un composant de l'ensemble complémentaire H de la frontière F de l'ensemble donné I* (page **267**).

Le plus grand sous-ensemble J de H qui est connexe et contient un point de I contient I, puisque I est un sous-ensemble connexe de H. Ainsi, J contient I. Si I n'était pas un composant de H, J contiendrait I plus un sous-ensemble non vide e de l'extérieur E de I. Il faudrait donc que I et e et par suite I et E fussent mutuellement connexes contrairement à la remarque ci-dessus.

X. Un ensemble G est connexe en un de ses points a, si dans tout voisinage V_a de a, il existe un voisinage W_a de a tel que deux points quelconques b, c de $G \cdot W_a$ appartiennent à un sous-ensemble connexe C de $G \cdot V_a$. On peut sans diminuer la généralité prendre $c = a$. Alors C est un sous-ensemble, connexe et comprenant a, de G. Donc C est un sous ensemble du composant K de G comprenant a. Un point b quelconques de $G \cdot W_a$ appartient ainsi, avec a, à un sous-ensemble connexe C de $K \cdot V_a$: K est connexe en a. La somme S des ensembles connexes C qui ont tous en commun a, est un ensemble connexe appartenant à G. De plus S contient a, est contenu dans V_a et contient $G \cdot W_a$. En particulier, si a est un point intérieur à G, on peut choisir pour V_a un sous-ensemble convenable de G. Et alors, S contenant tout W_a, on voit que a est intérieur à S, donc intérieur au composant de G qui contient a.

Ainsi lorsqu'un ensemble G est connexe en un de ses points intérieurs, soit a, a est aussi intérieur au composant de G contenant a.

Si maintenant G est localement connexe, c'est-à-dire connexe en chacun de ses points, chaque composant de G est aussi localement connexe; et si G est ouvert, tout point a de G est intérieur au composant de G contenant a. Par suite tout composant de G est ouvert. Il est d'autre part aussi localement connexe comme G. Ainsi, *tout composant d'un ensemble ouvert et localement connexe est lui-même ouvert et localement connexe* (page **269**).

XI. Supposons l'ensemble G précédent séparable. Il existe un suite dénombrable N de points $a_1, a_2, \cdots$ de G telle que G appartienne à $N + N'$. Si a est un point de G appartenant à N', comme a est intérieur à G, a est intérieur au composant, γ, de G contenant a, donc il y a au moins un point a_n de N appartenant à γ. Par suite γ est identique au composant γ_n de G contenant a_n. Ainsi, *les composants d'un ensemble séparable, ouvert et localement connexe forment une famille dénombrable* (page **272**$^{\text{bis}}$).

XII. Soit une famille F d'ensembles, dont les intérieurs existent et appartiennent à un même ensemble parfaitement séparable S. Lorsque ces intérieurs sont disjoints deux à deux, la famille F est dénombrable (page **275**).

En effet; par hypothèse, on peut prendre les voisinages des points de S parmi les ensembles d'une certaine suite dénombrable d'ensembles V_1, $V_2, \cdots, V_n, \cdots$.

Tout ensemble E de F a au moins un point intérieur a. Donc l'un des voisinages de a est un ensemble V_n appartenant à E. Soit $r(E)$ le rang du premier des ensembles $V_1, V_2, \cdots$ qui appartient à E. Pour deux ensembles distincts E, G appartenant à F, $r(E) \neq r(G)$, autrement les intérieurs de E, G ne seraient pas disjoints. Les ensemble E de F correspondent donc respectivement à des entiers $r(E)$ distincts. Donc F est dénombrable.

En particulier, *une famille de sous-ensembles ouverts et disjoints deux à deux, d'un ensemble parfaitement séparable est nécessairement dénombrable* (page **275**).

XIII. Les sous-ensembles ouverts d'un ensemble parfaitement séparable S sont chacun des sommes d'une famille d'ensembles $V_{n_1}, V_{n_2}, \cdots$ choisis dans la famille dénombrable $V_1, V_2, \cdots$. Chacun de ces sous-ensembles étant ainsi déterminé par une suite d'entiers, *la famille des sous-ensembles ouverts d'un ensemble parfaitement séparable S a au plus la puissance du continu* (page **275**).

XIV. Si F est un sous-ensemble fermé de S, $S—F$ est la partie commune à S à une des sommes $V_{r_1} + V_{r_2} + \cdots$. Donc: *la famille des sous-ensembles fermés d'un ensemble parfaitement séparable a au plus la puissance du continu* (page **275**).

XV. *Tout ensemble G parfaitement séparable est séparable* (page **279**$^{\text{bis}}$). Appelons en effet N l'ensemble dénombrable des points a_n obtenus en choisissant arbitrairement un point a_n dans chacun des voisinages V_n des points de G. On voit que G appartient à $N + N'$

XVI. *Fonctionnelles et transformations ponctuelles.* Nous avons défini (page **202**) la continuité des transformations ponctuelles de deux façons; nous allons démontrer qu'elles sont équivalentes.

Disons qu'une transformation $T(E) = F$ est continue au point a de E, si pour tout voisinage V_b du transformé b de a, il existe un voisinage V_a de a tel que $T(E \cdot V_a)$ appartienne à $F \cdot V_b$. Supposons maintenant qu'il existe un sous-ensemble, e, de E ayant a pour point d'accumulation. V_a contient au moins un point a' de e distinct de a. Par suite V_b contiendra le point b' transformé de a'. Ce point b appartient au transformé f de e. Donc: ou bien b' est confondu avec b ce qui nécessite que b appartienne à f, ou bien V_b qui est un voisinage arbitraire de b, contient un point de f distinct de b. Ainsi: *si $T(E) = F$ est une transformation continue au point a, quelque soit le sous-ensemble e de E ayant a pour point d'accumulation, le point b, transformé de a, appartient nécessairement, soit au transformé f de e, soit à son dérivé f'. La réciproque est vraie* (page **202**). Autrement, il existerait un voisinage V_b de b tel que tout voisinage V_a de a contienne un point a' de E dont le transformé b' n'appartient pas à V_b. Alors b' est distinct de b et par suite a' de a. L'ensemble e des points a' a a pour point d'accumulation. Et pourtant, le point b n'appartient, ni au transformé f de e, ni à son dérivé f', puisque f est disjoint de V_b.

XVII. Considérons une fonctionnelle $U(x)$ uniforme sur un ensemble E compact (en soi). Quel que soit n, il existe un point a_m de E tel que $U(a_n) > M—1/n$, si $U(x)$ a sur E une borne supérieure finie M ou tel que $U(a_n) > n$

dans le cas contraire. Alors il existe un point a (de E) dans tout voisinage V_a duquel se trouvent une infinité des points a_n. Finalement, *il existe au moins un point a (de E) tel que la borne supérieure (finie ou non) de $U(x)$ sur l'ensemble E compact (en soi), soit égale à la borne supérieure de $U(x)$ sur la partie de E appartenant à un quelconque des voisinages de a* (page 277).

XVIII. *Une fonctionnelle $U(x)$ continue sur un ensemble connexe C ne peut passer d'une valeur à une autre sans passer par toutes les valeurs intermédiaires* (page **277**). Soient a, b, deux points de C et α un nombre compris entre $U(a)$ et $U(b)$. Supposons, par exemple, $U(a) < \alpha < U(b)$. Si en aucun point x de C on n'avait $U(x) = \alpha$, on pourrait décomposer l'ensemble C en deux sous-ensembles disjoints et non vides

$$\begin{array}{lll} A & \text{où} & U(x) < \alpha, \\ B & \text{où} & U(x) > \alpha. \end{array}$$

Donc A et B seraient mutuellement connexes, c'est-à-dire qu'il y aurait, par exemple, un point c de A qui appartient au dérivé B' de B. Alors, dans tout voisinage V_c de c, il existerait au moins un point x de B. On aurait $U(x) > \alpha$. Or on peut prendre le voisinage tel que $| U(x) - U(c) | < \epsilon$, ϵ etant un nombre positif arbitraire et on pourrait prendre $\epsilon < \alpha - U(c)$. On aurait donc $U(x) < \alpha$ sur certains points de B, contrairement à sa définition.

XIX. *La limite $U(x)$ d'une suite de fonctionnelles $U_n(x)$ continues en a relativement à E, qui converge uniformément sur E, est continue en a relativement à E* (page **280**).

Soit ϵ un nombre positif arbitraire. Par hypothèse, il existe un nombre p tel que pour $n > p$ $| U(x) - U_n(x) | < \epsilon/3$ en tout point x de E et il existe un voisinage de a sur lequel l'oscillation de $U_n(x)$ est inférieure à $\epsilon/3$. Alors pour deux points y, z de ce voisinage et de E, on a

$$| U(y) - U(z) | \leqq | U(y) - U_n(y) | \\ + | U_n(y) - U_n(z) | + | U_n(z) - U(z) | < \epsilon.$$

XX. *La limite $U(x)$ d'une suite de fonctionnelles $U_n(x)$ également continues en a relativement à E et convergeant partout sur E est continue en a relativement à E* (page **281**).

Pour ϵ positif arbitraire, il existe par hypothèse un voisinage V_a de a sur lequel l'oscillation de chaque U_n est inférieure à $\epsilon/3$. D'autre part, pour x

assez grand ($n > n_x$), on a $| U(x) - U_n(x) | < \epsilon/3$. Pour deux points quelconques y, z de V_a, prenons $n > n_y$ et $n > n_z$; on aura

$$| U(y) - U(z) | < | U(y) - U_n(y) | \\ + | U_n(y) - U_n(z) | + | U_n(z) - U(z) | < \epsilon.$$

XXI. *Si, de plus, E est compact en soi et si la suite est également continue en chaque point de E, la convergence est nécessairement uniforme sur E* (page **281**).

En effet, dans le cas contraire, il y aurait un nombre $\epsilon > 0$ et quel que soit p, un entier $n_p > p$ et un point x_p de E tels que

$$| U(x_p) - U_{n_p}(x_p) | > \epsilon.$$

Si parmi les x_p, il n'y avait qu'un nombre fini de points distincts, une infinité des x_p seraient identiques à un même point a. On aurait donc $| U(a) - U_{n_p}(a) | > \epsilon$ avec $n_p > p$, ce qui est impossible puisque la suite converge en chaque point a de E. Dans le cas contraire, on peut ne retenir que les x_p distincts et alors, comme E est compact en soi, il existe un point a de E dans tout voisinage duquel se trouve une infinité des x_p. D'autre part, il existe un voisinage V_a de a où l'oscillation de tous les U_n est inférieure a $\epsilon/3$ et où, par suite, celle de U est $\leq \epsilon/3$. Pour une infinité de valeurs de p, on aura donc

$$| U(a) - U(x_p) | \leq \epsilon/3, \qquad | U_{n_p}(a) - U_{n_p}(x_p) | < \epsilon/3;$$

et pour p assez grand

$$| U(a) - U_{n_p}(a) | < \epsilon/3$$

On aurait donc $| U(x_p) - U_{n_p}(x_p) | < \epsilon$ pour une infinité de valeurs de p *contrairement à l'hypothèse.*

XXII. *La composition de deux transformations continues fournit une transformation continue* (page **283**). Soient deux transformations $y = A(x)$ et $z = B(y)$. Supposons $A(x)$ continue au point a relativement à un ensemble E; supposons que $B(y)$ soit continue au point $b = A(a)$ relativement à l'ensemble F transformé de E par A. Il s'agit de démontrer que la transformation $z = B[A(x)]$ est continue au point a relativement à E. Soit en effet $c = B(b)$. Les ensembles E, F et l'ensemble G, transformé de F par $B(x)$ appartiennent respectivement par hypothèse à trois espaces (V), distincts ou non. Soit V_c un voisinage arbitraire de c; il existe, par hypothèse, un voisinage V_b de b tel que le transformé par $B(x)$ de $F \cdot V_b$ appartienne à V_c. Il existe de même un voisinage V_a de a tel que le transformé par $A(x)$ de $E \cdot V_a$ appartienne à V_b. Donc il existe un voisinage V_a de a tel que le transformé par $B[A(x)]$ de $E \cdot V_a$ appartienne à V_c.

XXIII. *Toute transformation continue A transforme un ensemble connexe E en un ensemble F connexe* (page **283**).

Soit $F = H_1 + H_2$. Tout point de E est transformé, soit en un point de H_1, soit en un point de H_2; appelons G_1 le sous-ensemble des point de E transformés chacun en un point de H_1, $G_2 = E - G_1$. Si nous supposons H_1 et H_2 disjoints et non vides, G_1 et G_2 seront disjoints et non vides. Par suite $G_1 \cdot G_2' + G_2 \cdot G_1'$ n'est pas vide. Or le transformé de cet ensemble appartient à $H_1 \cdot (H_2 + H_2') + H_2 \cdot (H_1 + H_1') = H_1 \cdot H_2' + H_2 \cdot H_1'$, puisque H_1 et H_2 sont disjoints. Donc $H_1 \cdot H_2' + H_2 \cdot H_1'$ n'est pas vide.

Il faut observer que le raisonnement ne s'étend pas au cas où E serait seulement supposé bien enchaîné. Car un point a appartenant à $G_1 \cdot G_2' + G_2 \cdot G_1' + G_1' \cdot G_2'$ pourrait ne pas appartenir à E et alors on ne peut plus parler du transformé de a.

Non seulement la démonstration, mais l'énoncée ne s'étend pas. Par exemple, considérons la figure inverse D d'une circonférence S obtenue en prenant le pôle d'inversion p sur S. Prenons pour E un arc de S s'étendant de chaque coté de p mais privé du point p. Alors E est un ensemble non connexe mais bien enchaîné que l'inversion transforme dans un ensemble F composé de deux demi-droites situées sur D et séparées par un segment de D. L'ensemble F n'est ni connexe, ni bien enchaîné.

2. *Propriétés des espaces accessibles ou espaces* (H).

XXIII. *Pour un espace* (V) *satisfaisant aux condition* 1°, 2°, 3° *de F. Riesz* (page **206**), *il est équivalent de supposer que tout ensemble dérivé est fermé ou de supposer que l'intérieur de chaque ensemble est un ensemble ouvert* (page **247**).

[Dans l'un et l'autre cas, l'espace considéré sera donc un espace accessible ou espace (H)]. En effet, admettons la première condition et soit I l'intérieur d'un ensemble E. Si I n'était pas ouvert, il y aurait un point x intérieur à E sans être intérieur à I. C'est-à-dire qu'au moins un voisinage V_x de x appartiendrait entièrement à E et que tout voisinage de x contiendrait au moins un point y distinct de x et n'appartenant pas à I. L'ensemble des points y appartiendrait au complémentaire de I, donc au complementaire C de E ou à son dérivé C'. Et il aurait x pour point d'accumulation. Donc x appartiendrait à C' ou à C''. Or x appartenant à I ne peut appartenir à C'. Il y aurait donc un point x de C'' n'appartenant pas à C' contrairement à l'hypothèse.

Inversement, admettons la seconde condition. Si un ensemble dérivé F' n'était pas fermé il y aurait un point z de F'' n'appartenant pas à F'. Soit $f = F - z$; en vertu des trois conditions de F. Riesz, z sera aussi un point de f'' n'appartenant, ni à f', ni à f. Soit c l'ensemble complémentaire de f; z est intérieur à c. En vertu de la seconde condition, z est même intérieur à l'intérieur i de c. Il y a donc un voisinage V de z appartenant à i. Puisque z appartient à f'', il y aurait au moins un point t de V, distinct de z et appartenant à f'. Ce point t étant intérieur à c ne peut pourtant pas être point d'accumulation du complémentaire f de c.

XXIV. *Soit dans un espace accessible, ou espace* (H), *une famille F d'ensembles dont l'un au moins, E, est parfaitement compact et fermé. La condition nécessaire et suffisante pour qu'il existe au moins un point commun à tous les ensembles de F ou au moins un point commun aux dérivés de ces ensembles est qu'il en soit de même pour toute famille finie extraite de F* (page **272**).

La condition est évidemment nécessaire. Nous allons montrer que si elle est remplie, il est impossible que, pour tout point a de E, il existe un ensemble e_a de F disjoint de a et un ensemble E_a de F (distinct ou non de e_a) dont a n'est pas point d'accumulation. En effet, dans ce cas, il existerait un voisinage V_a de a n'ayant avec E_a aucun point commun distinct de a. Alors les V_a couvriraient E et, puisque cet ensemble est compact et fermé, E pourrait être couvert par un nombre fini des V_a : $V_{a_1}, \cdots, V_{a_n}$. Alors, si on considère les ensembles de F: $E, e_{a_1}, \cdots, e_{a_n}, E_{a_1}, \cdots, E_{a_n}$, qui sont en nombre fini, il existerait un point α commun à ces ensembles ou commun à leurs dérivés. Comme E est fermé, α appartient dans les deux cas à E. Par suite, il est intérieur à un des V_{a_i}, par exemple à V_{a_1}.

Dans le premier cas, α appartenant à E_{a_1} et V_{a_1} qui n'ont aucun point commun distinct de a_1, α coinciderait avec a_1. Or cela est impossible puisque α appartient à e_{a_1} tandis que a_1 ne lui appartient pas.

Dans le second cas, α étant intérieur à V_{a_1}, un des voisinages W_α de α appartient à V_{a_1}, et comme α est point d'accumulation de E_{a_1}, il y aurait un point au moins β de E_{a_1}, distinct de α, en commun avec W_α, donc avec V_{a_1}. Comme E_{a_1} et V_{a_1} n'ont en commun aucun point distinct de a_1, β coincide avec a_1. Puisque $E_{a_1} = a_1 + (E_{a_1} - a_1)$ a α pour point d'accumulation et puisque l'espace envisagé est accessible, α serait point d'accumulation de $E_{a_1} - a_1$ ou de l'ensemble réduit à a_1. Or $E_{a_1} - a_1$ étant disjoint de V_{a_1}, donc de W_α, ne peut avoir α pour point d'accumulation. Il faudrait donc que le point α fut point d'accumulation d'un ensemble composé d'un seul point, ce qui est impossible dans un espace accessible.

Il y a lieu de remarquer que la partie de la démonstration relative au premier cas reste valide pour un espace (V). Nous obtenons donc ici la démonstration du théorème V.

XXV. *La frontière F d'un ensemble E est un ensemble fermé* (page **286**). En effet, si C est l'ensemble complémentaire de E, on a, par définition,

$$F = C \cdot E' + C' \cdot E.$$

Dans un espace accessible, la dérivation est distributive, donc:

$$F' = C' \cdot E'' + C''E' + C'E';$$

et puisque tout ensemble dérivé est fermé,

$$F' \subset C'E' = C' \cdot E' \cdot E + C' \cdot E' \cdot C \subset C'E + E'C = F.$$

XXVI. *Tout ensemble séparable E appartient à un ensemble fermé séparable, par exemple à la fermeture $E + E'$ de E* (page **286**).

Il existe, par hypothèse, un sous-ensemble dénombrable N de E tel que

$$N \subset E \subset N + N'.$$

D'où

$$N' \subset E' \subset N' + N'' = N'.$$

Donc $E' = N'$, d'où $E + E' = N + N'$. Alors

$$E \subset N + N' \subset E + E'.$$

Et l'ensemble $E + E'$ est fermé et puisque $N \subset E + E' = N + N'$, $E + E'$ est aussi séparable.

XXVII. La formule $E' = N'$ prouve aussi que *tout ensemble séparable dense en soi appartient au derivé de l'un de ses sous-ensembles dénombrables et que tout ensemble séparable parfait est le dérivé de l'un de ses sous-ensembles dénombrables* (page **284**).

XXVIII. *Dans un espace accessible, tout ensemble parfaitement séparable et non clairsemé contient un ensemble dénombrable dense en soi* (page **274**).

L'ensemble E n'étant pas clairsemé contient un ensemble D dense en soi. Comme E est parfaitement séparable, D est aussi parfaitement séparable et par suite séparable (XV). Or tout ensemble séparable et dense en soi, D, appartient au dérivé d'un de ses sous-ensembles dénombrables (XXVII) soit N. Et comme $N \subset D \subset N'$, N est dense en soi.

XXIX. *Dans un espace accessible, tout ensemble parfaitement séparable E a au plus la puissance du continu* (page **285**). En effet, par hypothèse, il existe une famille dénombrable d'ensembles $V_1, V_2, \cdots, V_n, \cdots$ telle que la famille des voisinages de tout point x de E soit équivalente à la famille constituée de ceux des ensembles V_n, (soient $V_{n_1}, V_{n_2}, \cdots$) auxquels x est intérieur. Il n'y a d'ailleurs, puisque l'espace est accessible, qu'un point x commun aux voisinages $V_{n_1}, V_{n_2}, \cdots$ de x.

Tout point x détermine une suite d'entiers $n_1, n_2, \cdots$ telle que x soit intérieur aux ensembles $V_{n_1}, V_{n_2}, \cdots$ et à aucun autre des V_n.

Réciproquement, une telle suite d'entiers, c'est-à-dire une suite telle qu'il existe au moins un point x intérieur à $V_{n_1}, V_{n_2}, \cdots$ et à aucun autre des V_n détermine un point x de E *et un seul.*

Les points de E correspondent donc de façon biunivoque à certaines suites d'entiers $n_1, n_2, \cdots$.

Donc E a au plus la puissance du continu.

XXX. Dire qu'un point a d'un espace accessible est *à caractère dénombrable,* c'est dire qu'on peut supposer dénombrable la famille des voisinages de a. Soient donc $W_1, W_2, \cdots, W_n, \cdots$ ceux-ci. Appelons V_2 le premier des W_n qui est commun à W_1 et W_2. En posant $V_1 = W_1$, appelons en général V_p le premier des W_n qui est commun à $V_1, \cdots, V_{p-1}$ et W_p. On voit qu'on aura

$$V_1 \supset V_2 \supset \cdots \supset V_p \supset \cdots .$$

Comme les W, ces ensembles V ne pourront avoir en commun que le point a. Leur famille est équivalent à celle des W. Car chaque V est un W et chaque W_p comprend un V_p. Ainsi, *si un point a d'un espace accessible est à caractère dénombrable, on peut supposer que la famille de ses voisinages est une suite dénombrable d'ensembles V_p n'ayant en commun que le point a et dont chacun contient le suivant* (page **298**).

C'est en particulier ce que a lieu si a est un point quelconque d'un ensemble parfaitement séparable.

(Dans le cas exceptionel où la suite des V_p serait finie, la famille des voisinages de a serait même équivalente à la famille constituée par un seul ensemble: le dernier des V_p).

XXXI. *Toute transformation continue transforme un ensemble* (*parfaitement*) *compact en soi en un ensemble* (*parfaitement*) *compact en soi* (page **288**).

En effet, soit A une transformation continue d'un ensemble E d'un espace accessible en un ensemble E_1 d'un espace accessible (distinct ou non du premier). Soit e_1 un sous-ensemble infini de E_1. Chaque point y de e_1 est le transformé d'au moins un point x de E. Ne prenons qu'un point x de E pour chaque point y de e_1. Nous formerons ainsi un ensemble e de points distincts; e est un sous-ensemble infini de E et son transformé est e_1.

Si E est (parfaitement) compact en soi, il existera un point a de E dont tout voisinage comprend au moins un point de e distinct de a (comprend un ensemble de points de e ayant même puissance que e). Or si V_{a_1} est un voisinage arbitraire du transformé a_1 de a, il existe un voisinage V_a de a, tel que le transformé de $e \cdot V_a$ appartienne à V_{a_1}. Donc V_{a_1} comprend au moins un point de e_1 distinct de a_1 (comprend un ensemble de points de e_1 ayant une puissance au moins égale à celle de $e \cdot V_a$, c'est-à-dire à celle de e, c'est-à-dire à celle de e_1 et, comme la puissance de $e_1 \cdot V_{a_1}$ est au plus égale à celle de e_1, ces deux puissances sont égales).

XXXII. Démontrons une proposition qui constitue en quelque sorte une réciproque du théorème XXI.

Soit F une famille de fonctionnelles continues sur un ensemble E à caractère dénombrable (c'est-à-dire dont la famille des voisinages de chaque point de E est dénombrable). *Pour que de tout sous-ensemble infini de fonctionnelles de F, on puisse extraire une suite qui converge uniformément sur E, il faut qu'en chaque point de E, les fonctionnelles de F soient également bornées et également continues relativement à E* (page **288**).

1°. S'il existe au moins un point a de E où les fonctionnelles de F ne sont pas également bornées, c'est que, pour tout entier n, il existe une fonctionnelle $U_n(x)$ de F telle que $| U_n(a) | > n$. Cela est impossible car, par hypothèse, on pourrait extraire de la suite des $U_n(x)$ une suite qui converge uniformément sur E et en particulier qui converge en a. Cette démonstration est valable pour un ensemble E quelconque appartenant à un espace (V) quelconque. Il n'en est pas de même de la démonstration du second point.

2°. S'il existe un point a de E où les fonctionnelles de F ne sont pas également continues, il existe un nombre $\epsilon > 0$, tel que pour tout voisinage V de a, l'oscillation sur $E \cdot V$ de l'une au moins des fonctionnelles de F soit supérieure à ϵ. D'après ce qui précède, on peut supposer que les voisinages de a forment une suite dénombrable monotones d'ensembles $V_1, V_2, \cdots$ n'ayant en commun que le point a. Soit U_n une fonctionnelle de F dont l'oscillation sur V_n est supérieure à ϵ. On peut extraire par hypothèse de la suite des $U_n(x)$ une suite uniformément convergente sur E. Appelons $U(x)$ sa limite. Il y a un voisinage V_p de a sur lequel l'oscillation de $U(x)$ est $< \epsilon/6$. D'autre part, pour n assez grand, $| U_n(x) - U(x) | < \epsilon/6$ sur tout E. Par suite si x appartient à $E \cdot V_p$

$$| U_n(x) - U_n(a) | \leq | U_n(x) - U(x) | \\ + | U(x) - U(a) | + | U(a) - U_n(a) | < \epsilon/2.$$

Donc l'oscillation de $U_n(x)$ sur V_p serait $< \epsilon$, pour p fixé, n assez grand. Si donc on prend n assez grand et en particulier $> p$, l'oscillation de $U_n(x)$ sur V_n serait $< \epsilon$, contrairement à l'hypothèse.

3. *Espace régulier.*

XXXIII. *Dans un espace régulier* (page **240**), *tout ensemble ouvert parfaitement séparable, E, est la somme d'une famille dénombrable d'ensembles fermés* (page **299**).

L'ensemble E étant parfaitement séparable, soit S la suite dénombrable des voisinages $V_1, V_2, \cdots, V_n, \cdots$ des points de E. Pour chaque point a, l'un au moins des voisinages V_n auxquels a est intérieur appartient a E. Ainsi on peut couvrir E au moyen d'une famille f d'ensembles V_n appartenant chacun à E. Donc E est la somme des ensembles de cette famille. D'autre part, dire que l'espace est régulier, c'est dire qu'on peut supposer fermés les ensembles de S. La famille f est une famille dénombrable d'ensembles fermés.

4. *Propriétes des espaces* (L).

XXXIV. Nous avons appelé dans notre Thèse ensemble compact (en soi) de points d'un espace (L), un ensemble E dont tout sous-ensemble infini e donne lieu à au moins un point d'accumulation a (appartenant à E). Il y aura donc une suite de points de e: $a_1, a_2, \cdots, a_n, \cdots$ distincts et convergeant vers a. Comme a est aussi point limite de la suite $a_n, a_{n+1}, \cdots$, il y aura dans tout voisinage de a une infinité de points de e distincts. De sorte que E est aussi compact (en soi) au sens adopté dans notre livre pour les ensembles de points d'un espace (V).

Dans un espace (V) quelconque, un ensemble (parfaitement) compact et fermé est (parfaitement) compact en soi. Dans un espace (L), *la réciproque est vraie* (page **301**). En effet, dans un tel espace tout point d'accumulation a d'un ensemble E est point limite d'une suite convergente S extraite de E et formée d'éléments distincts. Si E est compact en soi (et, à fortiori, si E est parfaitement compact en soi), il existe un point de E qui est point d'accumulation de S. Comme a est le seul point d'accumulation de S, on voit que a appartient à E.

XXXV. *La condition nécessaire et suffisante pour qu'un ensemble E soit compact est, que pour toute infinie monotone de sous-ensembles fermés de E*

$$E_1 \supset E_2 \supset E_3 \supset \cdots \supset E_n \supset \cdots,$$

il existe au moins un point commun aux ensembles de cette suite (page **301**).

La condition est nécessaire, cela résulte du théorème III (ou **22**, p. 7). La condition est suffisante; en effet si E n'était pas compact, il existerait un sous-ensemble infini, e, de E, n'ayant aucun point d'accumulation. Soit, S, une suite infinie de points de e, distincts

$$a_1, a_2, \cdots, a_n, \cdots$$

et E_n l'ensemble des points $a_n, a_{n+1}, \cdots$. Il n'existe aucun point d'accumulation de E_n. La suite des E_n serait donc une suite monotone de sous-ensembles fermés de E. Il y aurait donc un point commun aux E_n, ce qui est évidemment impossible.

XXXVI. *Toute transformation continue d'un ensemble compact et fermé E appartenant à un espace (L) en un ensemble F appartenant à un espace (L)—distinct ou non du premier—fournit un ensemble compact et fermé* (pages **44** et **303**).

En effet, soit f un sous-ensemble infini de F. Comme au paragraphe XXXVI, on peut le considérer comme transformé d'un sous-ensemble infini e de E choisi de sorte qu'un point de f corresponde à un seul point de e. Il y a un point a de E qui est point d'accumulation de e. Le transformé b de a est peut-être aussi transformé d'un point a_1 de e distinct de a. Mais a est point d'accumulation de $e - a - a_2$. Alors b, ne pouvant appartenir à $f \cdot b$ transformé de $e - a - a_1$, sera point d'accumulation de f-b, donc de f. Et b appartient à F.

Donc f a au moins un point d'accumulation appartenant à F. Ainsi F est compact en soi, c'est-à-dire, d'après XXXIV, compact et fermé.

5. *Propriété des espaces* (S).

XXXVII. M. Hausdorff a démontré (X, page **265**) que tout espace de Hausdorff à caractère dénombrable est un espace (S).

La réciproque est vraie (page **248**). Comme tout espace (S) est un espace accessible, il suffit de démontrer qu'un espace (S) à caractère dénombrable possède la porpriété D) de M. Hausdorff. Dire qu'il ne la possède pas, ce serait dire qu'il existe deux points a, b distincts ne possédant pas deux voisinages disjoints. Soient $V_a^1, V_a^2, \cdots; V_b^1, V_b^2, \cdots$ les voisinages respectifs de a et de b. Il y aurait un point a_n commun à V_a^n et V_b^n quel que soit n. Comme a_n ne peut-être identique à la fois à a et b, l'un au moins des deux points a et b, a par exemple, est distinct d'un infinité des a_n. D'ailleurs, on peut supposer (voir XXX) que la suite $V_a^1, V_a^2, \cdots$ est monotone, de même que la suite $V_b^1, V_b^2, \cdots$. Donc, a, par exemple, est point d'accumulation de l'ensemble des a_n. C'est, par conséquent, la limite unique d'une certaine suite convergente de points distincts a_{n_1}, a_{n_2}, $\cdots$ extraite de la suite des a_n. Il ne peut donc y avoir dans cette suite qu'un nombre fini ou nul de points coincidant avec b. La suite formée par les autres points est formée de points distincts de a et de b, figurant dans une infinité des V_a^n et une infinité des V_b^n et par conséquent figurant dans tous les voisinages de a et tous les voisinages de b. C'est donc une suite convergeant vers a et qui aurait aussi b pour point d'accumulation.

6. *Propriétés des espaces* (D).

XXXVIII. *Tout ensemble complet E fait partie d'un ensemble fermé complet G extrait de ce même espace* (page **308**).

Nous appelons ensemble complet un ensemble E tel que sans altérer la convergence ou la non-convergence des suites de points de E, ni leurs limites, on puisse définir celles-ci par l'intermédiaire de distances vérifiant le critère de convergence de Cauchy. Autrement dit: une certaine distance $((x, y))$ étant définie pour tout couple de points x, y de l'espace (D) considéré, nous supposons qu'on peut en outre attacher à tout couple de points x, y de E une certaine distance (x, y) égale au non à $((x, y))$. Comme la distance $((x, y))$, cette distance (x, y) doit être $\geqq 0$, nulle seulement si x et y sont confondus, telle que

$$(x, y) \leqq (x, z) + (y, z)$$

quels que soient x, y, z de E; telle que si y de E tend vers x de E, (x, y) tend vers zéro et réciproquement. Mais, en outre, elle doit être telle que, si, pour une suite $x_1, x_2, \cdots, x_n, \cdots$ de points de E, les distances (x_n, x_{n+p}) sont infiniement petites avec $1/n$, cette suite converge (sa limite appartenant ou non à E); et inversement.

Et nous nous proposons de montrer qu'on peut attacher à tout couple de points x, y de l'ensemble fermé $F = E + E'$, une distance $[x, y]$ égale à (x, y) quand x et y appartiennent à E et satisfaisant relativement à F aux mêmes conditions que nons venons de spécifier pour (x, y) relativement à E.

Si cela est possible, on aura

$$| [x, y] - (x_n, y_n) | \leqq [x, x_n] + [y, y_n]$$

pour tout couple x, y de F et tout couple x_n, y_n de E. On pourra toujours prendre x_n et y_n de sorte qu'ils convergent respectivement vers x et y de F et, par suite, on aura:

$$[x, y] = \lim_{x_n \to x, y_n \to y} (x_n, y_n) \tag{5}$$

Inversement, ne supposons plus le problème résolu et posons, par définition de $[x, y]$, la relation (5) pour tout couple de points x, y de F et toute suite de couples x_n, y_n de points de E convergeant respectivement vers x, y.

Pour que la définition soit légitime, il faut que le second membre de (5) ait sûrement une limite et que celle-ci ne dépende que de x et y. On a d'abord

$$| (x_n, y_n) - (x_{n+p}, y_{n+p}) | \leq (x_n, x_{n+p}) + (y_n, y_{n+p}).$$

D'après les hypothèses faites, le second membre est, indépendamment de p, infiniment petit avec $1/n$. La suite de nombres $(x_1, y_1), \cdots, (x_n, y_n), \cdots$ est donc bien convergente.

Sa limite ne dépend que de x et de y. Soit, en effet, x_n', y_n', un couple de points de E convergeant respectivement vers x, y. Alors le couple x, y sera aussi la limite de la suite de couples de E

$$x_1, y_1;\ x_1', y_1';\ x_2, y_2;\ \cdots;\ x_n, y_n;\ x_n', y_n';\ \cdots.$$

Donc la suite des distances

$$(x_1, y_1);\ (x_1', y_1');\ (x_2, y_2);\ \cdots;\ (x_n, y_n);\ (x_n', y_n');\ \cdots$$

d'après ce qui précède, est aussi convergente, ce qui prouve que

$$\lim_{x_n' \to x,\ y_n' \to y} (x_n', y_n') = \lim_{x_n \to x,\ y_n \to y} (x_n, y_n),$$

Ainsi $[x, y]$ est un nombre bien déterminé pour tout couple de points de F. De plus, si x, y appartiennent à E, on peut prendre $x_n \equiv x$, $y_n \equiv y$, ce qui montre que dans ce cas

$$[x, y] = (x, y).$$

Si maintenant x, y sont deux points quelconques de F, on a évidemment

$$[x, y] \geq 0, \qquad [x, x] = 0.$$

Inversement, si $[x, y] = 0$, on a $\lim (x_n, y_n) = 0$.

Or, par hypothèse (x_n, x_{n+p}) tend indépendamment de p vers zéro avec $1/n$. Donc la suite de points

$$x_1, y_1;\ x_2, y_2;\ \cdots;\ x_n, y_n;\ \cdots$$

vérifie aussi les conditions de Cauchy relativement à la distance $(\cdots)$: elle converge et alors $x \equiv y$.

On a pour trois points x_n, y_n, z_n, quelconques sur E

$$(x_n, y_n) \leq (x_n, z_n) + (z_n, y_n);$$

donc à la limite, si ces points tendent vers x, y, z de F,

$$[x, y] \leq [x, z] + [z, x].$$

Montrons aussi que si $y^{(p)}$ de F tend vers y de F, on a $\lim\limits_{p\to\infty} [y^{(p)}, y] = 0$ et inversement.

Supposons d'abord que chaque $y^{(p)}$ appartienne à E. Alors le couple $y^{(p)}$, $y^{(p+n)}$ de E tend vers le couple $y^{(p)}$, y de F quand n croit indéfiniment. Donc

$$[y^{(p)}, y] = \lim_{n\to\infty} (y^{(p)}, y^{(p+n)}).$$

Puisque la suite des $y^{(p+n)}$ de E est convergente, les nombres $y^{(p)}$, $y^{(p+n)}$ sont infiniment petits avec $1/p$, indépendamment de n. Donc $[y^{(p)}, y]$ est infiniment petit avec $1/p$; on a bien dans ce cas

$$\lim_{p\to\infty} [y^{(p)}, y] = 0.$$

Passons au cas où $y^{(p)}$ appartient à F; il est limite d'une suite de points de E. On peut donc prendre, d'après ce qui précède, un point y_p de E tel que $[y_p, y^{(p)}] < 1/p$, et aussi $((y_p, y^{(p)})) < 1/p$. Comme $((y, y_p)) \leq ((y, y^{(p)})) + ((y^{(p)}, y_p))$, on voit que y_p de E tendra vers y de F. Donc

$$[y, y^{(p)}] \leqq [y, y_p] + [y_p, y^{(p)}] < [y, y_p] + 1/p.$$

Or

$$\lim_{p\to\infty} [y, y_p] = 0.$$

On a donc encore

$$\lim_{p\to\infty} [y, y^{(p)}] = 0.$$

Réciproquement, supposons que pour les points y, $y^{(p)}$ de F, on ait

$$\lim_{p\to\infty} [y, y^{(p)}] = 0.$$

Prenons des points z_p, $z^{(p)}$ de E; on peut, pour p fixe, les faire varier et les fair tendre vers y, $y^{(p)}$ respectivement. On peut donc, d'après ce qui précède, les prendre tels que $[z_p, y] < 1/p$; $[z^{(p)}, y^{(p)}] < 1/p$, et aussi

$$((z^{(p)}, y^{(p)})) < 1/p.$$

Or, on a

$$\begin{aligned}(z^{(p)}, z_p) = [z^{(p)}, z_p] \leq [z^{(p)}, y^{(p)}] + [y^{(p)}, y] + [y, z_p]& \\ < [y^{(p)}, y] + 2/p.&\end{aligned}$$

Donc

$$(6) \qquad \lim_{p\to\infty} (z^{(p)}, z_p) = 0.$$

On a d'autre part,

$$(z^{(p)}, z^{(p+n)}) \leq (z^{(p)}, z_p) + (z_p, z_{p+n}) + (z_{p+n}, z^{p+n}).$$

En vertu de l'égalité (6) et de la convergence de la suite des z_p, on voit, d'après cette inégalité, que $(z^{(p)}, z^{(p+n)})$ est infiniment petit avec $1/p$ indépendamment de n. La suite des $z^{(p)}$ est donc aussi convergente. Soit z sa limite, qui est un point de F. On a

$$[z, y] \leq [z, z^{(p)}] + [z^{(p)}, z_p] + [z_p, y] < [z, z^{(p)}] + [z^{(p)}, z_p] + 1/p,$$

et les trois termes du troisième membre tendent vers zéro avec $1/p$: le deuxième terme en vertu de (6), le premier d'après la proposition directe démontrée à l'instant. Donc $[z, y] = 0$, et on a vu que ceci implique $z \equiv y$; par suite $z^{(p)}$ tend vers y.

Enfin, on a

$$((y, y^{(p)})) \leq ((y, z^{(p)})) + ((z^{(p)}, y^{(p)})) \leq ((y, z^{(p)})) + 1/p.$$

Donc, on a bien

$$y = \lim_{p\to\infty} y^{(p)}.$$

Il nous reste à démontrer que la distance $[x, y]$, qu'on vient de définir pour tout couple de points de F, se prête à la généralisation du critère de Cauchy. Si $x_1, x, \cdots$ est une suite convergente de points de F, sa limite x appartient à F; et on a

$$[x_p, x_{n+p}] \leq [x, x_p] + [x, x_{n+p}].$$

Il résulte de ce qui précède que $[x_p, x_{n+p}]$ sera infiniment petit avec $1/p$ indépendamment de n.

Inversement, démontrons que la suite des points x_n de F est convergente quand elle satisfait au critère de Cauchy généralisé. Dans ce but, prenons un point y_p de E assez voisin de x_p pour que, quel que soit p,

$$[y_p, x_p] < 1/p.$$

On aura

$$\begin{aligned}(y_p, y_{n+p}) = [y_p, y_{n+p}] \leq [y_p, x_p] & \\ + [x_p, x_{n+p}] + [x_{n+p}, y_{n+p}] &< 2/p + [x_p, x_{n+p}];\end{aligned}$$

le second membre est infiniment petit avec $1/p$, indépendamment de n. Donc les distances (y_p, y_{n+p}) vérifient le critère de Cauchy: la suite des y_p est convergente. Soit y sa limite qui appartient à F. On aura

$$[x_p, y] \leq [x_p, y_p] + [y_p, y] < 1/p + [y_p, y].$$

Donc les x_p convergent (vers y).

XXXIX. *Dans un espace* (D), *tout ensemble compact et fermé possède la propriété de Borel-Lebesgue et réciproquement* (page **317**). J'ai démontré la réciproque pour l'espace (V) le plus général (**61**, page 15).

Le théorème direct a été démontré dans ma Thèse (**22**, page 26), moyennant une hypothèse restrictive dont on peut se débarasser suivant une remarque de M. Hildebrandt (**71**, page 353).

Mais cette démonstration repose sur le théorème analogue pour la propriété de Borel. Vue l'importance de la proposition énoncée ci-dessus, il est bon d'en donner une démonstration directe.

Supposons qu'une famille F d'ensembles I couvre un ensemble E. Tout point x de E étant intérieur à un ensemble de F, il y a au moins un nombre $\rho > 0$ tel que le sphéroïde de centre x, rayon ρ, appartienne entièrement à un même ensemble I de F. Soit ρ_x la borne supérieure de ρ pour un point x donné de E. (Si cette borne supérieure était infinie, tout E—et même tout l'espace—serait couvert par un seul des I). Soit S_x le sphéroïde de centre x, rayon $\rho_x/2$; il est *intérieur* à l'un au moins, I_x, des ensembles de F.

Pour démontrer le théorème, il suffit donc de démontrer qu'il existe un nombre fini de points x de E tels que tout point de E appartienne à l'un des sphéroïdes S_x, en nombre fini, correspondants.

Or, dans le cas contraire, soit a_1 un point quelconque de E; il existerait au moins un point a_2 de E n'appartenant pas à S_{a_1}. De même, il existerait au moins un point a_3 n'appartenant ni à S_{a_1}, ni à S_{a_2}. Si on a formé une suite de points $a_1, \cdots, a_n$ de E tels que a_n n'appartienne à aucun des ensembles $S_{a_1}, \cdots, S_{a_{n-1}}$, alors il existerait aussi un point a_{n+1} de E n'appartenant à aucun des ensembles $S_{a_1}, \cdots, S_{a_n}$. Ainsi le procédé devrait se continuer indéfiniment. Alors, on aurait une suite infinie T de points de E, $a_1, a_2, \cdots$ tous distincts et tels que, quel que soit n, a_n n'appartienne à aucun des ensembles $S_{a_1}, \cdots, S_{a_{n-1}}$.

Or puisque E est compact et fermé il y a au moins un point a de E qui est point d'accumulation de l'ensemble T. Il y a donc une infinité de points de T, distincts, aussi voisins que l'on veut de a. Prenons en particulier a_n et a_{p+n} à distances de a inférieures à $\rho_a/16$. Alors pour tout point x à distance de a_n au plus égale à $\rho_a/4$, on a

$$(x, a) \leq (x, a_n) + (a_n, a) < \rho_a/2.$$

Donc, tout point d'un sphéroïde de centre a_n, rayon $\rho_a/4$, appartient à S_a et, par suite, à I_a. Il résulte alors de la définition de ρ_{a_n} que $\rho_{a_n} \geq \rho_a/4$, et comme a_{n+p} n'appartient pas à S_{a_n}, on aurait

$$(a_{n+p}, a_n) \geq \rho_{a_n}/2;$$

alors que

$$(a_{n+p}, a_n) \leq (a_{n+p}, a) + (a_n, a) < \rho_a/8.$$

On remarquera que cette démonstration est aussi simple que les démonstrations qui ont été données pour les espaces euclidiens.

7. *Ensembles parfaitement compacts.*

XL. *Dans un espace* (D), *il a identité entre la notion d'ensemble compact* (*en soi*) *et celle d'ensemble parfaitement compact* (*en soi*) (page **317**).

Une première démonstration s'obtient par voie indirecte en s'appuyant sur ce que, dans un espace (D), tout ensemble compact et fermé possède la propriété de Borel-Lebesgue et sur le théorème de Chittenden d'après lequel dans tout espace (V), tout ensemble possédant cette propriété est parfaitement compact en soi.

J'ai aussi donné de ce fait, énoncé par M. R. L. Moore, une démonstration directe (**71**, page 348) mais où j'adoptais pour la notion d'ensemble parfaitement compact la définition de cet auteur. Il faut alors compléter par la démonstration de l'équivalence de cette définition et de la définition adoptée dans mon livre. Cette équivalence a été établie par MM. Alexandroff et Chittenden.

On peut aussi donner une démonstration ne faisant intervenir que la définition des ensembles parfaitement compacts dérivée de celle due à MM. Sierpinski et Kuratowski, définition adoptée dans mon livre et utilisée ci-dessus.

Il s'agit de démontrer que si E est un ensemble compact et si e est un sous-ensemble infini de E, il existe un point a dont tout voisinage comprend à son intérieur une infinité de points de e ayant la puissance de e. Or s'il existait un tel point, il appartiendrait à la fermeture $F = E + E'$ de E. Cet ensemble est compact et fermé (**34**, page 4). Si E n'était pas parfaitement compact, alors pour tout point x de F, il y aurait un sphéroïde S_x de centre x ne comprenant à son intérieur qu'un ensemble de points de e de puissance inférieure à celle de e. Or F ayant la propriété de Borel-Lebesgue (voir XXXIX), l'ensemble F est couvert par un nombre fini des S_x. Donc l'ensemble e est compris dans la somme des intérieurs d'un nombre fini d'ensembles, chacun de ces intérieurs ne contenant qu'un ensemble de points de e d'une puissance inférieure à celle de e. D'où la contradiction annoncée.

XLI. Soit $U(x)$ une fonctionnelle uniformément continue sur un ensemble E compact et fermé. C'est-à-dire que pour tout nombre $\epsilon > 0$, il existe un nombre $\eta > 0$, tel que $| U(x) - U(x') | < \epsilon$ pour tout couple de points x, x' de E dont la distance est $< \eta$. On peut couvrir E au moyen de sphéroïdes de rayon η. Puisque E est compact et fermé, on peut couvrir E au moyen d'un nombre fini de ces sphéroïdes. On peut donc couvrir E par un nombre fini d'ensembles en l'intérieur de chacun desquels l'oscillation de $U(x)$ soit inférieure à un nombre positif arbitraire ϵ, donné d'avance.

Réciproquement, supposons cette condition vérifiée. Alors, on peut couvrir E au moyen d'un nombre fini d'ensembles $E_1, \cdots, E_n$ dans l'intérieur de chacun desquels l'oscillation de U soit $< \epsilon/3$, par exemple. Soient I_r l'intérieur de E_r, p_{rs} la borne inférieure des distances des points de $E \cdot I_r$ à ceux de $E \cdot I_s$ et η la plus petite de celles des bornes $\rho_{r,s}$ (en nombre fini) qui ne sont pas nulles.* Si deux points a, b, de E sont à distance $(a, b) < \eta$, ou bien ils appartiennent au même I_r et alors $| U(a) - U(b) | < \epsilon/3 < \epsilon$, ou bien ils appartiennent à I_r et I_s tels que $\rho_{r,s} = 0$. Dans ce cas, il y a une suite de couples a_p, b_p de points de $E \cdot I_r$ et $E \cdot I_s$ respectivement, tels que (a_p, b_p) tende vers zéro et, par suite, puisque E est compact et fermé, on peut extraire de cette suite, une suite de couples a_p', b_p' convergeant vers le même point c de E.

Ce point c appartient à un des I, soit I_m (distinct ou non de I_r et de I_s). Puisque c est intérieur à E_m, les points a_p', b_p' seront aussi intérieurs à E_m à partir d'un certain rang q. Alors on aura

$$(1) \qquad | U(a_p') - U(b_p') | < \epsilon/3, \quad | U(a) - U(a_p') | < \epsilon/3, \quad | U(b_p') - U(b) | < \epsilon/3$$

pour les trois couples de points appartenant respectivement à $E \cdot I_m$, $E \cdot I_r$, $E \cdot I_s$. D'où

$$(2) \qquad | U(a) - U(b) | < \epsilon.$$

Ainsi, *la condition nécessaire et suffisante pour qu'une fonctionnelle U, définie sur un ensemble E compact et fermé, y soit uniformément continue est que, pour tour nombre $\epsilon > 0$, il existe un nombre fini d'ensembles couvrant E et dans l'intérieur de chacun desquels l'oscillation de U est inférieure à ϵ* (page **279**).

On pourrait se demander si l'on pourrait supprimer dans cet enoncé les mots "à l'intérieur de." On ne pourrait les supprimer purement et simplement, car alors toute fonctionnelle bornée serait uniformément continue, comme on le voit en considérant E comme la somme des ensembles E_α où $\alpha\epsilon \leq U(x) < (\alpha + 1)\epsilon$ et où α ne prend que des valeurs entières (> 0, < 0 ou $= 0$).

UNIVERSITÉ DE STRASBOURG,
December 29, 1926.

---

* Si tous les ρ_{rs} sont nuls, on prendra $\eta = 1$, par exemple.

On Certain Points in the Theory of Dirichlet Series.

By J. F. Ritt

The results of this paper, all of which deal with Dirichlet series, bear upon the following topics:

I. *Picard's Theorem.*
II. *Integral functions.*
III. *Asymptotic convergence.*

The results under each topic are given at the head of the section dealing with that topic.

I. Picard's Theorem.

1. *Statement of results.* We consider, in this section, a function $f(z)$, $(z = x + iy)$, analytic in a half-plane $x \leqq R$, and representable in a half-plane $x < T \leqq R$ by an absolutely convergent series

$$(1) \qquad a_0 + a_1 e^{\lambda_1 z} + \cdots + a_n e^{\lambda_n z} + \cdots,$$

where

$$0 < \lambda_1 < \cdots < \lambda_n < \cdots; \quad \lim_{n \to \infty} \lambda_n = \infty.$$

We use positive coefficients for z, rather than the customary negative ones, in order to heighten the similarity between our results and the corresponding known results on functions defined by Taylor series. No knowledge of T is needed other than that of its existence.

The value a_0 will be attributed to $f(z)$ for $z = -\infty$.

Theorems I and II, below, generalize respectively the theorems of Landau and of Schottky. We call particular attention to the fact that, in those theorems, the λ's which follow λ_1, and the extent of the region of convergence of (1), do not figure. The proofs, based upon the use of a modular function, are very parallel to the non-elementary proofs of the Landau and Schottky theorems.* Theorems IV and III, lemmas to Theorem II, are generalizations of Schwarz's lemma, and of the theorem that the modulus of a function analytic in a circle assumes its maximum on the circumference. A theorem like Theorem III, but dealing only with exponential polynomials, is given by Bohr.*

---

* More novelty of detail will probably be found in the proofs of Theorems III, VI, X, XI, XII than in the other proofs.

THEOREM I. *Let* $f(z)$, *analytic for* $x \leqq R$, *be representable, for* $x < T \leqq R$, *by an absolutely convergent series* (1). *There exists a function* $L(a_0, a_1, \lambda_1)$ *such that, if* $R > L(a_0, a_1, \lambda_1)$, *then* $f(z)$ *takes on at least one of the values* 0 *or* 1 *in the half-plane* $x \leqq R$.

THEOREM II. *Let* $f(z)$, *analytic for* $x \leqq R$, *be representable, for* $x < T \leqq R$, *by an absolutely convergent series* (1). *Let* θ *be a variable assuming all values greater than zero. There exists a function* $S(a_0, \lambda_1, \theta)$ *such that, if* $f(z)$ *is nowhere zero or unity for* $x \leqq R$, *we have* $|f(z)| < S$ *for* $x \leqq R - \theta$.

It should be noticed that S is independent of R.

THEOREM III. *Let* $f(z)$, *analytic, bounded and non-constant for* $x \leqq R$, *have an absolutely convergent development* (1) *for* $x < T \leqq R$. *Let* $M(x)$ *represent, for* $x \leqq R$, *the least upper bound of* $|f(z)|$ *on the vertical line of abscissa* x.† *Then* $M(x)$ *is an increasing function of* x *for* $x \leqq R$.‡

THEOREM IV. *Let* $f(z)$, *analytic, with* $|f(z)| \leqq h$, *(*h *arbitrary), for* $x \leqq R$, *have, for* $x < T \leqq R$, *an absolutely convergent development*

$$f(z) = a_1 e^{\lambda_1 z} + \cdots + a_n e^{\lambda_n z} + \cdots,$$

with

$$0 < \lambda_1 < \cdots < \lambda_n < \cdots; \quad \lim_{n \to \infty} \lambda_n = \infty.$$

Then, for $x < R$,

$$f(z) \leqq h\, e^{\lambda_1 (x-R)},$$

equality being possible only in the case of $f(z) = a_1 e^{\lambda_1 z}$.

2. *Proof of Theorem I.* Suppose that $f(z)$ is nowhere zero or unity for $x \leqq R$. Then $a_0 \neq 0, 1$.

Let $\mu(z)$ be a function, with $\mu(0) = a_0$, which maps the circle $|z| < 1$ upon the entire plane with the points 0, 1, ∞ removed. Let $\nu(z)$ be the inverse of $\mu(z)$.

Taking that branch of $\nu(z)$ for which $\nu(a_0) = 0$, let us form the function $\nu[f(z)]$. As $f(z)$ assumes none of the values 0, 1, ∞, which are the only singular points of $\nu(z)$, $\nu[f(z)]$ will be uniform, analytic and less in modulus than unity, for $x \leqq R$.

---

* *Acta Mathematica*, Vol. 47 (1926), p. 245.

† $M(x)$ is the least upper bound of $f(x + iy)$ for x fixed and $-\infty < y < +\infty$.

‡ If $x_1 < x_2$, $M(x_1) < M(x_2)$.

As $a_0 \neq 0, 1$, $\nu(z)$ is developable in a Taylor series in $z - a_0$. Because the development (1) of $f(z)$ is absolutely convergent, we can get a development for $\nu[f(z)]$, absolutely convergent in some half-plane, by substituting (1) for z, formally, in the expansion of $\nu(z)$ about a_0. We find thus, as $\nu(a_0) = 0$,

$$\nu[f(z)] = \nu'(a_0) a_1 e^{\lambda_1 z} + \cdots. \tag{2}$$

The coefficients of z which follow λ_1 are linear combinations of the λ's in (1) with positive integral coefficients.

As the development in (2) has a half-plane of absolute convergence, and as $| \nu[f(z)] |$ is bounded for $x \leqq R$, we have the following expression for the coefficient of $e^{\lambda_1 z}$ in (2):[*]

$$\nu'(a_0) a_1 = \lim_{\omega \to \infty} (1/2\omega i) \int_{R-\omega i}^{R+\omega i} \nu[f(z)] e^{-\lambda_1 z} dz,$$

The integration is performed along the line $x = R$.

Thus, as $| \nu[f(z)] | < 1$, we have

$$| a_1 \nu'(a_0) | < e^{-\lambda_1 R},$$

so that

$$R < -\{\log | a_1 \nu'(a_0) | \}/\lambda_1.$$

This proves our theorem.

3. *Proofs of Theorems III and IV.* Let $f(z)$ satisfy the hypothesis of Theorem III.

A recent theorem of Doetsch [†] informs us that $\log M(x)$ is a downward convex function of x for $x \leqq R$. By this, it is meant that if the points $(x, \log M(x))$ are plotted in Cartesian coordinates, then, if $x_1 < x_2 < x_3$, the point $(x_2, \log M(x_2))$ does not lie above the line joining $(x_1, \log M(x_1))$ and $(x_3, \log M(x_3))$.

Our first step will be to show that, for x close to $-\infty$, $M(x) > | a_0 |$. This is obvious when $a_0 = 0$. Suppose that $a_0 \neq 0$. We assume, without loss of generality, that $a_1 \neq 0$.

---

[*] Valiron, "Théorie générale des séries de Dirichlet." *Mémorial des Sciences Mathématiques.* Paris, 1926, p. 15.

[†] *Mathematische Zeitschrift*, Vol. 8 (1920), p. 237.

We have for $x < T$,

$$f(z) = a_0 + a_1 e^{\lambda_1 z} + e^{\lambda_1 z}(a_2 e^{(\lambda_2 - \lambda_1)z} + a_3 e^{(\lambda_3 - \lambda_1)z} + \cdots).$$

As the series in parentheses converges absolutely for $x < T$, and therefore, because of the nature of Dirichlet series, uniformly for $x < T - \delta$, $(\delta > 0)$, we have

$$f(z) = a_0 + a_1 e^{\lambda_1 z} + \epsilon(z) e^{\lambda_1 z},$$

where, as x approaches $-\infty$, $\epsilon(z)$ approaches zero uniformly with respect to y.

Let x be so close to $-\infty$ that $|\epsilon(z)| < |a_1|/2$. For any such x, we can find a y such that $a_1 e^{\lambda_1 z}$ has the same argument as a_0. Then

$$|f(x+iy)| > |a_0 + a_1 e^{\lambda_1 x}| - |a_1 e^{\lambda_1 x}|/2,$$

so that $M(x) > |a_0|$.

We now prove that $M(x)$ is an increasing function. Suppose that $a_0 \neq 0$. We shall show first that $M(x) > |a_0|$ for every $x \leqq R$. Suppose that, for some x_1, $M(x_1) \leqq |a_0|$. Let x_2 and $x_3 > x_2$ be two numbers, less than x_1 and close to $-\infty$, for which $M(x_3) > M(x_2)$. Such points exist, because $M(x)$ approaches $|a_0|$ from above as $x \rightarrow -\infty$. Then $\log M(x_3)$ exceeds both $\log M(x_1)$ and $\log M(x_2)$, so that $\log M(x)$ cannot be convex downward. Thus $M(x_1) > |a_0|$.

Let now x_1 and $x_2 > x_1$ be any two numbers, and suppose that $M(x_1) \geqq M(x_2)$. For x_3 close to $-\infty$, $M(x_3)$, being close to $|a_0|$, is less than $M(x_1)$. We see again that $\log M(x)$ is not convex downward. Thus $M(x_1) < M(x_2)$, so that $M(x)$ is an increasing function.

For the case $a_0 = 0$, it is at once obvious, because $\log M(x)$ is convex downward and approaches $-\infty$ with x, that $M(x)$ is an increasing function. Theorem III is proved.

We take now Theorem IV. Suppose that $f(z)$ is not $a_1 e^{\lambda_1 z}$. Then $g(z) = e^{-\lambda_1 z} f(z)$ satisfies the hypothesis of Theorem III. When $x = R$, $|g(z)| \leqq h\, e^{-\lambda_1 R}$. Hence, for $x < R$, $|g(z)| < h\, e^{-\lambda_1 R}$, so that

$$e^{-\lambda_1 x} |f(z)| < h\, e^{-\lambda_1 R},$$

or

$$|f(z)| < h\, e^{\lambda_1 (x-R)}.$$

When $f(z) = a_1 e^{\lambda_1 z}$, if we take $h = |a_1| e^{\lambda_1 R}$, we get an equation, rather than an inequality.

4. *Proof of Theorem II.* Suppose that $f(z)$ is neither zero nor unity

for $x \leqq R$. Then $| \nu[f(z)] | < 1$ for $x \leqq R$. This, and (2), make $\nu[f(z)]$ satisfy the hypothesis of Theorem IV. Then

$$\nu[f(z)] \leqq e^{-\lambda_1\theta}$$

for $x \leqq R - \theta$. As $f(z) = \mu\nu f(z)$, and as $\mu(z)$ is bounded for $| z | \leqq e^{-\lambda_1\theta} < 1$, we have an upper bound for $| f(z) |$ for $x \leqq R - \theta$, depending on a_0, λ_1 and θ. Theorem II is proved.

II. INTEGRAL FUNCTIONS.

5. *Statement of results.* We consider a non-constant function $f(z)$ defined by a series

$$a_0 + a_1 e^{\lambda_1 z} + \cdots + a_n e^{\lambda_n z} + \cdots, \tag{3}$$

with

$$0 < \lambda_1 < \cdots < \lambda_n < \cdots; \quad \lim_{n \to \infty} \lambda_n = \infty,$$

absolutely convergent for all values of z.

Of course, $f(z)$ is bounded in any left half-plane, and therefore along any vertical line. Let $M(x)$ be the least upper bound of $| f(z) |$ along a vertical line of abscissa x. We know from Theorem III that $M(x)$ increases with x. Then $M(x)$ must increase indefinitely, for $f(z)$ cannot be bounded in the entire plane. With respect to the growth of $M(x)$, we prove, below, the

THEOREM V. *If $M(x) \leqq e^{lx}$ $(l \geqq 0)$, for all sufficiently large values of x, then $f(z)$ is an exponential polynomial in which the greatest coefficient of z does not exceed l, that is, $a_n = 0$ for $\lambda_n > l$.*

We shall say that $f(z)$ is of order ρ if, for every $\epsilon > 0$, we have, for x sufficiently large,

$$M(x) < e^{e^{x(\rho+\epsilon)}}$$

and if, besides, there exists a sequence of values of x tending towards $+ \infty$ for which

$$M(x) > e^{e^{x(\rho-\epsilon)}}$$

We shall say that a function $f(z)$, of order $\rho > 0$, belongs to the type σ of the order ρ if, for every $\epsilon > 0$,

$$M(x) < e^{(\sigma+\epsilon)e^{\rho x}},$$

whereas, for a sequence of x's tending towards $+ \infty$,

$$M(x) > e^{(\sigma-\epsilon)e^{\rho x}}.$$

We shall study the influence of the coefficients in (3) on the rate of increase of $M(x)$, placing upon the λ's the quantitative restriction

$$\liminf_{n\to\infty} \lambda_n/\log n > 0. \tag{4}$$

This assumption, frequently made in investigations on Dirichlet series, holds, for instance, for ordinary Dirichlet series $\sum a_n n^z$.

The problem of determining whether $f(z)$ belongs to a certain order, differs in no material respect from the corresponding problem for integral functions defined by power series. This is due largely to the crudeness with which the order of a function measures its modulus. One has only to follow the power series investigation,[*] using (5), below, to establish the following result:

For $f(z)$, *defined by the everywhere absolutely convergent series* (3), *for which* (4) *holds, to be of order* ρ, *it is necessary and sufficient that*

$$\rho = -\limsup_{n\to\infty} (\lambda_n \log \lambda_n)/\log |a_n|.$$

On the other hand, the determination of conditions for $f(z)$ to belong to a certain type of a given order, presents essentially new features over the power series investigation. We prove, in this connection, the

THEOREM VI. *For* $f(z)$, *defined by the everywhere absolutely convergent series* (3), *with* (4) *holding, to belong to the type* σ *of the order* ρ, *where* $\sigma > 0$, *it is necessary and sufficient that*

$$\limsup_{n\to\infty} n^{1/\rho} \left(\sum_{\lambda_i = n}^{\lambda_i < n+1} |a_i| \right)^{1/n} = (\sigma e \rho)^{1/\rho}.$$

The summation is over all i's for which $n \leqq \lambda_i < n+1$.

For $\sigma = 0$, the condition is necessary, but it is sufficient only if $f(z)$ is not of order less than ρ.

6. *Proof of Theorem V.* Let $f(z)$ satisfy the hypothesis of Theorem V. We have, for any R,

$$a_n = \lim_{\omega\to\infty} (1/2\omega i) \int_{R-\omega i}^{R+\omega i} f(z)\, e^{-\lambda_n z}\, dz. \tag{5}$$

Thus, for x large,

$$|a_n| \leqq e^{(l-\lambda_n)x},$$

so that $a_n = 0$ if $\lambda_n > l$.

---

[*] Bieberbach, *Lehrbuch der Funktionentheorie*, Vol. 2, p. 228.

7. *Proof of Theorem VI.* Let $f(z)$ be defined by the everywhere absolutely convergent series (3) for which (4) holds. Let $\xi(z)$ represent the series

$$1 + e^{-\lambda_1 z} + \cdots + e^{-\lambda_n z} + \cdots,$$

which, because of (4), converges for $x > 0$. Let

$$(6) \qquad g(z) = |a_0| + |a_1| e^{\lambda_1 z} + \cdots + |a_n| e^{\lambda_n z} + \cdots.$$

We shall prove that, for $h > 0$ and x arbitrary,

$$g(x - h) \leqq \xi(h) M(x),$$

where $M(x)$ is the least upper bound of $|f(z)|$ on the vertical line of abscissa x.

In short,[*] by (5), $|a_n| \leqq M(x) e^{-\lambda_n x}$. Thus, by (6),

$$g(x - h) \leqq M(x)\ (1 + e^{-\lambda_1 h} + \cdots) = \xi(h) M(x).$$

We shall show that *if $f(z)$ belongs to the type σ of the order ρ, then $g(z)$ belongs to the type σ of the order ρ.* Let $f(z)$ belong to the type σ of the order ρ. The least upper bound of $|g(z)|$ on a vertical line of abscissa x is $g(x)$. Now, for $\epsilon > 0$, $h > 0$, and for x large,

$$\begin{aligned}\log g(x) &\leqq \log \xi(h) + \log M(x+h) < \log \xi(h) + (\sigma + \epsilon) e^{\rho(x+h)} \\ &= \log \xi(h) + e^{\rho h}(\sigma + \epsilon) e^{\rho x}.\end{aligned}$$

An ϵ having been selected, let h be taken so small that

$$e^{\rho h}(\sigma + \epsilon) < \sigma + 2\epsilon.$$

Then

$$\log g(x) < \log \xi(h) + (\sigma + 2\epsilon) e^{\rho x}.$$

If, now, keeping ϵ and h fixed, we make x large, we have

$$\log g(x) < (\sigma + 3\epsilon) e^{\rho x},$$

so that $g(z)$ satisfies the first condition for belonging to the type σ of the order ρ. Furthermore, as $g(x) \geqq |f(z)|$, the second condition is also satisfied by $g(z)$.

Let

$$c_n = \sum_{\lambda_i = n}^{\lambda_i < n+1} |a_i|, \qquad (n = 0, 1, \cdots).$$

---

[*] We put $\lambda_0 = 0$.

Then, for $x > 0$,

$$g(x) \geqq c_0 + c_1 e^{x} + \cdots + c_n e^{nx} + \cdots,$$
$$g(x) < c_0 e^{x} + c_1 e^{2x} + \cdots + c_n e^{(n+1)x} + \cdots.$$

From the first inequality, we see that the integral function $\sum c_n z^n$ satisfies the first condition for belonging to the type σ of the order ρ in the ordinary sense of the theory of integral functions. The second inequality shows that $\sum c_n z^{n+1}$ satisfies the second condition. But if either power series satisfies one of the conditions, the other does also. Hence, to ask that $f(z)$ belong to the type σ of the order ρ, in the sense of this paper, is to ask that $\sum c_n z^n$ belong to the type σ of the order ρ in the crdinary sense. This gives our theorem immediately.*

III. ASYMPTOTIC CONVERGENCE.

8. *Statement of results.* We deal with series of the type

$$a_0 + a_1 e^{-\lambda_1 z} + a_2 e^{-\lambda_2 z} + \cdots + a_n e^{-\lambda_n z} + \cdots, \tag{7}$$

with

$$0 < \lambda_1 < \cdots < \lambda_n < \cdots; \quad \lim_{n\to\infty} \lambda_n = \infty,$$

making no assumptions as to the convergence of the series.

Consider any domain containing points of arbitrarily large x, and a function $f(z)$ defined for all points of this domain whose x's exceed some value. We shall say that $f(z)$ is *asymptotically equivalent* to (7) in the given domain if, for every n,

$$f(z) = a_0 + a_1 e^{-\lambda_1 z} + \cdots + a_n e^{-\lambda_n z} + \epsilon_n(z)\, e^{-\lambda_n z},$$

where $\epsilon_n(z)$ approaches zero, uniformly with respect to y, as x increases indefinitely. We shall write, putting $\lambda_0 = 0$,

$$f(z) \sim \sum_{0} a_n e^{-\lambda_n z}.$$

For domains, we shall use generally, in this paper, sectors of the type $|\operatorname{amp} z| \leqq \phi \leqq \pi/2$. We shall call such a sector a *sector of opening* 2ϕ. It will be understood that $f(z)$ is analytic for all sufficiently distant points of the sector.

Our results follow.

---

* Bieberbach, *loc. cit.*, p. 231.

THEOREM VII. *If the series* $\sum a_n e^{-\lambda_n z}$ *has a domain of convergence, then the function* $f(z)$ *represented by the series is asymptotically equivalent to the series in any sector of opening less than* π.

THEOREM VIII. *If* $f(z) \sim \sum a_n e^{-\lambda_n z}$ *and* $g(z) \sim \sum b_n e^{-\lambda_n z}$, *then* $f(z) + g(z)$ *and* $f(z)\, g(z)$ *are respectively equivalent, asymptotically, to the sum and to the Dirichlet product of the series for* $f(z)$ *and* $g(z)$.

It is unnecessary to specify a domain in Theorem VIII. The proofs of both items of that theorem are analogous to the proofs of the corresponding results for power series, and will be omitted. The result on multiplication has a special interest, in view of the fact that the product of two Dirichlet series may diverge at points for which both converge.

THEOREM IX. *If* $f(z) \sim \sum a_n e^{-\lambda_n z}$ *in a sector of opening* 2ϕ, *then* $f'(z) \sim \sum - \lambda_n a_n e^{-\lambda_n z}$ *in any sector of opening less than* 2ϕ.

Theorem IX is analogous to one which we proved several years ago for power series.[*] It will be seen that the proof really applies to more general domains than sectors.

THEOREM X. *Let*

$$\liminf \lambda_n / \log n > 0.$$

Let $f(z) \sim \sum a_n e^{-\lambda_n z}$ *in a sector of opening* π. *Then* $\sum a_n e^{-\lambda_n z}$ *converges absolutely to* $f(z)$ *in some half-plane.*

Theorem X, more hidden than those which precede, generalizes the theorem that a power series which is asymptotically equivalent to an analytic function for a complete neighborhood of ∞ converges for large values of the variable.

THEOREM XI. *If* $f(z)$ *and* $g(z)$ *are both equivalent asymptotically to* $\sum a_n e^{-\lambda_n z}$, *with unrestricted* λ's, *in a sector of opening* π, *then* $f(z) \equiv g(z)$.

THEOREM XII. (Existence theorem.) *Let* $\lambda_1, \lambda_2, \cdots$, *be any sequence of positive numbers, increasing towards* ∞, *and* $a_1, a_2, \cdots$, *be any numbers. The series*

$$\sum_{n=1}^{\infty} a_n e^{-\lambda_n z} \left(1 - e^{b_n z \log z}\right), \tag{8}$$

where the b's *are so taken that, for every* n, $b_n < 0$ *and*

$$|a_n b_n| < 2^{-n},$$

---

[*] *Bulletin of the American Mathematical Society*, Vol. 24 (1918), p. 225.

converges, for points of sufficiently large modulus, in any sector of opening less than π, *to a function asymptotically equivalent, in that sector, to* $\sum a_n e^{-\lambda_n z}$.

The coefficient of i in $\log z$ is understood to be taken between $-\pi$ and π.

The series (8) is similar to a series which we used some time ago for the construction of a function with any desired derivatives at a point.[*]

9. *Proof of Theorem VII.* The proof of Theorem VII is contained, in principle, in the proof of the familiar theorem that the series for $f(z)$ converges uniformly in any sector of opening less than π.[†]

No generality is lost in assuming that the series converges for $z = 0$. In fact, if the series converges for $z = R > 0$, we can deal with the function $f(z + R)$. The equivalence of $f(z+R)$ to $\sum a_n e^{-\lambda_n R} e^{-\lambda_n z}$ will imply that of $f(z)$ to $\sum a_n e^{-\lambda_n z}$. This is because the distant points of any sector of opening $2\phi < \pi$, with vertex at the origin, are contained within any sector of opening greater than 2ϕ with vertex at R.

We take any sector, with vertex at the origin, of opening $2\phi < \pi$.

Let

$$r_n(z) = \sum_{n+1}^{\infty} a_i e^{-\lambda_i z}, \quad A_n = \sum_{n+1}^{\infty} a_i.$$

Then, by partial summation,

$$r_n(z) = A_n e^{-\lambda_{n+1} z} + \sum_{n+1}^{\infty} A_i (e^{-\lambda_{i+1} z} - e^{-\lambda_i z}).$$

Now

$$\left| e^{-\lambda_{i+1} z} - e^{-\lambda_i z} \right| = \left| \int_{\lambda_i}^{\lambda_{i+1}} z\, e^{-uz}\, du \right| \leqq |z| \int_{\lambda_i}^{\lambda_{i+1}} e^{-ux}\, du$$
$$= (|z|/x)\, (e^{-\lambda_i x} - e^{-\lambda_{i+1} x}).$$

As the series for $f(z)$ converges for $z = 0$, there exists an $h > 0$ which exceeds every $|A_i|$. Then

$$\begin{aligned} |r_n(z)| &< h\, e^{-\lambda_{n+1} x} + (h\,|z|/x) \sum_{n+1}^{\infty} (e^{-\lambda_i x} - e^{-\lambda_{i+1} x}) \\ &= h\,(1 + |z|/x)\, e^{-\lambda_{n+1} x} \\ &< h\,(1 + \sec\phi)\, e^{-\lambda_{n+1} x}. \end{aligned}$$

Because $\lambda_{n+1} > \lambda_n$, it is plain that $e^{\lambda_n z} r_n(z) \rightarrow 0$ as $x \rightarrow \infty$ in our sector. This proves the theorem.

---

[*] *Annals of Mathematics*, Vol. 18 (1916), p. 18.

[†] Valiron, *loc. cit.*, p. 5.

10. *Proof of Theorem IX.* We have

$$f'(z) = -\lambda_1 a_1 e^{-\lambda_1 z} - \cdots - \lambda_n a_n e^{-\lambda_n z} + e^{-\lambda_n z}\left(-\lambda_n \epsilon_n(z) + \frac{d}{dz}\epsilon_n(z)\right).$$

Consider any ζ of large modulus in a sector of opening less than 2ϕ. The modulus of the derivative of $\epsilon_n(z)$ for $z = \zeta$, does not exceed the maximum modulus of $\epsilon_n(z)$ on the circle $|z - \zeta| = 1$. Hence the derivative of $\epsilon_n(z)$ approaches zero as $z \to \infty$ in the sector of opening less than 2ϕ. This proves our theorem.

5. *Proofs of Theorems X and XI.* Let $f(z)$ satisfy the hypothesis of Theorem X. There is an R such that $f(z)$ is analytic and bounded for $x \geqq R$. As we may replace $f(z)$ by $f(z+R)$, we assume that $R = 0$.

We shall prove, without using our assumption relative to the λ's, that, for every n,

$$a_n = \lim_{\omega \to \infty} (1/2\omega i) \int_{-\omega i}^{\omega i} f(z)\, e^{\lambda_n z} dz, \tag{9}$$

the integration being performed along the line $x = 0$.

As

$$f(z) = a_0 + a_1 e^{-\lambda_1 z} + \cdots + a_n e^{-\lambda_n z} + \epsilon_n(z)\, e^{-\lambda_n z},$$

where $\epsilon_n(z) \to 0$ as $x \to \infty$, we have

$$\int_{-\omega i}^{\omega i} f(z)\, e^{\lambda_n z} dz = \int_{-\omega i}^{\omega i} (a_0 + \cdots + a_{n-1} e^{-\lambda_{n-1} z}) e^{\lambda_n z}\, dz + 2\omega i a_n + \int_{-\omega i}^{\omega i} \epsilon_n(z)\, dz. \tag{10}$$

The first integral in the second member of (10) stays bounded as $\omega \to \infty$. If, then, we can show that

$$\lim_{\omega \to \infty} (1/2\omega i) \int_{-\omega i}^{\omega i} \epsilon_n(z)\, dz = 0, \tag{11}$$

our formula for a_n will be established.

Let σ be any positive number. We have

$$\int_{-\omega i}^{\omega i} \epsilon_n(z) dz = \int_{-\omega i}^{\sigma + \omega i} \epsilon_n(z)\, dz + \int_{\sigma - \omega i}^{\sigma + \omega i} \epsilon_n(z)\, dz + \int_{\sigma + \omega i}^{\omega i} \epsilon_n(z) dz, \tag{12}$$

every path of integration being rectilinear.

Any $\eta > 0$ being taken, let a σ be chosen so that $|\epsilon_n(z)| < \eta$ for $x \geqq \sigma$. Because $f(z)$ is bounded for $x \geqq 0$, $\epsilon_n(z) e^{-\lambda_n z}$ is bounded for $x \geqq 0$.

Hence $\epsilon_n(z)$ is bounded for $0 \leqq x \leqq \sigma$.[*] Let N be an upper bound for $|\epsilon_n(z)|$ in this region.

Then the first and third integrals in the second member of (12) have moduli not exceeding σN, whereas the modulus of the second integral is less than $2\omega\eta$. Hence

$$(13) \qquad \left| (1/2\omega i) \int_{-\omega i}^{\omega i} \epsilon_n(z)\, dz \right| < \sigma N/\omega + \eta.$$

If ω is large, the second member of (13) is less than 2η. This proves (11), and gives our formula for a_n.

Let

$$(14) \qquad r_n(z) = f(z) - a_0 - a_1 e^{\lambda_1 z} - \cdots - a_n e^{\lambda_n z}.$$

We shall prove that there exists an $S > 0$ such that $r_n(z) \to 0$ as $n \to \infty$, if $x > S$. This will prove that the asymptotic development converges to $f(z)$ in some half-plane. Furthermore, it is known that, for a Dirichlet series with lim inf $\lambda_n/\log n > 0$, the existence of a half-plane of convergence implies the existence of a half-plane of absolute convergence.[†] Thus, our theorem will be proved.

Let M be an upper bound for $|f(z)|$ on the imaginary axis. We see from (9) that $|a_n| \leqq M$. It follows from (14) that $|r_n(z)|$ has $(n+1)M$ for an upper bound on the imaginary axis.

The function $\epsilon_n(z) = e^{\lambda_n z} r_n(z)$ is bounded for $x \geqq 0$. We know from Doetsch's theorem that the least upper bound of $\log|\epsilon_n(z)|$, on a vertical line, is a downward convex function. Since $\log|\epsilon_n(z)| \to -\infty$, $x \to \infty$, the upper bound of $\log|\epsilon_n(z)|$ must decrease as x increases. Then the upper bound of $|\epsilon_n(z)|$ also decreases as x increases.

For $x = 0$, $|\epsilon_n(z)| = |r_n(z)|$, so that $|\epsilon_n(z)| \leqq (n+1)M$. Hence, for $x > 0$,

$$|r_n(z)| = e^{-\lambda_n x} |\epsilon_n(z)| < e^{-\lambda_n x}(n+1)M.$$

Our hypothesis with respect to λ_n implies the existence of a $\tau > 0$ such that $\lambda_n > \tau \log n$ for every n. Then, for every n,

$$|r_n(z)| < M(n+1)/n^{\tau x}.$$

It follows that, for $x > 1/\tau$, $r_n(z) \to 0$ as $n \to \infty$. This proves Theorem X.

We now prove Theorem XI. Because $(f(z) - g(z))e^{\lambda_n z}$ approaches zero as $x \to \infty$, the upper bound of the modulus of that function on a vertical line decreases as x increases. The upper bound for $x = 0$[*] is, for any λ_n, the upper bound of $f(z) - g(z)$. Then, if $x > 0$, $f(z) - g(z)$ must be zero, since $| e^{\lambda_n x} | \to \infty$ as $n \to \infty$.

---

[*] As well as for $x > \sigma$.

[†] Valiron, *loc. cit.*, p. 3.

12. *The existence theorem.* We consider the series

$$\sum_{n=1}^{\infty} a_n e^{-\lambda_n z} \left(1 - e^{b_n z \log z}\right) \tag{15}$$

where, for every n, $b_n < 0$ and

$$| a_n b_n | < 2^{-n}. \tag{16}$$

We take any sector of opening $2\phi < \pi$, $(\phi > 0)$. For any z in this sector, let $z = r\, e^{i\theta}$, with $| \theta | < \phi$. Then

$$\begin{aligned} \mathcal{R}(z \log z) &= \mathcal{R}[(x + iy)\ (\log r + i\theta)] \\ &= x \log r - y\theta \\ &> x(\log r - \phi \tan \phi). \end{aligned} \tag{17}$$

Thus, when z is sufficiently large,

$$\mathcal{R}(z \log z) > 0. \tag{18}$$

Again, if z is a point of large modulus, in our sector,

$$| z \log z | < | z |^2 = x^2 \sec^2 \theta < x^2 \sec^2 \phi,$$

so that certainly, for $| z |$ large,

$$| z \log z | < e^x. \tag{19}$$

Let (18) and (19) both hold, in our sector, for $x > \sigma$. We say that (15) converges uniformly in the sector, for $x > \sigma$.

Let z be a point in the sector, with $x > \sigma$. Consider any n for which

$$| a_n | \leqq 2^{-n} | z \log z |. \tag{20}$$

We have, by (18) and (19),

$$\begin{aligned} | a_n e^{-\lambda_n z} (1 - e^{b_n z \log z}) | &< 2^{-n} | z \log z | \cdot e^{-\lambda_n x} (1 + 1) \\ &< 2^{-n+1} e^{-(\lambda_n - 1) x}. \end{aligned} \tag{21}$$

Consider any n for which (20) does not hold. We have

$$\begin{aligned} | 1 - e^{b_n z \log z} | &\leqq | b_n z \log z | + (1/2!)\ (| b_n z \log z |)^2 + \cdots \\ &< | b_n z \log z |\ e^{| b_n z \log z |}. \end{aligned} \tag{22}$$

---

[*] It is evidently legitimate to assume $f(z)$ and $g(z)$ analytic and bounded for $x \geqq 0$.

Using (16), and the fact that (20) does not hold, we have

$$|\, b_n z \log z \,| < 2^{-n} \left| \frac{z \log z}{a_n} \right| < 1,$$

so that, by (22),

$$(23) \qquad |\, 1 - e^{b_n z \log z} \,| < |\, b_n z \log z \,| \cdot e.$$

Then, by (19) and (23),

$$(24) \qquad |\, a_n e^{-\lambda_n z} (1 - e^{b_n z \log z}) \,| < e \cdot |\, a_n b_n \,| \cdot e^{-(\lambda_n - 1)x} < 2^{-n} e \cdot e^{-(\lambda_n - 1)x}.$$

From (21) and (24) we see that, for $x > \sigma$, in our sector, the terms of (15) are all less in modulus than the corresponding terms of

$$(25) \qquad \sum_{n=1}^{\infty} 2^{-n} e \cdot e^{-(\lambda_n - 1)x}.$$

But (25) is evidently uniformly convergent for $x > 0$. Hence (15) is uniformly convergent, in our sector, for $x > \sigma$.

We show now that the function $f(z)$ to which (15) converges is asymptotically equivalent to $\sum a_n e^{-\lambda_n z}$.

It follows from (17) (last item), that as $x \to \infty$, with z in our sector,

$$(26) \qquad \lim e^{\lambda z} \cdot e^{b_n z \log z} = 0$$

for every $\lambda > 0$. We see from (25) that the remainder after n terms in (15) is of the form $\epsilon_n(z)\, e^{-(\lambda_n - 1)z}$, where $\epsilon_n(z) \to 0$ as $x \to \infty$.

Taking any n, let p be such that $\lambda_p > \lambda_n + 1$. Then

$$f(z) = \sum_{i=1}^{p} a_i e^{-\lambda_i z} (1 - e^{b_i z \log z}) + \epsilon_p(z) e^{-(\lambda_p - 1)z}.$$

For any i such that $n < i \leqq p$, we have

$$a_i e^{-\lambda_i z} (1 - e^{b_i z \log z}) = \eta_i(z)\, e^{-\lambda_n z},$$

where $\eta_i(z) \to 0$ as $x \to \infty$. Hence

$$f(z) = \sum_{i=1}^{n} a_i e^{-\lambda_i z} (1 - e^{b_i z \log z}) + \delta_n(z)\, e^{-\lambda_n z},$$

where $\delta_n(z) \to 0$ as $x \to \infty$. Then

$$f(z) = \sum_{i=1}^{n} a_i e^{-\lambda_i z} - e^{-\lambda_n z} \sum_{i=1}^{n} a_i e^{(\lambda_n - \lambda_i)z} \cdot e^{b_i z \log z} + \delta_n(z)\, e^{-\lambda_n z}$$

and this, by (26), proves that $f(z) \sim \sum a_n e^{-\lambda_n z}$ in our sector.

COLUMBIA UNIVERSITY,
NEW YORK, N. Y.

On a Theorem of Severi.

By Oscar Zariski.

1. In one of his memoirs published in the *Mathematische Annalen*,[*] Professor Severi undertook to prove the following theorem:

Every correspondence between the points of the general curve of a linear system $| C |$, ∞^2 *at least, drawn on a regular surface, has a valence. In particular every plane curve varying in a linear system* ∞^2 *at least has only correspondences with valence* (*loc. cit.*, p. 516-517).

Severi gives two proofs of this theorem, one transcendental, based upon the investigation of the abelian integrals of the varying curve C of $| C |$, and the other purely geometric. But in each of them the theorem is proved only under certain conditions which the author points out explicitly. Thus, in the transcendental treatment Severi assumes that when a curve of the system acquires a new multiple point, this point will generally be a nodal double point, while in the geometric treatment of the question he makes the hypothesis that the system $| C |$, supposed to be ∞^r, should not have ∞^{r-1} reducible curves. Since the two conditions seem to him to be independent, he draws the conclusion that in the definite statement of the theorem both can be omitted and therefore the theorem is true without restrictions. Investigating closer the geometrical proof of Severi, certain doubts as to the rigor of the procedure were brought forth. There is a single weak point in Severi's reasoning. When this is eliminated, the necessary and sufficient conditions under which the result is true, can be completely formulated. The theorem thus limited still retains its importance, especially in that it is still able to prove the non-existence of singular correspondences on a curve of genus $p > 0$ with general modulus, which application was the first motive to Severi's investigation.

I formulate Severi's theorem thus:

I. *If in a net* $| C |$ *of degree* n *drawn on an algebraic regular surface the general curve* C *possesses a singular correspondence, the correspondence* $(n-1, n-1)$ *defined on* C *by the characteristic series* g_n^1 *is broken up into several irreducible correspondences, and at least one of them is singular.*

I also show:

---

[*] "Le corrispondenze singolari fra i punti di una curva variabile in un sistema lineare sopra una superficie regolare," *Mathematische Annalen*, Vol. 74 (1913).

II. *The above mentioned correspondences into which the correspondence* $(n-1, n-1)$ *is broken up form a base*[*] *on the general curve* C, *i. e. designating them by* $M_1, \cdots, M_k$, *for every other correspondence* M *on* C

$$\lambda M + \lambda_1 M_1 + \cdots + \lambda_k M_k \equiv 0,$$

where $\lambda, \lambda_1, \cdots, \lambda_k$ *are integers, and* $\lambda \neq 0$.

It may happen that the k correspondences M_i are not independent. Then the number of correspondences forming a base is reduced.

I further show that Severi's theorem is absolutely true for *simple* linear systems (consequently at least ∞^3), i. e., such that the curves of the system which pass through a given point P must not necessarily pass through other points varying with P. We have then the following theorem:

III. *On the general curve* C *of a simple linear system, drawn on a regular surface, all correspondences have a valence.*

2. Let $|C|$ be a linear system, ∞^2 at least, drawn on a regular surface F. and let us suppose that there exists on the general curve C a singular correspondence T, which can therefore vary continuously when C varies continuously in $|C|$. In general the correspondence T will not be determined rationally on every C, i. e. while C circulates in $|C|$, starting from an initial position and returning to it, T although preserving its indices will be transformed into another correspondence T', different from T. Severi shows that starting from T one can construct another correspondence also singular, which is rationally determined on every C. Possessing such a correspondence, which we shall call T, Severi reasons as follows: Let the curve C vary in a net H of degree n. Let x be any point on the surface F and let us consider the curve D_x described by all the points y which are homologous to x in the correspondences T relative to all the ∞^1 curves C passing through x. Let D_x have a multiplicity s at the point x and the multiplicities t—necessarily equal due to consideration of symmetry—at each of the $n-1$ points z in which the curves C passing through x intersect outside x and the base points of the net. D_x will also pass through the base points of the net with certain multiplicities which we can neglect because, when x varies on a fixed C, they give rise to fixed intersections of C with D_x. Varying x on a given C we obtain a continuous system ∞^1 of curves D_x, and since the surface is regular, the curves

---

[*] On the concept of a base, see F. Severi: "Sulle corrispondenze tra i punti di una curva algebrica e sopra certe classi di superficie," *Memorie della Reggia Accademia di Torino*, Vol. 64 (1903); "Sulla totalita delle curve algebriche tracciate sopra una superficie algebrica," *Mathematische Annalen*, Vol. 62 (1906).

D_x will be contained in one and the same linear system, by a well-known theorem of Enriques.* The curves D_x will consequently intersect C in equivalent groups of points. One such group, for a given D_x, consists of the point x, counted s times, of the group Y of the points homologous to x in the correspondence T on C, and of the group Z of the $n-1$ points z, each of which is counted t times. We have therefor

$$sx + Y + tZ \equiv sx' + Y' + tZ',$$

where x and x' are any two arbitrary points of C. But since $x + Z \equiv x' + Z'$, we deduce

$$Y + (s-t)x \equiv Y' + (s-t)x',$$

which proves that T has a valence $s-t$, which contradicts the hypothesis. The objection which may be raised against Severi's reasoning is that it is not necessarily true that D_x will pass through all the $n-1$ points z of the group Z with the same multiplicity. It would have been so, if, when x circulates on C starting out from an initial position and returning to it, any point z could be transformed into any one of the remaining $n-2$ points of the group Z, in other words, when the correspondence $(n-1, n-1)$ associating x and the points of the group Z is irreducible. But if this correspondence breaks up into several correspondences, and if we call $z_1, z_2, \cdots, z_r$ $(r < n-1)$ the points homologous to x in one of these correspondences, all we can say is that D_x will pass with the same multiplicity through the points $z_1, z_2, \cdots, z_r$, but the multiplicities relative to two different correspondences may well be different.

Suppose that the correspondence $(n-1, n-1)$ is broken up into k correspondences $M_1, \cdots, M_k$ and let $Z_1, \cdots, Z_k$ respectively be the groups of points homologous to x in the correspondences $M_1, \cdots, M_k$. If we denote by t_i the multiplicity with which the curve D_x passes through the points of the group Z_i, we will have instead of Severi's relation the following:

$$sx + Y + t_1Z_1 + \cdots + t_kZ_k \equiv sx' + Y' + t_1Z_1' + \cdots + t_kZ_k'.$$

We conclude that the correspondence

$$sI + T + t_1M_1 + \cdots + t_kM_k,$$

where I denotes the identical correspondence, has a valence, in symbols

$$sI + T + t_1M_1 + \cdots + t_kM_k \equiv 0. \tag{1}$$

---

* F. Enriques, "Una proprietà delle serie continue di curve appartenenti ad una superficie algebrica regolare," *Rendiconti del Circolo Matematico di Palermo*, Vol. 13 (1899).

From this we conclude that if all the correspondences T_i $(i=1, 2, \cdots, k)$ have a valence, then T has a valence, and this proves our theorem I (§ 1). It is easily shown that the correspondences $M_1, \cdots, M_k$, or a certain number of them in the case where they are not independent, form a base, i. e. for any other correspondence M on C we have

$$(2) \qquad \lambda M + \lambda_1 M_1 + \cdots + \lambda_k M_k \equiv 0,$$

where $\lambda, \lambda_1, \cdots, \lambda_k$ are integers and $\lambda \neq 0$. In fact, if M is rationally determined on C then (2) is satisfied for $\lambda = 1$, in virtue of (1).[*] If, however, M is not rationally determined, we can construct, by Severi's method, another correspondence M', which is rationally determined on every C. The correspondence M', constructed thus, is related to M by $M' \equiv \mu M$, where μ is a positive integer. Since

$$M' + \lambda_1 M_1 + \cdots + \lambda_k M_k \equiv 0$$

we have also

$$\mu M + \lambda_1 M_1 + \cdots + \lambda_k M_k \equiv 0,$$

which proves the theorem II (§ 1).

3. In fact, one can give examples of nets of curves on regular surfaces, for instance, on the plane, in which the general curve possesses singular correspondences. There exist in the plane nets of harmonics or equianharmonic cubics,[†] and every such cubic possesses singular correspondences. Such nets of cubics have been studied recently by Miss Rossi (Rome) in her thesis. Miss Rossi finds an interesting class of nets of equianharmonic cubics with six base points, for which the characteristic series g_3^1 is cyclic, i. e. is generated by a birational cyclic transformation of third order. This is in accordance with our general theorem I, since in this case the characteristic series gives rise to a correspondence (2, 2) which, as it has to be reducible, breaks up necessarily into two singular birational transformations, one the inverse of the other. Note that these nets are generally deprived of a pencil of reducible curves.

In the examples mentioned above, the curves of the net have the same modulus. We can show that this is general for nets of elliptical curves, i. e. *if in a net of elliptical curves the general curve possesses a singular corre-*

---

[*] Note that the correspondence I can be eliminated from (1), since evidently $I + M_1 + M_2 + \cdots + M_k \equiv 0$.

[†] See O. Chisini, "Sui fasci di cubiche a modulo costante," *Rendiconti del Circolo Matematico di Palermo*, Vol. 40 (1916).

spondence, the curves of the net have the same modulus. This is seen by considering the transcendental representation of a singular correspondence T on an elliptical curve C. Let $y^{(1)}, y^{(2)}, \cdots, y^{(\beta)}$ be the homologous points of x in T. Then we have

$$\sum_{1}^{\beta}{}_{\nu}\, u(y^{\nu}) \equiv \pi u(x) + \pi_1 (\text{mod } 1, \omega),$$

where u is an abelian integral of the first kind, the periods of which are 1 and ω, and π, π_1 are constants. There exist also a known relation between π and ω:

$$\pi = h + g\omega, \quad \pi\omega = H + G\omega,$$

where h, g, H, G are integers and $g \neq 0$, if T is singular. The intergers h, g, H, G are called by Severi the characteristic integers of the correspondence. The period ω satisfies an equation of second degree

$$g\omega^2 + (h - G)\omega - H = 0. \tag{3}$$

If the curve C varies in a net in such a way that also the correspondence T varies continuously, the characteristic integers of T must remain unaltered, since they cannot vary continuously. Equation (3) shows that the period ω, which is also the modulus of C, does not vary with C. Q. E. D.

4. We will show now that Severi's theorem is applicable without restrictions for simple linear systems (consequently at least ∞^3). We have only to make a preliminary observation. Let us have on an irreducible curve C a linear involution g_n^1. We will designate by G the monodromic group of the g_n^1, i. e. the monodromic group of the n-valued algebraic function $y(x)$, which we obtain cutting the g_n^1 on C by the lines of the pencil $x = \text{const.}$ It is known that G is transitive when C is irreducible. Let us consider the correspondence $(n-1, n-1)$ defined by the g_n^1. If this correspondence is broken up into several correspondences, what can we say about the group G? It is evident that in this case the substitutions of G which leave one letter unaltered, do not permute transitively the remaining $n-1$ letters, i. e. the group G is only simply transitive. If, on the other hand, the correspondence $(n-1, n-1)$ is irreducible, the group G is at least doubly transitive. In particular, when the group G is the symmetrical group on n letters, then the correspondence $(n-1, n-1)$ is certainly irreducible.

Let us take now on a regular surface F a linear system $|C|$, at least ∞^3 and simple. With this hypothesis we are certain that taking in $|C|$ a general net, the characteristic series g_n^1 of a general C of the net is simple,

i. e. its groups are not composed of the sets of points of an involution of lower order, and therefore its monodromic group G will be primitive. On the other hand, if two curves of $|C|$ are tangent to each other at a point P, they will have a simple contact there, and will not have another point of tangency. Therefore the monodromic group G of the above mentioned characteristic series g_n^1 will contain a transposition. The group being primitive, we deduce by means of a well known theorem in the theory of groups, that it will be the symmetrical group on n letters. That is sufficient to affirm that the correspondence $(n-1, n-1)$ on the general curve taken in a general net of $|C|$ is irreducible, and hence, by the general theorem I, C does not contain singular correspondences.

The application of this result to the proof of the non-existence of singular correspondences on a curve of genus p with general modulus is immediate. It is sufficient to show that there exist particular curves of genus p, which do not contain singular correspondences. This follows simply from the fact that for every value of p there exist simple linear systems of plane curves of genus p. In fact, given p, we determine the integer n such that

$$(n-1)(n-2)/2 < p \leq n(n-1)/2.$$

The curves of order $n+1$ with $n(n-1)/2 - p \leq n-2$ fixed double points form evidently such a linear system.

In fact, the system contains degenerate curves ϕ_{n+1} broken up in $d = n(n-1)/2 - p$ lines forming a polygon whose vertices are at the d fixed double points and in $n+1-d \geqq 3$ arbitrary lines, hence the system is simple. The above mentioned degenerate curves ϕ_{n+1} relative to different polygons do not possess a common part and other common double points outside the d fixed points. Moreover the curves of the system are not composed of the curves of a pencil, due to the fact that these degenerate curves contain $n+1-d$ arbitrary lines. Hence, by the theorem of Bertini, the system is irreducible and of genus equal precisely to p.

THE JOHNS HOPKINS UNIVERSITY.

On Gierster's Classnumber Relations.

By J. V. Uspensky.

In the fifth of the papers published under the common title: "Sur les relations entre les nombres des classes des formes quadratiques binaires et positives"[*] I gave an elementary and purely arithmetical proof of Gierster's classnumber relations. The exposition could have been greatly simplified had I known at that time the explicit expressions for certain numerical functions. Lacking these expressions I had to resort to indirect means to eliminate the numerical function whose expression was not known *à priori,* which made my method unnecessarily complicated. Recently I discovered the lacking expressions, which are by no means obvious, and this makes it possible for me now to present my proof in its simplest form. My method is based on certain very general identities which may be useful on many occasions, and I deem it advisable to reproduce those identities with their proof, inasmuch as the proof is very simple and does not require much space.

1. *The Fundamental Identities.* Let $F(x, y, z)$ be an arbitrary function defined for integral values of its arguments and satisfying the following conditions of parity:

$$F(-x, y, z) = F(x, y, z)\,;\; F(x, -y, -z) = -F(x, y, z)\,,$$

and we suppose

$$F(x, 0, z) = 0$$

whenever the middle argument y can assume the value 0. On the other hand, n being an arbitrary positive integer, we consider two modes of partition of this number, the first one being indicated by the equation

$$n = i^2 + 2d\delta \tag{1}$$

where i may have any integral value, while d and δ are supposed to be positive integers. The second mode of partition is indicated by the equation

$$n = h^2 + \Delta\Delta' \tag{2}$$

where again h may be any integer, while Δ and Δ' are supposed to be positive integers, satisfying the additional conditions

$$\Delta < \Delta', \quad \Delta' \equiv \Delta \pmod 2.$$

---

[*] *Bulletin de l'Académie des Sciences de l'USSR,* No. 8-9, 1926.

This being so, let us consider the sum

$$S=\sum\{F[(\Delta'+\Delta)/2,(\Delta'-\Delta)/2,\ \Delta-h]\ -F[h,\ (\Delta'-\Delta)/2,\ \Delta-h]\}$$

extended over all the partitions (2). This sum may be naturally split into three parts S_1, S_2, S_3 according to the positive, negative, or zero value of $\Delta'-\Delta+2h$. It is easy to see that $S_3=0$ except when n is a perfect square $=s^2$ in which case

$$S_3=\sum\{F(s,j,s)-F(j,j,s)\},\quad (j=1,2,3,\cdots,s-1).$$

As to the part S_2, it vanishes in every case. For, supposing that h, Δ, Δ' run over the set of partitions (2) for which $\Delta'-\Delta+2h<0$ it is obvious that the numbers defined by

$$h_1=-(\Delta'+\Delta)/2,\quad \Delta_1=(\Delta-\Delta')/2-h,\ \Delta_1'=(\Delta'-\Delta)/2-h$$

reproduce exactly the same set. We have therefore

$$S_2=\sum\{F[(\Delta_1'+\Delta_1)/2,\ (\Delta_1'-\Delta_1)/2,\ \Delta_1-h_1]\\ -F[h_1,\ (\Delta_1'-\Delta_1)/2,\ \Delta_1-h_1]\}.$$

But as

$$F[(\Delta_1'+\Delta_1)/2,\ (\Delta_1'-\Delta_1)/2,\ \Delta_1-h_1]\\ =F[-h,\ (\Delta'-\Delta)/2,\ \Delta-h]=F[h,\ (\Delta'-\Delta)/2,\ \Delta-h],$$
$$F[h_1,\ (\Delta_1'-\Delta_1)/2,\ \Delta_1-h_1]=F[-(\Delta'+\Delta)/2,\ (\Delta'-\Delta)/2,\ \Delta-h)\\ =F[(\Delta'+\Delta)/2,\ (\Delta'-\Delta)/2,\ \Delta-h],$$

we get the equation

$$S_2=-S_2;\quad \text{that is } S_2=0.$$

It remains to consider the part S_1 extended over the partitions satisfying the inequality $\Delta'-\Delta+2h>0$. Suppose we let h, Δ, Δ' run over all the partitions of this kind. It is easy to see that the numbers i, d, δ defined either by

$$i=\Delta-h,\quad d=\Delta,\quad \delta=(\Delta'-\Delta+2h)/2$$

or by

$$i=h-\Delta,\quad d=(\Delta'-\Delta+2h)/2,\quad \delta=\Delta$$

reproduce all the solutions of the equation (1) satisfying either the condition $\delta+i-d>0$ or the condition $\delta+i-d<0$. As in the first case we have

$$F(\delta+i,\ \delta+i-d,\ i)=F[(\Delta'+\Delta)/2,\ (\Delta'-\Delta)/2,\ \Delta-h],$$

and in the second

$$F(\delta+i,\ \delta+i-d,\ i)=-F[h,\ (\Delta'-\Delta)/2,\ \Delta-h],$$

it follows from the preceding remarks

$$S_1 = \sum F(\delta + i,\ \delta + i - d,\ i)$$

the sum being extended over all the partitions (1). The three parts of the sum S being thus transformed, we finally get the following important identity:

$$\text{(I)}\qquad \sum F(\delta + i, \delta + i - d, i) = \sum \{F[(\Delta' + \Delta)/2,\ (\Delta' - \Delta)/2,\ \Delta - h] \\ - F[h,\ (\Delta' - \Delta)/2,\ \Delta - h]\} + T,$$

where the sums are extended over the partitions (1) and (2) respectively and

$$T = \{\sum (F(j, j, s) - F(s, j, s))\}, \\ (j = 1, 2, 3, \cdots, s - 1,\ n = s^2,\ s > 0),$$

using the same abbreviated notation as in our paper: "On Jacobi's arithmetical theorems concerning the simultaneous representation of numbers by two different quadratic forms."[*] Repeating word by word the same reasoning, we can establish the second fundamental identity involving the function which satisfies the conditions:

$$F(-x, y, z) = -F(x, y, z)\,;\ F(x, -y, -z) = F(x, y, z) \\ F(x, 0, z) = 0.$$

This second identity is

$$\text{(II)}\quad \sum F(\delta + i, \delta + i - d, i) = \sum \{F[(\Delta' + \Delta)/2,\ (\Delta' - \Delta)/2,\ \Delta - h] \\ + F[h,\ (\Delta' - \Delta)/2,\ \Delta - h]\} + T,$$

where

$$T = \{\sum (F(j, j, s) - F(s, j, s)), \\ (j = 1, 2, 3, \cdots, s - 1,\ n = s^2,\ s > 0),$$

the extent of summations being the same.

The identities (I) and (II) constitute the essential foundation of what follows. There are other important identities of similar nature which we hope to publish in connection with our investigations on incomplete numerical functions.

2. *Important Relation derived from* (I) *and* (II). *Let* $f(x)$ denote an absolutely arbitrary *even* function. We can take in the identities (I) and (II)

$$F(x, y, z) = (-1)^{z/2}[(-1)^x + (-1)^y] z f(x), \\ F(x, y, z) = (-1)^{z/2}[(-1)^x + (-1)^y] x f(x),$$

---

[*] *Transactions of the American Mathematical Society*, Vol. 30 (1928).

respectively, provided y is different from 0; but in both cases

$$F(x, 0, z) = 0.$$

Substituting furthermore $4n$ for n after a simple discussion we reach two following equations.

$$\sum_{(a)} (-1)^{i+\delta}(\delta + 2i)f(\delta + 2i) = \sum_{(b)} (-1)^{h+\Delta'}(\Delta' + \Delta)f(\Delta' + \Delta) -$$

$$- \sum_{(c)} (-1)^{(\Delta-1)/2+(h-1)/2} hf(h) + 2 \sum_{(d)} (-1)^{j+k} jf(2j) + T,$$

$$\sum_{(a)} (-1)^{i+\delta}(\delta + 2i) = \sum_{(b)} (-1)^{h+\Delta'} 2\Delta f(\Delta' + \Delta)$$

$$- \sum_{(b)} (-1)^{h}\Delta[(-1)^{\Delta} + (-1)^{\Delta'}]f(2h) -$$

$$- \sum_{(c)} (-1)^{(\Delta-1)/2+(h-1)/2} hf(h) + 2 \sum_{(d)} (-1)^{j+k}(j - k)f(2j) + T'.$$

The sums bearing the signs (a), (b), (c), (d) are extended over the partitions indicated below by the same signs, namely

$$(a)\ \ n = i^2 + d\delta; \quad (b)\ \ n = h^2 + \Delta\Delta',\ \Delta < \Delta';$$

$$(c)\ \ 4n = h^2 + \Delta\Delta,\ \Delta < \Delta',\ h \text{ odd}; \quad (d)\ \ n = j^2 + k^2,\ j > 0,\ k > 0,$$

while T and T' are both zeros except when n is a perfect square: $n = s^2$, $s > 0$, in which case

$$T = \sum [(-1)^{s+j} jf(j) - (-1)^{s}(1 + (-1)^{j})sf(2s)],$$

$$T' = \sum [(-1)^{s+j} 2sf(j) - (-1)^{s}(1 + (-1)^{j})sf(2s)],$$

the sums being extended over $j = 1, 2, 3, \cdots, 2s - 1$. Subtracting now the second equation from the first we come to an important relation

$$\text{(III)} \quad \sum_{(b)} (-1)^{h+\Delta'}(\Delta' - \Delta)[f(\Delta' + \Delta) - f(2h)]$$

$$= \sum_{(a)} (-1)^{i+d} d[f(d + 2i) - f(2i)] + U,$$

the extent of summations being as above and with the usual symbolism

$$U = \{\sum_{j=1}^{2s-1} (-1)^{s+j}(2s - j)f(j) + (-1)^{s} sf(0)\},\ \ n = s^2,\ s > 0.$$

The preceding equation can be exhibited in a different form. Denote as usual by $G(n)$ the classnumber of all quadratic forms corresponding to a given determinant $-n$, this number being diminished by ½ when n is a perfect square, and by ⅔ when n is a triple of a perfect square; moreover $G(0) = -1/12$. The notation $F(n)$ will be used to denote the classnumber of those forms which can represent odd numbers, this number being diminished by ½ when n is a square of an *odd* number, while $F(0) = 0$. Putting

$$T(n) = 3G(n) - 2T(n), \quad T(0) = -\tfrac{1}{4}$$

it follows from the well-known Kronecker's classnumber relations

$$(-1)^m \sum T(m-k^2) = 2\sum (-1)^{\Delta'}(\Delta'-\Delta), \quad m = \Delta\Delta', \quad \Delta < \Delta'$$

the first sum being extended over *all* integers k whose squares *do not exceed* an arbitrary positive integer m. By means of the last equation it is possible to present the equation (III) in the following final form: put for brevity

$$\psi(m) = \sum (-1)^{d'}(d'-d)f(d'+d) - \tfrac{1}{2}(-1)^m \sum T(m-h^2)f(2h),$$

the first sum being extended over all conjugate divisors of m satisfying the condition $d < d'$ while the second is extended over all integers h whose squares *do not exceed* m; with this notation adopted we have

$$\text{(IV)} \quad \sum (-1)^i \psi(m-i^2) = \sum (-1)^{i+d} d[f(d+2i) - f(2i)] + W,$$

the first sum being extended over all integers i satisfying the inequality $i^2 < n$ while the second sum is extended over all the solutions of the indeterminate equation

$$n = i^2 + d\delta$$

with positive integers d and δ. As to W we have

$W = 0$, when n is not a perfect square,

$$W = (-1)\sum_{k=1}^{2s-1}(-1)^k(2s-k)f(k) + (-1)^s sf(0) \\ + [(-1)^s/4]\,[f(2s) - f(0)]$$

when $n = s^2$, $s > 0$.

For particular choice of an arbitrary even function $f(x)$ the equation (IV) gives means to obtain different classnumber relations. In particular by taking $f(x) = \cos(2\pi x/3)$ or $f(x) = \cos(2\pi x/5)$ we obtain Gierster's classnumber relations of the third and fifth degrees (dritter und fünfter Stufe) respectively.

3. *The Equation derived from* (IV) *when* $f(x) = \cos(2\pi x/3)$. By taking $f(x) = \cos(2\pi x/3)$ in the fundamental equation (IV) we obtain, as a consequence of this assumption, an equation from which Gierster's relations of the third degree, and even more general, can be easily derived. To represent the right-hand member of (IV) in a suitable form it should be remembered that the number of representations of any integer n by the form

$$x^2 + y^2 + 3z^2 + 3t^2$$

is given by the sum

$$(-1)^n\,{}^{16}\!/_3 \sum (-1)^d\, d \sin^2 (2\pi d/3)$$

extended over all the divisions of n. Hence, it is easy to see that the right-hand member of (IV) may be represented thus:

$$-\,{}^3\!/_8 \sum (-1)^h \sum (-1)^{i+j+k+l} \cos(2\pi i/3)$$

where the inner sum is extended over all solutions of the equation

$$n - h^2 = i^2 + j^2 + 3k^2 + 3l^2$$

while h ranges over all integers whose squares are *less* than n. From the equation

$$\sum (-1)^h \psi(n - h^2) = -\,{}^3\!/_8 \sum (-1)^h \sum (-1)^{i+j+k+l} \cos (2\pi i/3)$$

holding true for *every* n it follows necessarily

$$\psi(n) = -\,{}^3\!/_8 (-1)^n \sum \cos(2\pi i/3), \quad n = i^2 + j^2 + 3k^2 + 3l^2,$$

that is

$$\begin{aligned} &\sum T(n - h^2) \cos (2\pi h/3) \\ &= (-1)^n 2 \sum (-1)^{d'} (d' - d) \cos 2\pi (d' + d)/3 + {}^3\!/_4 \sum \cos (2\pi i/3). \end{aligned}$$

On the other hand, we have

$$\sum T(n - h^2) = 2(-1)^n \sum (-1)^{d'} (d' - d), \quad n = dd',\ d < d'$$

whence it follows

$$\begin{aligned} &\sum [3G(n - 9h^2) + 2F(n - 9h^2)] \\ &\qquad = 2(-1)^n \sum (-1)^{d'} (d' - d) + {}^1\!/_2 \sum \cos (2\pi i/3), \end{aligned} \tag{3}$$

the first sum in the right-hand member being extended now over conjugate divisors of n which satisfy the conditions:

$$d < d', \quad d' + d \equiv 0 \pmod 3. \tag{4}$$

To the equation (3) we can adjoin another equation

$$\begin{aligned} &3 \sum [2F(n - 9h^2) - G(n - 9h^2)] \\ &\qquad = {}^1\!/_4\, N(n = 9x^2 + y^2 + z^2 + t^2) \end{aligned} \tag{5}$$

as immediate consequence of Gauss' theorem concerning the number of representations by the sum of three squares. Now from (3) and (5) we can derive the expression for both sums

$$\sum F(n - 9h^2), \quad \sum G(n - 9h^2),$$

provided we can express their right-hand members in terms of the known numerical functions. But this makes it necessary to consider separately the three cases:

$$n \equiv 0, \quad n \equiv 2, \quad n \equiv 1 \pmod 3.$$

4. *Discussion of the case* $n \equiv 0 \pmod 3$. In this case the condition $d' + d \equiv 0 \pmod 3$ requires that d' and d should be separately divisible by 3 so that we have to evaluate the sum

$$\sigma = (-1)^n 6 \sum (-1)^{d'} (d' - d), \quad n = 9dd', \ d < d'.$$

Using the notations adopted by Kronecker

$$\Phi(n) = \sum d, \qquad n = dd',$$
$$\Psi(n) = \sum (d' - d), \quad n = dd', \ d < d',$$

we find easily: for an *odd* n

$$\sigma = 6\,\Psi(n/9)$$

and for an *even* n

$$\sigma = 12\Psi(n/36) - 12\Phi(n/36) + 6 \sum (-1)^d d, \quad n = 9dd'.$$

On the other hand, we obviously have

$$\sum \cos (2\pi i/3) = N(n = i^2 + j^2 + 3k^2 + 3l^2);$$

and it is easy to verify that

$$N(n = 9x^2 + y^2 + z^2 + t^2) = \tfrac{1}{4} N(n = x^2 + y^2 + z^2 + t^2) + \\ + \tfrac{3}{4} N(n/9) = x^2 + y^2 + z^2 + t^2).$$

Thus we see that the right-hand members of (3) and (5) may be expressed by means of the known numerical function and there is no further difficulty in obtaining the following final expressions: for an *odd* n

$$\sum F(n - 9h^2) = \tfrac{1}{2}\Phi(n) - \Phi(n/3) + \tfrac{3}{2}\Psi(n/9), \tag{6}$$

$$\sum G(n - 9h^2) = \Phi(n) - \tfrac{8}{3}\Phi(n/3) + 3\Psi(n/9), \tag{7}$$

and for an *even* n

$$\sum F(n - 9h^2) = \mathrm{X}(n/3) - 3\Phi(n/36) + 3\Psi(n/36), \tag{8}$$

$$\sum G(n - 9h^2) = 2\Phi(n/6) - 6\Phi(n/36) + 6\Psi(n/36), \tag{9}$$

where $\mathrm{X}(n)$ is defined by

$$\mathrm{X}(n) = \sum d, \quad n = d\delta, \quad \delta \ \textit{odd}.$$

5. *Discussion of the case* $n \equiv 2 \pmod 3$. In this case the condition $d' + d \equiv 0 \pmod 3$ is always verified and we obtain for the sum

$$\sigma = 2(-1)^n \sum (-1)^{d'}(d' - d), \ n = dd', \ d < d', \ d' + d \equiv 0 \pmod 3$$

the following expressions: for an *odd n*

$$\sigma = 2\Psi(n)$$

and for an *even n*

$$\sigma = 4\Phi(n/4) - 4\Phi(n/4) + 2\sum (-1)^d d, \quad n = dd'.$$

Furthermore

$$\sum \cos(2\pi i/3) = -\tfrac{1}{2}N\ (n = i^2 + j^2 + 3k^2 + 3l^2),$$

and

$$N\ (n = 9x^2 + y^2 + z^2 + t^2) = \tfrac{1}{2}N\ (n = x^2 + y^2 + z^2 + t^2).$$

It is easy now to arrive at the following final expressions: for an *odd n*

$$\sum F(n - 9h^2) = \tfrac{1}{2}\Psi(n), \tag{10}$$

$$\sum G(n - 9h^2) = \Psi(n) - \tfrac{1}{3}\Phi(n); \tag{11}$$

and for an *even n*

$$\sum F(n - 9h^2) = \Psi(n/4) - \Phi(n/4) + \tfrac{1}{2}\mathrm{X}(n), \tag{12}$$

$$\sum G(n - 9h^2) = 2\Psi(n/4) - \Phi(n/4) + \tfrac{1}{2}\mathrm{X}(n). \tag{13}$$

6. *Discussion of the case* $n \equiv 1 \pmod 3$. The condition $d' + d \equiv 0 \pmod 3$ is never satisfied in this case, so that we have $\sigma = 0$ using the same notation as above. Moreover

$$\sum \cos(2\pi i/3) = \tfrac{1}{4}N(n = i^2 + j^2 + 3k^2 + 3l^2)$$

but the expression for the number of solutions of the equation

$$n = 9x^2 + y^2 + z^2 + t^2$$

is not so obvious in this case. However, there is no great difficulty in finding this number when n is *even*. For if n be even both the equations

$$4n = 9x^2 + y^2 + z^2 + t^2, \quad n = 9x^2 + y^2 + z^2 + t^2$$

have exactly the same number of solutions, and it suffices to consider the first of them. Suppose we consider the totality of repreesntations of $4n$ by the sum of 4 squares. Let P_1 denote the number of representations where none of the squares is divisible by 3, P_2 being the number of the remaining representations (all the squares but one being divisible by 3). We have

$$P_1 + P_2 = N(4n + x^2 + y^2 + z^2 + t^2),$$
$$\tfrac{3}{4}P_2 = N(4n = 9x^2 + y^2 + z^2 + t^2),$$

and therefore the required number of solutions will be found as soon as we can find P_1. But we can obtain all the solutions of the equation

$$4n = x^2 + y^2 + z^2 + t^2,$$

where none of the numbers x, y, z, t is divisible by 3, by means of the formulas

$$x = \xi + \eta, \quad y = \xi - \eta, \quad z = \zeta + \theta, \quad t = \zeta - \theta,$$

where ξ, η, ζ, θ range over all the solutions of the equation

$$2n = \xi^2 + \eta^2 + \zeta^2 + \theta^2,$$

subject to the condition that neither $\xi + \eta$ nor $\zeta + \theta$ is divisible by 3. But it is easy to see that there are exactly

$$\tfrac{2}{3}N(2n = \xi^2 + \eta^2 + \zeta^2 + \theta^2)$$

solutions of this kind. Therefore

$$P_1 = \tfrac{2}{3}N(n = x^2 + y^2 + z^2 + t^2)$$

and

$$P_2 = \tfrac{1}{3}N(n = x^2 + y^2 + z^2 + t^2);$$

whence

$$N(4n = 9x^2 + y^2 + z^2 + t^2) = \tfrac{1}{4}N(n = x^2 + y^2 + z^2 + t^2).$$

It is easy now to find in case of an *even* $n \equiv 1 \pmod 3$ the following expressions for our sums:

$$\sum F(n - 9h^2) = \tfrac{1}{2}\mathrm{X}(n/2), \tag{14}$$

$$\sum G(n - 9h^2) = \tfrac{1}{2}\Phi(n/2). \tag{15}$$

We encounter greater difficulties in finding the number of solutions of the equation

$$n = 9x^2 + y^2 + z^2 + t^2,$$

when n is *odd.* It follows from the preceding that the number of *all* solutions of the equation

$$4n = 9x^2 + y^2 + z^2 + t^2$$

is $6\Phi(n)$ so that it remains to find the number of solutions of the equation

$$4n = 9i^2 + j^2 + k^2 + l^2$$

in *odd* numbers i, j, k, l. For this purpose we write first the equation (3) in the form

$$\sum [3G(n-9h^2)-2F(n-9h^2)] = \tfrac{1}{4}N(n=9x^2+y^2+3z^2+3t^2)$$

and, this being true for every $n \equiv 1 \pmod 3$, it follows

$$N(n=y^2+3z^2+3t^2) = 4[3G(n)-2F(n)], \quad n \equiv 1 \pmod 3.$$

If, in addition to this $n \equiv 3 \pmod 8$, then $G(n) = \tfrac{4}{3}F(n)$ and

$$N(n=y^2+3z^2+3t^2) = 8F(n)$$

or, what is the same,

$$N(n=y^2+z^2+t^2) = N(n=y^2+3z^2+3t^2).$$

By means of this important relation we find

$$\begin{aligned} R = N(4n &= 9i^2+j^2+k^2+l^2) \\ &= N(4n=9i^2+j^2+3k^2+3l^2;\ i \text{ odd},\ j \text{ even}). \end{aligned}$$

Now, the total number of solutions of the equation

$$4n = 9x^2+y^2+3z^2+3t^2,$$

where x and y are of different parity, being $8\Phi(n)$ it remains to find the sum

$$S = \sum (-1)^x$$

extended over all the solutions of the preceding equation with *odd* value of $x+y$ and this can be done by means of the equation (28) of our paper "On Jacobi's arithmetical theorems, etc." which in the present case may be exhibited as follows:

$$\sum (-1)^x = \{4s(s/3)\}, \quad k=s^2, \quad s>0$$

the sum being extended over the solutions of the equation

$$4k = 9x^2+y^2+3z^2 \text{ with } z \textit{ odd.}$$

Applying this result we find

$$S = 8\sum (s/3)s = 8\sigma$$

where the summation is extended over all solutions of the indeterminate equation

$$n = s^2+3t^2, \quad s>0.$$

The sum S being thus determined, we get

$$R = 4\Phi(n) - 4\sigma,$$
$$N(n = 9x^2 + y^2 + z^2 + t^2) = 2\Phi(n) + 4\sigma,$$

and finally

$$\Sigma F(n - 9h^2) = \tfrac{1}{4}\Phi(n) + \tfrac{1}{4}\sigma, \tag{16}$$

$$\Sigma\, G(n - 9h^2) = \tfrac{1}{3}\Phi(n) + \tfrac{1}{6}\sigma. \tag{17}$$

It is hardly necessary to add that the relations we have established contain Gierster's relations of the third degree, which involve the numerical function

$$H(n) = G(n) - F(n)$$

representing the number of classes corresponding to a given negative *discriminant* $-n = b^2 - 4ac$.

7. *The equation derived from* (IV) *when* $f(x) = \cos(2\pi x/5)$. Before giving the fundamental equation we can derive from (IV) by taking $f(x) = \cos(2\pi x/5)$ it is necessary to recall certain known results concerning the number of representations by some quaternary forms. Let us denote by $N_1(n)$ and $N_1^*(n)$ respectively the numbers of solutions of the equations

$$n = x^2 + y^2 + z^2 + 5t^2, \quad n = x^2 + 5y^2 + 5z^2 + 5t^2.$$

Following expressions may be considered as known

$$N_1(n) = (-1)^n \Sigma\, (-1)^d\,[5(d'/5) + (d/5)]d,$$
$$N_1^*(n) = (-1)^n \Sigma\, (-1)^d\,[(d'/5) + (d/5)]d,$$

both sums being extended over conjugate divisions of n. Considering the equation

$$2n = x^2 + 2y^2 + 5z^2 + 10t^2$$

and denoting by $N_2(n)$ the number of its solutions we have

$$N_2(n) = (-1)^n\, 2\, \Sigma\, (-1)^d (d/5)^2 d$$

Using now the expressions for $N_1(n)$, $N_1^*(n)$ and $N_2(n)$ it is easy to find the equation

$$\Sigma\, (-1)^d d[1 - \cos(2\pi d/5)]$$
$$= (-1)^n[\tfrac{5}{8}N_2(n) + (5^{1/2}/16)N_1(n) - (5^{3/2}/16)N_1^*(n)]$$

holding true for every n. This equation will serve us to transform the right-hand number of (IV) where we assume $f(x) = \cos(2\pi x/5)$. Proceeding exactly as in § 3 we can obtain the following fundamental equation:

$$\Sigma\, [3G(n - h^2) - 2F(n - h^2)] \cos(4\pi h/5)$$
$$= \tfrac{5}{4}\sigma_1 + \tfrac{5}{8}\sigma_2 - (5^{3/2}/8)\sigma_3 + 2\sigma_4$$

where σ_1, σ_2, σ_3, σ_4 denote the following sums the extent of summation being shown behind the expression of the corresponding sum:

$$\begin{aligned}
\sigma_1 &= \sum \cos(4\pi y/5), \quad n = x^2 + y^2 + z^2 + 5t^2 \\
\sigma_2 &= \sum \cos(4\pi x/5), \quad n = x^2 + y_2 + z^2 + 5t^2 \\
\sigma_3 &= \sum \cos(4\pi x/5), \quad n = x^2 + 5y^2 + 5z^2 + 5t^2 \\
\sigma_4 &= (-1)^n \sum (-1)^{d'}(d - d') \cos[(2\pi/5)(d' + d)], \\
&\qquad\qquad n = dd, \quad d < d'.
\end{aligned}$$

Now, putting

$$\sigma_5 = \sum \cos(4\pi x/5), \quad n = x^2 + y^2 + z^2 + t^2$$

we have by Gauss theorem

$$\sum [2F(n - h^2) - G(n - h^2)] \cos(4\pi h/5) = \tfrac{1}{12}\sigma_5$$

whence and from the preceding equation it follows

$$\text{(V)}\quad 4\sum F(n - h^2)\cos(4\pi h/5) = \tfrac{5}{4}\sigma_1 + (5^{1/2}/8)\sigma_2 - (5^{3/2}/8)\sigma_3 + \tfrac{1}{4}\sigma_5 + 2\sigma_4.$$

What remains now is to express different sums in the right-hand member of this equation by means of certain standard numerical functions. This can be easily done in case when $n \equiv 0 \pmod 5$ both the remaining cases $n \equiv \pm 1$, $n \equiv \pm 2 \pmod 5$ requiring more elaborate investigation.

8. *Discussion of the case* $n \equiv 0 \pmod 5$. First it is obvious that for n divisible by 5

$$\sigma_1 = N(2n = x^2 + 2y^2 + 5z^2 + 10t^2) = N_2(n).$$

Second, in this case it is easy to find the number of solutions of the equation

$$n = x^2 + y^2 + z^2 + 5t^2$$

where x satisfies one of the congruences: $x \equiv 0, \pm 1, \pm 2 \pmod 5$, and to compute the sums σ_2 and σ_3. We find

$$\sigma_2 - 5\sigma_3 = -4N_1{}^*(n).$$

As to the sum σ_4, it has different expressions according to the parity of n, namely for an *odd* n

$$\begin{aligned}
\sigma_4 = \tfrac{25}{2}\Psi(n/25) - \tfrac{1}{2}\Psi(n) + (5^{1/2}/2)N_1{}^*(n) - 5^{1/2}\sum (d' + d)/5; \\
n = dd', \quad d < d',
\end{aligned}$$

while for an *even* n

$$\sigma_4 = 25\Psi(n/100) + {}^{25}\!/_2\Phi(n/50) - \Psi(n/4) - \tfrac{1}{2}\Phi(n/2)$$
$$+ (5^{1/2}/2)N_1^*(n) + 2(5^{1/2}) \sum [(d' + d)/5]d; \quad n = 4dd',\ d < d'.$$

To evaluate the sum σ_5 we notice first that in the equation

$$n = x^2 + y^2 + z^2 + t^2$$

either all numbers x, y, z, t are divisible by 5, or only two of them or none. Let us denote by Q the number of solutions where

$$x^2 \equiv 1, \quad y^2 \equiv -1, \quad z \equiv 0, \quad t \equiv 0 \pmod 5,$$

and by Q_1 the number of solutions where

$$x^2 \equiv 1, \quad y^2 \equiv 1, \quad z^2 \equiv -1, \quad t^2 \equiv -1 \pmod 5.$$

The obvious relation

$$12Q + 6Q_1 = N(n = x^2 + y^2 + z^2 + t^2) - N(n/25 = x^2 + y^2 + z^2 + t^2)$$

enables us to express Q_1 in terms of Q and known numerical functions. After a simple discussion we find

$$\sigma_5 = {}^{15}\!/_8 Q + {}^{5}\!/_2\Phi(n/25) - \tfrac{1}{2}\Phi(n) \text{ for an } odd\ n,$$
$$\sigma_5 = {}^{15}\!/_8 Q + {}^{15}\!/_2\phi(n/25) - \tfrac{3}{2}\Phi(n) \text{ for an } even\ n,$$

denoting by $\Phi(n)$ the sum of *odd* divisions of n.

Introducing these various expressions into the equation (V) and separating rational and irrational parts, we get first

(18) $$\sum (h/5)\, F(n - h^2) = \sum [(d' + d)/5]d, \quad n = dd', \quad d < d',$$

for an *odd* n and

(19) $$\sum (h/5)\, F(n - h^2) = -2 \sum [(d' + d)/5]d, \quad n = 4dd', \quad d < d'$$

for an *even* n.

The equation obtained by equating rational parts, combined with Kronecker's relations

$$\sum F(n - h^2) = \tfrac{1}{2}\Phi(n) + \tfrac{1}{2}\Psi(n) \text{ for an odd } n,$$
$$\sum F(n - h^2) = \mathrm{X}(n) - \Phi(n/4) + \Psi(n/4) \text{ for an } even\ n,$$

gives in case of an *odd* n

$$\sum F(n - 25h^2) = \tfrac{1}{2}[\Phi(n) - 5\Phi(n/5)]$$
$$+ {}^{5}\!/_2\Psi(n/25) + \tfrac{1}{2}\Phi(n/25) + {}^{3}\!/_8 Q.$$

But the sum

$$\tfrac{1}{2}\Phi(n/25)+\tfrac{3}{8}Q$$

represents 1/16 of all solutions of the equation

$$n=x^2+y^2+z^2+25t^2,$$

so that, denoting this number by $R(n)$, we have

$$\sum F(n-25h^2)=\tfrac{1}{2}[\Phi(n)-5\Phi(n/5)]+\tfrac{1}{16}R(n)+\tfrac{5}{2}\Psi(n/25)$$

or, which is the same,

$$(20)\quad \sum F(n-25h^2)=\tfrac{1}{4}N\ (n)+\tfrac{1}{16}R(n)+\tfrac{5}{2}\Psi(n/25).$$

For an *even* n the expression for the corresponding sum is

$$(21)\quad \sum F(n-25h^2)=\tfrac{1}{4}N_2(n)+\tfrac{1}{16}R(n)+\tfrac{5}{2}\Phi(n/50)+5\Psi(n/100).$$

The equations (20) and (21) involve the numerical function $R(n)$ together with other simple functions depending only on divisors. It seems very difficult to reduce $R(n)$ *in case of an odd* n to some simpler functions; nevertheless the equation (20) even in this imperfect form may be of great use. For an *even* n it is possible to find a simple expression for $R(n)$. For an *even* n we have

$$\sum\cos(2\pi\xi/5)=\sum\cos(2\pi x/5)\cos(2\pi y/5)$$

the sums being extended over solutions of the equations

$$2n=\xi^2+\eta^2+\zeta^2+\theta^2,\quad n=x^2+y^2+z^2+t^2.$$

Now it is possible to express the left-hand member in terms of $Q(2n)$, $Q_1(2n)$ and the known numerical functions, and the right-hand member may be in the same way expressed by means of $Q(n)$ and $Q_1(n)$. Thus we get the relation

$$3Q(2n)-Q_1(2n)=-\tfrac{1}{3}Q(n)-\tfrac{1}{6}Q_1(n),$$

or, changing n into $2n$,

$$9Q(n)-3Q_1(n)=-Q(2n)-\tfrac{1}{2}Q_1(2n).$$

But

$$-Q(2n)-\tfrac{1}{2}Q_1(2n)=-2[\phi(n)-\phi(n/25)];$$

therefore

$$3Q(n)+Q_1(n)=-\tfrac{2}{3}[\phi(n)-\phi(n/25)],$$

and

$$2Q(n)+Q_1(n)=4[\phi(n)-\phi(n/25)];$$

whence it follows that

$$Q(n) = \tfrac{1}{4}Q_1(n) = \tfrac{2}{3}[\phi(n) - \phi(n/25)],$$

and

$$R(n) = 4[\phi(n) + 5\phi(n/25)] = 24\phi(n/5).$$

It remains to introduce this expression into (21) to get after all reductions the final expression for an *even n*:

$$(22)\quad \sum F(n - 25h^2) = \Phi(n/4) - 4\Phi(n/20) + 2\phi(n/5) + 5\Psi(n/100).$$

As soon as we have (20) and (22) it is easy to find the expressions for the sum

$$\sum F(n - h^2)$$

where h ranges separately either over quadratic residus or non-residus modulo 5. But it is hardly necessary to give final expressions. In the following reference will be made to the number of representation of an even number by the form

$$x^2 + y^2 + 5z^2 + 5t^2$$

which can be easily derived from what has been established. For, starting from the relation

$$N(5n = x^2 + y^2) + N(n/5 = x^2 + y^2) = 2N(n = x^2 + y^2)$$

we find first

$$\begin{aligned} &2N(n = x^2 + y^2 + 5z^2) \\ &\qquad = N(5n = x^2 + y^2 + 25z^2) + N(n/5 = x^2 + y^2 + z^2) \end{aligned}$$

and then

$$\begin{aligned} &2N(n = x^2 + y^2 + 5z^2 + 5t^2) \\ &\qquad N(5n + x^2 + y^2 + 25z^2 + 25t^2) + N(n/5 = x^2 + y^2 + z^2 + t^2) \end{aligned}$$

whatever n may be. Now supposing *n even* we have

$$N(5n = x^2 + y^2 + 25z^2 + 25t^2) = 2Q(5n) + 24\phi(n/5);$$

and using the preceding expression for $Q(5n)$,

$$N(n = x^2 + y^2 + 5z^2 + 5t^2) = \tfrac{2}{3}[\phi(5n) - \phi(n/5)] + 24\phi(n/5),$$

or supposing $n = 5^{\beta}m$, m non-divisible by 5,

$$N(n = x^2 + y^2 + 5z^2 + 5t^2) = 2[5^{\beta+1} - 3]\phi(m).$$

9. *Subsidiary developments.* Trying to evaluate the right-hand mem-

ber of (V) when $m \equiv \pm 1$ or $\equiv \pm 2 \pmod 5$, we meet with greater difficulties. These difficulties arise from the fact that the indeterminate equations

$$2n = x^2 + 2y^2 + 5z^2 + 10t^2,$$
$$n = x^2 + y^2 + z^2 + 5t^2,$$
$$n = x^2 + y^2 + z^2 + t^2,$$

possess different types of solutions and it does not seem easy to determine the number of solutions belonging to these various types. The equation

$$2n = x^2 + 2y^2 + 5z^2 + 10t^2,$$

supposing $n \equiv \pm 1$ or $\pm 2 \pmod 5$, may have solutions where x or y are divisible by 5. Let us denote the number of such solutions by $T(n)$, the number of the remaining solutions being denoted by $T'(n)$, so that

$$T(n) + T'(n) = N_2(n).$$

Similarly the equation

$$n = x^2 + y^2 + z^2 + 5t^2$$

may have solutions where none of the numbers x, y, z is divisible by 5, while in the remaining solutions one [in case $n \equiv \pm 2 \pmod 5$] or two of these numbers [in case $n \equiv \pm 1 \pmod 5$] are divisible by 5. Let us introduce the notation $U(n)$ for the number of solutions of the equation

$$n = x^2 + 25y^2 + 25z^2 + 5t^2$$

in case $n \equiv \pm 1 \pmod 5$, and of the equation

$$n = x^2 + y^2 + 25z^2 + 5t^2$$

in case $n \equiv \pm 2 \pmod 5$. Finally the equation

$$n = x^2 + y^2 + z^2 + t^2$$

possesses three different types of solutions, namely, solutions where none of the numbers x, y, z, t is divisible by 5, or only one of them or more than one. Corresponding to the case $n \equiv \pm 1 \pmod 5$ $Q(n)$, $Q_1(n)$, $Q_2(n)$ will denote the number of solutions satisfying the respective conditions:

$$\left.\begin{array}{llll} x^2 \equiv \mp 1, & y^2 \equiv \mp 1, & z^2 \equiv \mp 1, & t^2 \equiv \mp 1 \\ x^2 \equiv \pm 1, & y^2 \equiv \pm 1, & z^2 \equiv \mp 1, & t^2 \equiv 0 \\ x^2 \equiv \pm 1, & y^2 \equiv 0, & z^2 \equiv 0, & t^2 \equiv 0 \end{array}\right\} \pmod 5$$

the upper and lower signs corresponding to those in the congruence

$n \equiv \pm 1 \pmod 5$. In the case $n \equiv \pm 2 \pmod 5$ the same notations will be used to denote the number of solutions satisfying the conditions

$$\left.\begin{array}{llll} x^2 \equiv \pm 1, & y^2 \equiv \pm 1, & z^2 \equiv \pm 1, & t^2 \equiv \mp 1 \\ x^2 \equiv \mp 1, & y^2 \equiv \mp 1, & z^2 \equiv \mp 1, & t^2 \equiv 0 \\ x^2 \equiv \pm 1, & y^2 \equiv \pm 1, & z^2 \equiv 0, & t^2 \equiv 0 \end{array}\right\} \pmod 5.$$

For an *even* n we have the relation

$$Q(n) + 12Q_1(n) + 4Q_2(n) = 24\phi(n) \tag{23}$$

when $n \equiv \pm 1 \pmod 5$ or the relation

$$2Q(n) + 2Q_1(n) + 3Q_2(n) = 12\phi(n) \tag{24}$$

when $n \equiv \pm 2 \pmod 5$. Moreover, starting from the relation $N(n = x^2 + 5y^2 + 5z^2) = \tfrac{1}{3}N(n = x^2 + y^2 + z^2)$ when $n \equiv \pm 2 \pmod 5$ which can be easily verified, we find

$$\begin{aligned} Q_1(n) + Q_2(n) &= \tfrac{1}{2}N(n = x^2 + y^2 + 5z^2 + 5t^2) \quad \text{when } n \equiv \pm 1 \pmod 5 \\ Q(n) + Q_2(n) &= N(n = x^2 + y^2 + 5z^2 + 5t^2) \quad \text{when } n \equiv \pm 1 \pmod 5 \end{aligned}$$

so that for an *even* n

$$\begin{aligned} Q_1(n) + Q_2(n) &= 2\phi(n), \quad n \equiv \pm 1 \pmod 5 \\ Q(n) + Q_2(n) &= 4\phi(n), \quad n \equiv \pm 2 \pmod 5. \end{aligned}$$

Combined with the preceding relations these relations permit to express $Q(n)$ and $Q_1(n)$ in terms of $Q_2(n)$ when n is even, but $Q_2(n)$ remains unknown, and it seems very difficult to reduce it to some simpler numerical functions. For an *even* n we have the obvious equation

$$\sum \cos(2\pi x/5) = \sum \cos(2\pi\xi/5)\cos(2\pi\eta/5)$$

the sums being extended over solutions of the respective equations

$$2n = x^2 + y^2 + z^2 + t^2, \quad n = \xi^2 + \eta^2 + \zeta^2 + \theta^2.$$

Now it is possible to express the left-hand member of the preceding equation as a linear function of $Q(2n)$, $Q_1(2n)$, $Q_2(2n)$ while the right-hand member is a linear function of $Q(n)$, $Q_1(n)$, $Q_2(n)$. Equating the rational and irrational parts, we get two linear equations from which we can eliminate $Q(2n)$ by means of (23) or (24) and solve the resulting equations for $Q_1(2n)$, $Q_2(2n)$.

This way leads to the following results: for an *even* $n \equiv \pm 1 \pmod 5$

$$Q(2n) = 2Q_1(n) = 4\phi(n) - 2Q_2(n),$$
$$Q_1(2n) = Q_1(n) = 2\phi(n) - Q_2(n),$$
$$Q_2(2n) = 4\phi(n) - 2Q_1(n) = 2Q_2(n),$$

while for an *even* $n \equiv \pm 2 \pmod 5$

$$Q(2n) = 4Q_2(n),$$
$$Q_1(2n) = 2\phi(n) - \tfrac{1}{2}Q_2(n),$$
$$Q_2(2n) = \tfrac{1}{2}Q_2(n).$$

The equation connecting $Q_2(2n)$ and $Q_2(n)$ may be also written as follows:

$$Q_2(2n) = \tfrac{1}{4}[5 + 3(n/5)]Q_2(n) \tag{25}$$

provided n is even non-divisible by 5. In a similar way we can find the expressions for $Q_1(4n)$, $Q_2(4n)$ in terms of $Q_1(n)$, $Q_2(n)$ for an *odd* n. The final expressions are as follows: for $n \equiv \pm 1 \pmod 5$

$$Q_1(4n) = 3Q_1(n), \quad Q_2(4n) = 2\phi(n) - 3Q_1(n),$$

and for $n \equiv \pm 2 \pmod 5$

$$Q_1(4n) = 2\phi(n) - \tfrac{3}{2}Q_2(n), \quad Q_2(4n) = 3Q_2(n).$$

10. *Expression for* $T(n)$. Our next aim is to obtain an expression for $T(n)$ involving only elementary functions and $Q_2(n)$. For this purpose we make use of a certain fundamental identity which may be found in our paper "Sur les relations entre les nombres des classes des formes quadratiques binaires et positives. Premier mémoire" as well as in the paper "On Jacobi's arithmetical theorems, etc." Let $F(x, y, z)$ denote an arbitrary function defined for integral values of x, y, z and satisfying the conditions:

$$F(-x, y, z) = -F(x, y, z)\,; \quad F(x, -y, -z) = F(x, y, z)\,;$$
$$F(0, y, z) = 0.$$

For such a function we have the following important relation

$$\sum F(\Delta + \Delta', h, \Delta - \Delta') = 2\sum F(\delta - 2i, d + i, 2d + 2i + \delta) + \mathrm{T} \tag{VI}$$

where the sums are extended over solutions of the indeterminate equations

$$n = h^2 + \Delta\Delta', \quad n = i^2 + d\delta;$$

here h and i may have any integral values, while Δ, Δ', d, δ are supposed to be positive integers. As to the term T it is zero except when n is a perfect square $= s^2$ in which case

$$T = \sum_{j=1}^{2s-1} [F(2s,\ s-j,\ 2s-2j) - 2F(2s-j,\ s,\ 2s-j)].$$

By means of a simple device it is possible to obtain another relation which may be useful on many occasions. Denoting by p any odd number which is prime or composed of different primes, let us take pn instead of n in (VI) and furthermore let us suppose that $F(x, y, z) = 0$ whenever

$$2y - z \not\equiv 0 \text{ or } y \equiv 0 \pmod{p}$$

which is consistent with the conditions imposed on $F(x, y, z)$. Under these circumstances the right-hand member of (VI) becomes

$$2 \sum F(p\delta - 2pi,\ d + pi,\ 2d + 2pi - p\delta) + T'$$

where the sum is now extended over solutions of the equation

$$n = pi^2 + d\delta,$$

d being not divisible by p and $T' = 0$ unless $n = ps^2$ in which case

$$T' = \sum F(ps,\ ps - j,\ 2ps - 2j),$$

j ranging over numbers of the series $1, 2, 3, \cdots$ which are *not divisible* by p. The left-hand member sum is now extended over solutions of the equation

$$pn = h^2 + \Delta\Delta',$$

subject to the conditions

$$h \not\equiv 0, \quad 2h + \Delta' - \Delta \equiv 0 \pmod{p},$$

which are entirely equivalent to

$$h \not\equiv 0, \quad h \equiv \Delta \pmod{p}.$$

Now we can disregard the condition $h \equiv \Delta \pmod{p}$ if we introduce the function $\omega(h, \Delta)$ whose value is 0 unless $h \equiv \Delta \pmod{p}$, in which case $\omega(h, \Delta) = 1$. Such a function is given, for instance, by

$$\omega(h, \Delta) = 1/p \Big[1 + 2\cos[2\pi(h - \Delta)/p] + 2\cos[4\pi(h - \Delta)/p] + \cdots + 2\cos\{[(p-1)/2]\, 2\pi(h - \Delta)/p\}\Big].$$

We can represent now the right-hand member as follows

$$\sum \omega(h, \Delta)\, F(\Delta + \Delta',\ h,\ \Delta - \Delta')$$

where the sum is extended over solutions of the equation

$$pn = h^2 + \Delta\Delta'$$

with h *non-divisible by* p and the function $F(x, y, z)$ being of the same general character as in (VI). This way we finally arrive at the important relation

$$\text{(VII)}\quad \sum \omega(h, \Delta)\, \mathrm{F}(\Delta + \Delta', h, \Delta - \Delta') \\ = 2 \sum F(p\delta - 2pi,\ d + pi,\ 2d + 2pi - p\delta + T'$$

the extent of summations being explained above. If we suppose that instead of the single condition

$$F(x, -y, z) = F(x, y, z)$$

we have two conditions

$$F(x, -y, z) = F(x, y, -z) = F(x, y, z)\ ,$$

we can take simply

$$\omega(h, \Delta) = 1/p\left[1 + 2\cos\,(2\pi h/p)\cos\,(2\pi\Delta/p) + 2\cos\,(4\pi h/p)\cos 4\pi\Delta/p) \\ + \cdots + 2\cos\{[(p-1)/2]\,2\pi h/p\}\cos\{[(p-1)/2]\,2\pi\Delta/p\}\right].$$

Taking $p = 3$ we have

$$\omega(h, \Delta) = \tfrac{1}{2}$$

so that for any function $F(x, y, z)$ odd with respect to x and even with respect to y and z we have

$$\text{(VIII)}\quad \tfrac{1}{2} \sum F(\Delta + \Delta',\ h,\ \Delta - \Delta') \\ = 2 \sum F(3\delta - 6i,\ d + 3i,\ 2d + 6i - 3\delta) + T'$$

sums being extended over solutions of the equations with h and d not divisible by 3. For $p = 5$ we have

$$\omega(h, \Delta) = \tfrac{1}{4}\ [1 + (h\Delta/5)]$$

and the following relation holds true for every function $F(x, y, z)$ odd with respect to x and even with respect to y and z:

$$\text{(IX)}\quad \tfrac{1}{4} \sum [1 + (h\Delta/5)]\, F(\Delta + \Delta',\ h,\ \Delta - \Delta') \\ = 2 \sum F(5\delta - 10i,\ d + 5i,\ 2d + 10\mathrm{i} - 5\delta) + T'$$

sums being extended over solutions of the equations

$$5n = h^2 + \Delta\Delta', \quad n = 5i^2 + d\delta$$

with h and d not divisible by 5. The equation (IX) is of fundamental importance for what follows. We assume in it

$$F(x,\ y,\ z) = (y/5)\sin\,(\pi x/4)\cos\,(\pi z/4)$$

and suppose $n \equiv \pm 2 \pmod 5$. The result of this substitution is

$$(26)\quad \tfrac{1}{4}\sum (h/5)\sin(\pi\Delta/2) + \tfrac{1}{4}\sum (\Delta/5)\sin(\pi\Delta/2)$$
$$= \sum (d/5)\sin(\pi d/2) - 2\sum (-1)^{i}\cos(\pi d/2)(\delta/5)\sin(\pi\delta/2).$$

Now the sum

$$\sum \sin(\pi d/2)$$

extended over divisors of any number m is equal to one-fourth of the number of representations of m by the form $x^2 + y^2$. Similarly, the sum

$$\sum (d/5)\sin(\pi d/2)$$

extended over divisors of m is equal to one-half of the number of representations of m by two forms: $x^2 + 5y^2$ and $2x^2 - 2xy + 3y^2$ the first being unable to represent numbers $\equiv \pm 2$ and the second those $\equiv \pm 1 \pmod 5$. The first sum in the left-hand member of (26) may be, therefore, represented as the sum

$$\tfrac{1}{16}\sum (h/5)$$

extended over solutions of the equation

$$5n = h^2 + k^2 + l^2$$

with h not divisible by 5. But it is obvious that such a sum has zero for its value. The second sum in the left-hand member represents $\tfrac{1}{8}$ of the number of solutions of the equation

$$5n = h^2 + k^2 + 5l^2$$

with h not divisible by 5 and this is equal to

$$\tfrac{1}{4}N(n = x^2 + y^2 + z^2).$$

The first sum in the right-hand member of (26) is

$$\tfrac{1}{2}N(n = 5i^2 + 2j^2 + 2jk + 3k^2),$$

while the second sum has different expressions for an *odd* or *even* n, namely

$$-(-1)^{n-1/2}N(n = 5i^2 + 2j^2 - 2jk + 3k^2,\ i \text{ odd}),\ n \textit{ odd}$$
$$(-1)^{n/2}N(n = 5i^2 + 2j^2 - 2jk + 3k^2,\ i \text{ even}),\ n \textit{ even}$$

or, which is the same,

$$-(-1)^{n-1/2}N(n = 5i^2 + 2j^2 + 10k^2) \text{ for an } \textit{odd } n,$$
$$(-1)^{n/2}N(n = 5i^2 + 2j^2 + 10k^2) \text{ for an } \textit{even } n.$$

After substituting these various expressions into (26) we finally get

$$(27)\quad N(2n = x^2 + 5y^2 + 10z^2) + 2(-1)^{n-1/2} N(n = 2x^2 + 5y^2 + 10z^2) = \tfrac{1}{2}N(n = x^2 + y^2 + z^2)$$

$$(28)\quad N(2n = x^2 + 5y^2 + 10z^2) - 2(-1)^{n/2} N(n = 2x^2 + 5y^2 + 10z^2) = \tfrac{1}{2}N(n = x^2 + y^2 + z^2)$$

for an *odd* and *even* $n \equiv \pm 2 \pmod 5$ respectively. Let now n denote any number not divisible by 5. We substitute $2n - t^2$ instead of n in (27) and (28), choosing $t \equiv 0 \pmod 5$ when $n \equiv \pm 1$, $t \equiv \pm 1$ when $n \equiv 2$ and $t \equiv \pm 2$ when $n \equiv -2 \pmod 5$. As a result we obtain a single equation independently of whether t is odd or even:

$$N(4n - 2t^2 = x^2 + 5y^2 + 10z^2) - 2(-1)^n N(2n - t^2 = 2x^2 + 5y^2 + 10z^2) = \tfrac{1}{2} N(2n - t^2 = x^2 + y + z^2).$$

We let t now assume all the values, subject to the preceding limitations, which make $2n - t^2$ positive, and add all the resulting equations. What we get this way, attention paid to the definition of $T(n)$, $T'(n)$, $Q(n)$, $Q_1(n)$, $Q_2(n)$ given in § 9, may be presented as follows:

$$T(2n) - (-1)^n 2T(n) = \tfrac{1}{2}(3Q_2(2n) + Q_1(2n)) \text{ in case } n \equiv \pm 1 \pmod 5,$$

$$T'(2n) - (-1)^n 2T'(n) = \tfrac{1}{2}(Q(2n) + 3Q_1(2n)) \text{ in case } n \equiv \pm 2 \pmod 5.$$

Now we can express $T'(2n)$, $T'(n)$ by $T(2n)$, $T(n)$; and $Q(2n)$, $Q_1(2n)$ by $Q_2(2n)$. All simplifications performed, we obtain:

$$\begin{aligned} T(2n) - 2T(n) &= \phi(n) + \tfrac{5}{2}Q_2(n), & n &\equiv \pm 1 \pmod 5, & &n \text{ even} \\ T(2n) + 2T(n) &= \phi(n) + \tfrac{5}{4}Q_2(2n), & n &\equiv \pm 1 \pmod 5, & &n \text{ odd}, \\ T(2n) - 2T(n) &= 3\phi(n) - \tfrac{5}{4}Q_2(n), & n &\equiv \pm 2 \pmod 5, & &n \text{ even}, \\ T(2n) + 2T(n) &= 3\phi(n) - \tfrac{5}{2}Q_2(2n), & n &\equiv \pm 2 \pmod 5, & &n \text{ odd}, \end{aligned}$$

which also may be written as follows:

$$(29)\quad T(2n) - 2T(n) = [2 - (n/5)]\phi(n) + \tfrac{5}{8}[1 + 3(n/5)]Q_2(n), \quad n \text{ even};$$

$$(30)\quad T(2n) + 2T(n) = [2 - (n/5)]\phi(n) - \tfrac{5}{8}[1 - 3(n/5)]Q_2(2n), \quad n \text{ odd}.$$

To these equations we can add (25):

$$Q_2(2n) = \tfrac{1}{4}[5 + 3(n/5)]Q_2(n), \quad n \text{ even}.$$

Now it is easy to see that for an odd n

$$T(2n) = T(n),$$

and therefore we can find from (30) the expression for $T(2n)$, n being odd. Using then (29) and (25) we can find the expression for $T(n)$, n being any *even* number $= 2^k m$, m odd. The final results are as follows:

$$(31) \quad T(n) = [2^k - 2 + (2^k + 1/3)]\phi(n) - \tfrac{5}{6}Q_2(n), \quad n \equiv \pm 1 \pmod 5,$$

$$(32) \quad T(n) = [2^k - 2 + 2^k - 1/3)]\phi(n) + \tfrac{5}{12}Q_2(n), \quad n \equiv \pm 2 \pmod 5.$$

As to *odd* values of n, we have

$$(33) \qquad T(n) = 2\phi(n) - \tfrac{5}{2}Q_1(n), \quad n \equiv \pm 1 \pmod 5,$$
$$n \equiv \pm 1 \pmod 5,$$

$$(34) \qquad T(n) = \phi(n) - \tfrac{5}{4}Q_2(n), \quad n \equiv \pm 2 \pmod 5.$$

11. *Expression for $U(n)$.* Let $f(x)$ denote an odd function and n a number non-divisible by 5. Taking in (VI)

$$F(x, y, z) = 0 \text{ when } x \equiv 0 \pmod 2 \text{ or } y \not\equiv 0 \pmod 5,$$
$$F(x, y, z) = f(x) \text{ otherwise},$$

we obtain

$$\sum f(\Delta + \Delta') = \sum f(\delta - 2i)$$

where the first sum is extended over solutions of the equation

$$n = 25h^2 + \Delta\Delta'$$

with *odd* Δ and *even* Δ' while the second over solutions of

$$n = i^2 + d\delta$$

with *odd* δ and $i + d$ divisible by 5. In the same way as before, denoting by $\omega(d, i)$ the sum

$$\omega(d, i) = \tfrac{1}{5}\Big[1 + 2\cos\,[2\pi(d + i)/5 + 2\cos[4\pi(d + i)/5]\Big],$$

we can write the preceding equation as follows

$$\sum f(\Delta + \Delta') = \sum \omega(d, i)\, f(\delta - 2i)$$

the right-hand member sum this time being extended over solutions of

$$n = i^2 + d\delta$$

with δ *odd* and i and d non-divisible by 5. Now we take

$$f(x) = (-1)^{(x-1)/2}$$

and replace n by $4n$; after evident simplifications we get

$$(35)\qquad \sum (-1)^{(\Delta-1)/2} = \tfrac{1}{4} \sum \left[1 + (id/5)\right] (-1)^{(\delta-1)/2+i}$$

the sums being extended over solutions of the equations

$$\begin{aligned} n &= 25h^2 + \Delta\Delta', \ \Delta \text{ odd}, \\ 4n &= i^2 + d\delta, \ \delta \text{ odd}, \ i \text{ and } d \not\equiv 0 \pmod 5. \end{aligned}$$

Now for any number m

$$\begin{aligned} &\sum (-1)^{(\delta-1)/2} = \tfrac{1}{4} N(m = x^2 + y^2) \\ &\sum (-1)^{(\delta-1)/2} (d/5) = \tfrac{1}{2} \left[N(m = x^2 + 5y^2) - N(2m = x^2 + 5y^2)\right] \end{aligned}$$

both sums being extended over all the representations of m in the form

$$m = d\delta, \quad \delta \text{ odd}$$

whence it is easy to see that the relation (35) implies the following equations:

$$\begin{aligned} &5N(n = x^2 + y^2 + 25z^2) - N(n = x^2 + y^2 + z^2) \\ &\qquad = \mp 2N(n = x^2 + y^2 + 5z^2) - 2C, \quad n \equiv \pm 2 \pmod 5, \\ &5N(n = x^2 + 25y^2 + 25z^2) - \tfrac{1}{3} N(n = x^2 + y^2 + z^2) \\ &\qquad = \mp \tfrac{1}{2} N(n = x^2 + y^2 + 5z^2) - C, \quad n \equiv \pm 1 \pmod 5, \end{aligned}$$

where C stands for the sum

$$\sum (-1)^i (i/5)$$

extended over all solutions of the equation

$$8n = 2i^2 + j^2 + 5k^2 \ .$$

Let us introduce now the sums

$$A = \sum_{(a)} (-1)^i, \qquad B = \sum_{(b)} (-1)^i$$

extended over solutions of the same equation subject to the following limitations:

$$\begin{aligned} &(a) \quad i^2 \equiv \pm 1 \text{ in case } n \equiv \pm 1; \qquad i \equiv 0 \text{ in case } n \equiv \pm 2 \pmod 5, \\ &(b) \quad i^2 \equiv \mp 1 \text{ in case } n \equiv \pm 1; \qquad i^2 \equiv \pm 1 \text{ in case } n \equiv \pm 2 \pmod 5. \end{aligned}$$

It is obvious that

$$\begin{aligned} \pm C &= A - B \text{ in case } n \equiv \pm 1 \pmod 5, \\ \pm C &= \phantom{A -{}} B \text{ in case } n \equiv \pm 2 \pmod 5. \end{aligned}$$

There exists a relation between A and B which can be easily derived from (IX). To obtain this relation we suppose for a moment that n is any number, then take $4n$ instead of n in (IX) and define the function $F(x, y, z)$ as follows:

$$F(x, y, z) = 0 \text{ for even } x,$$
$$F(x, y, z) = \sin \pi x/4 \text{ for odd } x.$$

A simple discussion of the relations thus obtained leads to the following result:

$$[(-1)^n/4] \sum_{(a)} (-1)^h - [(-1)^n/4] \sum_{(b)} (-1)^h (h/5)$$
$$= - \sum_{(c)} (-1)^z + \sum_{(d)} (-1)^z,$$

the extent of various summations being as follows:

(a) $5n = h^2 + k^2 + 2l^2;$ $h \not\equiv 0 \pmod 5$
(b) $5n = h^2 + k^2 + 10l^2;$ $h \not\equiv 0 \pmod 5$
(c) $n = x^2 + 2y^2 + 5z^2$
(d) $n/5 = x^2 + 2y^2 + z^2.$

Without discussing all possible cases we confine ourselves to considering only two of them. First suppose n *odd* and non-divisible by 5. In this case we have obviously

$$\sum_{(a)} (-1)^h = 0, \qquad \sum_{(b)} (-1)^z = 0$$

and the resulting equation

$$\sum_{(c)} (-1)^z = -\tfrac{1}{4} \sum_{(d)} (-1)^h (h/5)$$

being true for all odd n non-divisible by 5 leads to the remarkable conclusion:

$$\sum (-1)^z = -\tfrac{1}{4} \sum (-1)^h (h/5)$$

sums now being extended over solutions of the equations

$$n = x^2 + 5z^2, \quad 5n = h^2 + k^2$$

respectively. This theorem belongs to the same type as those well known theorems given by Jacobi.

Next we replace n by $8n$. This time we have

$$\sum_{(b)} (-1)^h (h/5) = 0$$

and the final result may be expressed thus:

$$(36) \quad \sum (-1)^z = \tfrac{5}{4} N(n/5 = x^2 + y^2 + z^2) - \tfrac{1}{4} N(5n = x^2 + y^2 + z^2)$$

or, which is the same,

$$\sum (-1)^y = -\tfrac{3}{2} N(n = x^2 + y^2 + 5z^2) + 2N(n/5 = x^2 + y^2 + z^2)$$

the sum being extended over all solutions of the equation

$$8n = x^2 + 2y^2 + 5z^2.$$

Supposing now again n non-divisible by 5 we see that the last equation is equivalent to

$$A + B = -\tfrac{3}{2}N(n = x^2 + y^2 + 5z^2).$$

Thus we can express B by A and finally arrive at the following relations interesting for many reasons:

$$(37)\quad \pm\,[5N(n = x^2 + y^2 + 25z^2) - N(n = x^2 + y^2 + z^2)]$$
$$= N(n = x^2 + y^2 + 5z^2) + 2A, \quad n \equiv \pm 2 \pmod 5,$$

$$(38)\quad \pm\,[5N(n = x^2 + 25y^2 + 25z^2) - \tfrac{1}{3}N(n = x^2 + y^2 + z^2)]$$
$$= -2N(n = x^2 + y^2 + 5z^2) - 2A, \quad n \equiv \pm 1 \pmod 5.$$

It follows from these relations:

$$(39)\quad \pm\,[5U(n) - N_1(n)] = n = x^2 + y^2 + 5z^2 + 5t^2) + 2W,$$
$$n \equiv \pm 2 \pmod 5$$

$$(40)\quad \pm\,[5U(n) - \tfrac{1}{3}N_1(n)] = -2N(n = x^2 + y^2 + 5z^2 + 5t^2) - 2W,$$
$$n \equiv \pm 1 \pmod 5$$

where W stands for the sum

$$W = \Sigma\,(-1)^i$$

extended over solutions of the equation

$$8n = 2i^2 + j^2 + 5k^2 + 40l^2,$$

subject to the following limitations:

$$i^2 \equiv \pm 1 \text{ when } n \equiv \pm 1 \pmod 5,$$
$$i^2 \equiv 0 \quad \text{when } n \equiv \pm 2 \pmod 5.$$

All which remains now is to find the appropriate expression for W. To this end we resort again to the equations (27), (28) in which we replace n by $4n - t^2$, multiply both parts by $(-1)^t$ and let t assume all integral values making $4n - t^2$ positive and satisfying the conditions

$$t \equiv 0 \quad \text{when } n \equiv \pm 2 \pmod 5,$$
$$t^2 \equiv \pm 1 \text{ when } n \equiv \pm 1 \pmod 5.$$

Adding all the equations thus obtained and performing obvious transformations, we finally find

$$W = 2T(n) - \tfrac{1}{2}T(4n) + \tfrac{1}{4}\sum(-1)^t, \quad n \equiv \pm 2 \pmod 5,$$
$$W = 2T'(n) - \tfrac{1}{2}T'(4n) + \tfrac{1}{4}\sum(-1)^t, \quad n \equiv \pm 1 \pmod 5,$$

the sum being extended over solutions of the equation

$$4n = x^2 + y^2 + z^2 + t^2,$$

where

$$t \equiv 0 \quad \text{when } n \equiv \pm 2 \pmod 5,$$
$$t^2 \equiv \pm 1 \text{ when } n \equiv \pm 1 \pmod 5.$$

The latter sum can be expressed in terms of $Q_1(n)$, $Q_2(n)$, the expression for $T(n)$ is given by (31), (32), (33), (34) and that for $T'(n)$ can easily be derived. So we can find the expression for W in terms of $Q_1(n)$, $Q_2(n)$ and elementary numerical functions; substituting it into (39) and (40) we find the following expressions for $U(n)$:

(41) $\pm 5U(n) = -2\phi(n) + \tfrac{5}{2}Q_2(n) \pm N_1(n), \quad n \equiv \pm 2 \pmod 5,$ n *eve*

(42) $\pm 5U(n) = 2\phi(n) - \tfrac{15}{2}Q_2(n) \pm N_1(n), \quad n \equiv \pm 2 \pmod 5,$ n *odd*

(43) $\pm 5U(n) = 2\phi(n) - 5Q_2(n) \pm \tfrac{1}{3}N_1(n), \quad n \equiv \pm 1 \pmod 5,$ n *ev*

(44) $\pm 5U(n) = 8\phi(n) - 15Q_1(n) \pm \tfrac{1}{3}N_1(n), \quad n \equiv \pm 1 \pmod 5,$ n *odd*

12. *Discussion of the case* $n \equiv \pm 2 \pmod 5$. The really difficult part of all this investigation was in finding the expressions for $T(n)$ and $U(n)$. Now that this has been done the rest does not present difficulties any more. Returning to the equation (V) and considering first the case $n \pm \equiv 2 \pmod 5$ we obtain two relations by equating rational and irrational parts. The equation obtained by equating rational parts combined with Kronecker's expression for the sum

$$\sum F(n - h^2)$$

gives in all cases when $n \equiv \pm 2 \pmod 5$:

$$\sum F(n - 25h^2) = \tfrac{1}{2}\omega(n) + \tfrac{1}{16}(Q_1(n) + 3Q_2(n)) - \tfrac{1}{16}N_2(n) \pm \tfrac{1}{32}N_1(n)$$

where

$$\omega(n) = \tfrac{5}{16}(2T(n) \mp U(n)).$$

The equation resulting by equating irrational parts is

$$\pm \sum (h/5)\, F(n - h^2) = \tfrac{1}{2}\omega(n) \pm \tfrac{1}{32}N_1(n) - \tfrac{5}{16}N_2(n) + \tfrac{1}{16}(2Q(n) + 3Q_2(n) - 3Q_1(n)) + \tfrac{1}{2}\phi(n/2) + \Psi(n/4)$$

for an *even* n and

$$\pm \sum (h/5)\, F(n-h^2) = \tfrac{1}{2}\omega(n) \pm \tfrac{1}{32}N_1(n) - \tfrac{5}{16}N_2(n) + \tfrac{1}{16}(2Q(n) + 3Q_2(n) - 3Q_1(n)) + \Psi(n)$$

for an *odd* n. Now we can eliminate $Q(n)$ in any case, express $Q_1(n)$ by $Q_2(n)$ in case of an *even* n and substitute for $T(n)$, $U(n)$, $N_2(n)$ their expressions. This way we finally get

$$\sum F(n-25h^2) = \tfrac{1}{3}\Phi(n/4) + \tfrac{1}{6}\phi(n) + \tfrac{5}{24}Q_2(n)$$

when n is even and

$$\sum F(n-25h) = \tfrac{1}{8}\Phi(n) + [2Q_1(n) + Q_2(n)]/32$$

when n is odd. Similarly,

$$\pm \sum (h/5)\, F(n-h^2) = \Psi(n/4) - \tfrac{2}{3}\Phi(n/4) + \tfrac{1}{6}\phi(n) + \tfrac{5}{24}Q_2(n)$$

when n is even and

$$\pm \sum (h/5)\, F(n-h^2) = \tfrac{1}{2}\Psi(n) - \tfrac{1}{8}\Phi(n) - \tfrac{5}{32}(2Q_1(n) + Q_2(n))$$

when n is odd.

To obtain Gierster's relations involving only elementary functions depending on divisors, we notice that for an *even* n

$$\sum [2F(n-25h^2) - G(n-25h^2)] = \tfrac{1}{6}\phi(n) + \tfrac{5}{24}Q_2(n),$$

$$\pm \sum (h/5)\, [2F(n-h^2) - G(n-h^2)] = \tfrac{1}{6}\phi(n) + \tfrac{5}{24}Q_2(n),$$

so that, putting

$$H(n) = G(n) - F(n),$$

we have

$$\pm \sum (h/5)\, H(n-h^2) = \Psi(n/4) - \tfrac{2}{3}\Phi(n/4)$$

$$\sum H(n-25h^2) = \tfrac{1}{3}\Phi(n/4)$$

whence, changing n into $4n$ we finally get

$$\sum H(4n-25h^2) = \tfrac{1}{3}\Phi(n)$$

and

$$\sum_{h \equiv \pm 1 \pmod 5} H(4n-h^2) = \tfrac{2}{3}\Phi(n)$$

$$\sum_{h \equiv \pm 2 \pmod 5} H(4n-h^2) = \Psi(n)$$

for $n\equiv 2 \pmod 5$, while

$$\sum_{h\equiv\pm 1 \pmod 5} H(4n-h^2)=\Psi(n)$$

$$\sum_{h\equiv\pm 2 \pmod 5} H(4n-h^2)=\tfrac{2}{3}\Phi(n)$$

for $n\equiv -2 \pmod 5$.

13. *Discussion of the case* $n\equiv \pm 1 \pmod 5$. The equation obtained by equating rational parts in (V) combined with Kronecker's expression for the sum

$$\sum F(n-h^2)$$

yields in this case two relations, which, introducing two functions $\Psi_{+1}(n)$ and $\Psi_{-1}(n)$ defined by

$$\Psi_{+1}(n)=\sum\,[d'+(d/5)d], \quad n=dd',\ d<d'$$

$$\Psi_{-1}(n)=\sum\,[d'-(d/5)d], \quad n=dd',\ d<d',$$

may be presented as follows

$$\sum F(n-25h^2)=\tfrac{1}{4}\mathrm{X}(n)+\Psi(n/4)-\Psi_{\pm 1}\ (n/4)$$

for an *even* $n\equiv\pm 1 \pmod 5$, and

$$\sum F(n-25h^2)=\tfrac{1}{8}\Phi(n)+\tfrac{1}{2}\Psi(n)-\tfrac{1}{2}\Psi_{\mp 1}\ (n)+\tfrac{3}{16}(Q_1(n)+Q_2(n))$$

for an *odd* $n\equiv\pm 1 \pmod 5$. Similarly, by equating irrational parts we get

$$\pm\sum\,(h/5)\,F(n-h^2)=\omega(n)+\Psi(n/4)-\Psi_{\mp 1}\ (n/4)$$
$$+\tfrac{5}{16}(3Q_1(n)+Q_2(n)-\tfrac{3}{16}N_2(n)-\tfrac{3}{2}\phi(n)\pm\tfrac{1}{48}N_1(n)$$

for an *even n,* while

$$\pm\sum\,(h/5)\,F(n-h^2)=\omega(n)-\Phi(n)+\tfrac{1}{2}\Psi_{\mp 1}\ (n)$$
$$+\tfrac{5}{16}(3Q_1(n)+Q_2(n)-\tfrac{3}{16}N_2(n)\pm\tfrac{1}{48}N_1(n),$$

for an *odd n.* In both cases $\omega(n)$ stands for

$$\omega(n)=\tfrac{5}{16}(2T(n)\mp U(n)).$$

Now it remains to substitute for $T(n)$ and $U(n)$ their expressions and to eliminate $Q_1(n)$ in case of an *even n* to get finally

$$\pm\sum\,(h/5)\,F(n-h^2)=-\tfrac{5}{6}\Phi(n/4)-\Psi_{\mp 1}\ (n/4)+\tfrac{1}{2}\phi(n)-\tfrac{5}{6}Q_2(n)$$

for an *even* n and

$$\pm \sum (h/5)\, F(n - h^2) = -\tfrac{5}{8}\Phi(n) + \tfrac{1}{2}\Psi_{\mp 1}(n) + \tfrac{5}{16}(Q_1(n) + Q_2(n))$$

for an *odd* n. Now taking $4n$ instead of n in the first of these relations and introducing

$$H(n) = G(n) - F(n)$$

we find easily

$$\pm 3 \sum (h/5)\, H(4n - h^2) = \tfrac{5}{2}\Phi(n) - 3\Psi_{\mp 1}(n).$$

There is no difficulty any more to establish the following system of Gierster's relations corresponding to $n \equiv \pm 1 \pmod 5$:

$$2 \sum H(4n - 25h^2) = \Phi(n) + 2\Psi(n) - 2\Psi_{\mp 1}(n)$$

$$6 \sum_{h \equiv \pm n^{1/2} \pmod 5} H(4n - h^2) = 4\Phi(n)$$

$$6 \sum_{h \equiv \pm 2n^{1/2} \pmod 5} H(4n - h^2) = -\Phi(n) + 6\Psi_{\mp 1}(n).$$

AUGUST 4, 1927.

Note on a Theorem of Bôcher.[*]

BY GRIFFITH C. EVANS.

We prove the following theorem:

THEOREM 1. *If* $u(M)$ *is a potential function for its gradient vector* $\nabla u = (D_1 u, D_2 u)$, *and if the equation*

$$\int \nabla_n u \, ds = 0 \tag{1}$$

is satisfied on almost all segments[†] *in an open connected plane region* Σ, *then the function* $u(M)$ *has merely unnecessary discontinuities in* σ, *and when these are removed by changing the value of* $u(M)$ *at most in the points of a set of superficial measure* 0, *the resulting function has continuous derivatives of all orders and satisfies Laplace's equation*

$$\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2 = 0, \tag{2}$$

at every point in Σ.

The theorem is required for an adequate treatment of Poisson's and related equations. It derives originally from the well known theorem of Bôcher's to the effect that if $u(M)$ and its partial derivatives $\partial u/\partial x$ and $\partial u/\partial y$ are continuous in Σ and if the equation

$$\int (\partial u/\partial n) \, ds = 0 \tag{3}$$

is satisfied for all circles, then $u(M)$ is harmonic in Σ. Less remotely it derives from a theorem of the author,[‡] based on the ideas of generalized derivatives and potential, and thus more general than Bôcher's theorem, but proved in substantially the same manner after these ideas have been developed.

We generalize slightly the idea of potential function [§] in the following fashion.

---

[*] Presented to the American Mathematical Society, Sept. (1927).

[†] That is, all rectangles in Σ formed from lines $x = a$, $y = b$ except possibly those which correspond to values of a and b constituting sets of zero measure.

[‡] "Fundamental Points of Potential Theory," *Rice Institute Pamphlet*, Vol. 7 (1920), pp. 252-329; see also the author's recent book *Logarithmic Potential—Discontinuous Dirichlet and Neumann Problems*, New York (1927), Chapter VII.

[§] Evans, *Rice Institute Pamphlet, loc. cit.*, p. 274. The curve w is of class Γ if it possesses a constant Γ such that

$$\int_w \frac{|\cos n, \overline{MP}|}{\overline{MP}} \, ds_P < \Gamma$$

independently of the position of M in Σ. For Theorem 1 only rectangles and circles are needed.

Let $u(M)$, D_1u, D_2u be three functions summable over every closed two-dimensional region in Σ, E some fixed set in Σ of superficial measure 0, and w an arbitrary regular closed curve of class Γ which contains of E no subset which has positive linear measure on w. Then if

$$(4)\qquad \begin{aligned} \int_w u\,dy &= \int_\sigma D_1u\,d\sigma \\ \int_w u\,dx &= -\int_\sigma D_2u\,d\sigma, \end{aligned}$$

σ being the region bounded by w, we say that $u(M)$ is the potential function of its generalized or vector derivative or gradient vector $\nabla u = (D_1u, D_2u)$. Any vector whose components are identical with these except on a set of superficial measure 0 is equally a gradient vector for $u(M)$.

It follows immediately that

$$\int_w u\,d\alpha' = \int_\sigma D_\alpha u\,d\sigma,$$

where

$$D_\alpha u = D_1u\cos x\alpha + D_2u\cos y\alpha$$

and α' is the direction $\pi/2$ in advance of the arbitrary direction α, and also [*]

$$(5)\qquad \int_w u\,d\theta = \int_\sigma (1/r)\,D_r u\,d\sigma$$

if w does not contain the pole of the system of coordinates (r, θ).

The proof of the author's theorem, already cited, applies to functions which are the potential functions of their vector derivatives in the more general sense given above, and even when (1) holds merely for almost all circles which have a common center M_0, for every M_0 in Σ. But in the theorem now to be proved the hypothesis does not require the vanishing of this total flux on a continuous or "almost continuous" family of curves about a point which are level loci of a harmonic function with an isolated singularity at the point. Hence Bôcher's device is not sufficient for the proof and an indirect method must be used.

Let then $u_\mu(M)$ be the μ-approximation [†]

$$u_\mu(M) = (1/4\mu^2)\int_{-\mu}^{\mu}\int_{-\mu}^{\mu} u(x+\xi, y+\eta)\,d\xi\,d\eta.$$

---

[*] Evans, *loc. cit.*, p. 282 and p. 287. For formula (23) p. 282 and that on the bottom of page 279 there should be required an additional hypothesis, as the author has remarked elsewhere. This does not affect the validity of the Theorem of Art. 5. 33, requiring merely an obvious modification of the first part of the proof.

[†] H. E. Bray, "Proof of a Formula for an Area," *Bulletin of the American Mathematical Society*, Vol. 29 (1923), pp. 264-270.

We know that $u_\mu(M)$ is bounded and continuous in any given closed region in Σ, μ small enough; similarly for the quantities

$$\partial u_\mu/\partial x = (1/4\mu^2) \int_{-\mu}^{\mu} \int_{-\mu}^{\mu} D_1 u(x+\xi, y+\eta)\, d\xi\, d\eta,$$

$$\partial u_\mu/\partial y = (1/4\mu^2) \int_{-\mu}^{\mu} \int_{-\mu}^{\mu} D_2 u(x+\xi, y+\eta)\, d\xi\, d\eta,$$

since $u(M)$ is a potential function of its vector derivative. In fact these statements result immediately from the properties (3), (6), (7) of Bray's paper. Incidentally, of course, u_μ is a potential function of its vector derivative.

Consider now an arbitrary circumference C in Σ, and take μ less than the distance from C to the frontier of Σ. Let s be a finite plurisegment interior to C, no point of which is distant from C by more than some δ which will eventually be arbitrarily small. We have

$$I_s = \int_s (\partial u_\mu/\partial n)\, ds = \int_s -(\partial u/\partial x)\, dy + (\partial u/\partial y)\, dx$$

$$= (1/4\mu^2) \int_{-\mu}^{\mu} d\xi \int_{-\mu}^{\mu} up \int_s (-D_1 u(x+\xi, y+\eta)\, dy + D_2 u(x+\xi, y+\eta)\, dx),$$

by means of Bray's theorem (6). Hence

$$I_s = (1/4\mu^2) \int_{-\mu}^{\mu} d\xi \int_{-\mu}^{\mu} d\eta \int_{s_{\xi,\eta}} (-D_1 u(\phi, \psi)\, d\psi + D_2 u(\phi, \psi)\, d\phi),$$

where $s_{\xi,\eta}$ is the locus obtained by displacing s in the amount (ξ, η). But the inside integral vanishes for almost all (ξ, η), and, therefore,

$$I_s = 0.$$

If we compare I_s with the integral

$$I = \int_C (\partial u_\mu/\partial n)\, ds,$$

we see that the latter vanishes also. In fact I and I_s are merely Riemann integrals of continuous functions, and I_s may be made to differ by as little from I as desired. Hence by Bôcher's theorem, the function $u_\mu(M)$ is harmonic in Σ.

We return now to the function $u(M)$, by letting μ approach zero. Let σ_R be the region contained between two arbitrary concentric circles in Σ of radii r_0 and R, respectively, $r_0 < R$. We have

$$\lim \int_{\sigma_R} (1/r)(\partial u_\mu/\partial r)\, d\sigma = \int_{\sigma_R} (1/r) D_r u\, d\sigma$$

by Bray's theorem (8). But the integral of the left hand member vanishes, as is seen when it is expressed as an iterated integral, and therefore

$$0 = \int_{\sigma_R} (1/r) D_r u \, d\sigma = \int_{r_0}^{R} (dr/r) \int_{C_r} D_r u \, ds,$$

where C_r is the concentric circle of radius r. Hence

$$\int_{C_r} D_r u \, ds = 0, \text{ for almost all } r. \tag{6}$$

This result is however sufficient, as noted above, since $u(M)$ is the potential of its vector derivative, to imply that $u(M)$ has merely removable discontinuities at a set of points of zero measure, and, when these are removed, becomes harmonic.

In a similar fashion we may prove a more general theorem, useful also in the treatment of equations of elliptic type. We shall say that a *translatable* regular net consists of the curves in Σ obtained by arbitrary parallel displacements of a regular net[*]; and an equation in functions of curves will be said to hold almost everywhere on such a net if, given a cell w, it holds for almost all the homologous cells obtained by displacing it parallel to itself in the x and y directions.[†] We may state then the theorem:

THEOREM II. *Let $u(M)$ be a potential function for its vector derivative $(D_1u, D_2u) = \nabla u$ in Σ, and let the equation*

$$\int_w \nabla_n u \, ds = 0$$

be satisfied almost everywhere on a translatable regular net of class Γ in Σ. Then the conclusion of Theorem I is valid.

It is hardly necessary to say that for theorems of this sort the case $n = 2$ is typical of n dimensions.[‡] Some five years ago Professor Bray and the author did however work over the corresponding part of the article on "Fundamental Points of Potential Theory" for the case $n = 3$, without finding any essential novelty; the set corresponding to E' of Art. 5. 32 of that paper is perhaps a little more difficult to define. Lately, among other theorems, Professor Raynor has given Bôcher's theorem for three dimensions.[§]

---

[*] H. E. Bray and G. C. Evans, "A Class of Functions Harmonic in a Sphere," *American Journal of Mathematics*, Vol. 49 (1927), pp. 153-180; see p. 169. Here the regular net may be regarded as given over the entire plane, and the number of cells in any lattice containing points of a bounded region will be finite.

[†] Or, instead of the group of translations T we may for instance use throughout the group of transformations $S^{-1}TS$ where S is a fixed conformal transformation—this of course only in the plane.

[‡] See the *Cambridge Colloquium Lectures*, Part I, New York (1918), p. 77, and G. Bouligand, "Fonctions harmoniques. Principes de Picard et de Dirichlet," *Mémorial des sciences mathématiques*, fasc. XI, Paris (1926), pp. 7-9.

[§] *Bulletin of the American Mathematical Society*, Vol. 52 (1926), pp. 654-658.

Generalized Neumann Problems for the Sphere.*

By Griffith C. Evans.

Introduction. Let $v(M)$ be harmonic within a sphere S of unit radius, and let w_r be a simple closed regular curve, with interior region ω_r, on the concentric sphere S_r of radius $r < 1$, whose projection on S is w. We consider the problem of determining $v(M)$ within S in such a way that the quantity

$$\int_{\omega r} (\lambda v + \partial v/\partial r)\, d\sigma$$

takes on given values $H(w)$ as r approaches 1, $H(w)$ being a bounded additive function of regular curves on S, with regular discontinuities.

For the class of functions for which $\int |\, \partial v/\partial r \,|\, d\sigma$, extended over the sphere S_r, remains bounded as r approaches 1, we shall show that there is a unique solution of the problem provided that λ is not one of a set of special parameter values, this set being the collection of negative integers and zero. For the value $\lambda = 0$ there is a solution, provided $H(S) = 0$, unique except for an additive constant; in this last case if $H(w)$ is absolutely continuous the problem reduces essentially to the Neumann problem with boundary values summable in the Lebesgue sense.

We shall write $v(M)$ as the potential of a simple distribution of mass and obtain the mass function in terms of $H(w)$ by solving a Stieltjes integral equation. In a supplementary note, we discuss briefly the situation in n dimensions, with special reference to the irregular case, $n = 2$.

1. Consider first the condition that

$$\int_{S_r} |\, \partial v/\partial r + \lambda v \,|\, d\sigma$$

remains bounded as r approaches 1, and let

$$\partial v/\partial r + \lambda v = \xi(M).$$

We have

$$v(r, P) = e^{-\lambda r} \int_{r_0}^{r} \xi(r, P) e^{\lambda r} dr + e^{\lambda(r_0 - r)} v(r_0, P),$$

---

[*] Preliminary report presented to the American Mathematical Society, April, 1927.

where v is given in terms of r and the radial projection P of M on S. Hence

$$\int_{S_r} | v(M) | \, d\sigma = r^2 \int_S | v(r, P) | \, d\sigma_P$$

$$\leq e^{-\lambda r} \int_{r_0}^{r} dr' e^{\lambda r'} \int_S | \xi(r', P) | \, d\sigma_P + e^{\lambda(r_0 - r)} \int_S | v(r_0, P) | \, d\sigma_P,$$

which is bounded, from the given condition. But this implies, even when λ is 0, that $\int_{S_r} | v(M) | \, d\sigma$ is bounded, $r < 1$.

It is evident then that *the two conditions*

$$\text{(j)} \qquad \int_{S_r} | \partial v/\partial r + \lambda v | \, d\sigma < N_1, \quad \int_{S_r} | \partial v/\partial r | \, d\sigma < N, \quad r < 1,$$

are equivalent, if the functions v, $\partial v/\partial r$ are continuous within S.

2. Let us assume that (j) holds. Then both the quantities

$$\int_{S_r} | r\partial v/\partial r | \, d\sigma, \qquad \int_{S_r} | r\partial v/\partial r + v/2 | \, d\sigma$$

remain bounded within S; and since $r\partial v/\partial r$ and $r\partial v/\partial r + v/2$ are both harmonic, $r < 1$, we have [*]

$$\text{(1)} \qquad r \frac{\partial v}{\partial r} = \frac{1}{4\pi} \int_S \frac{(1 - r^2)}{(1 + r^2 - 2r \cos \alpha)^{3/2}} \, dG(s_P)$$

$$\text{(2)} \qquad r \frac{\partial v}{\partial r} + \frac{v}{2} = \frac{1}{4\pi} \int_S \frac{(1 - r^2)}{(1 + r^2 - 2r \cos \alpha)^{3/2}} \, dF(s_P)$$

where $F(w)$, $G(w)$ are bounded additive functions of regular curves on S (in particular, of segments s), with regular discontinuities; and also

$$\text{(2')} \qquad \lim_{r=1} \int_{\omega_r} r \frac{\partial v}{\partial r} \, d\sigma = G(w), \quad \lim_{r=1} \int_{\omega_r} \left(r \frac{\partial v}{\partial r} + \frac{v}{2}\right) d\sigma = F(w),$$

$$v(0) = \frac{F(S)}{2\pi}.$$

In these formulae, α is the arc PQ on the unit sphere, Q being the projection of M, and $MP^2 = 1 + r^2 - 2r \cos \alpha$.

Consider now the quantity, which is also harmonic within S:

$$X(M) = \frac{1}{2\pi} \int_S \frac{dF(s)}{(1 + r^2 - 2r \cos \alpha)^{1/2}}$$

---

[*] Bray and Evans, "A Class of Functions Harmonic within the Sphere," *American Journal of Mathematics;* Vol. 49 (1927), pp. 153-180.

An obvious calculation yields the result

$$r\partial X/\partial r + X/2 = r\partial v/\partial r + v/2$$

$$r^{1/2}X = r^{1/2}v + \psi(\theta,\phi)$$

where $\psi(\theta,\phi)$ is a continuous function of the coordinates θ,ϕ on the sphere. Hence, since the left hand member of the above equation vanishes when $r=0$, we have $\psi(\theta,\phi)=0$, and the functions X and v are identical. Hence

$$(3) \qquad v(M) = \frac{1}{2\pi}\int_s \frac{dF(s)}{(1+r^2-2r\cos\alpha)^{1/2}} \,.$$

From (2) we have

$$G(w) = F(w) - \lim_{r=1}\frac{1}{4\pi}\int_S dF(s')\int_{\omega_r}\frac{d\sigma}{(1+r^2-2r\cos\alpha)^{1/2}}$$

$$= F(w) - \frac{1}{4\pi}\int_S dF(s')\int_\omega \frac{d\sigma}{[2(1-\cos\alpha)]^{1/2}}$$

on account of the uniform convergence of the improper integral. Hence

$$(4) \qquad F(w) = G(w) + \frac{1}{8\pi}\int_S dF(s')\int_\omega \frac{d\sigma}{\sin\alpha/2} \,.$$

By the same sort of calculation we have also

$$(5) \qquad F(w) = H(w) + \frac{1-2\lambda}{8\pi}\int_S dF(s')\int_\omega \frac{d\sigma}{\sin\alpha/2}$$

where, in virtue of the condition (j),

$$(6) \qquad r\frac{\partial v}{\partial r} + \lambda v = \frac{1}{4\pi}\int_s \frac{(1-r^2)\,dH(s)}{(1+r^2-2r\cos\alpha)^{3/2}}$$

$$(6') \qquad \lim_{r=1}\int_{\omega r}(r\partial v/\partial r + \lambda v)\,d\sigma = \lim_{r=1}\int_{\omega r}(\partial v/\partial r + \lambda v)\,d\sigma = H(w),$$

the function $H(w)$ being again a bounded additive function of regular curves, with regular discontinuities. Accordingly we state the following theorem.

THEOREM 1. *That $v(M)$ be harmonic within the sphere and that also (j) hold is a necessary and sufficient condition that $v(M)$ be given by a formula (3), in which $F(w)$ is a bounded additive function of regular curves, with regular discontinuities. In this case the $G(w)$, defined by (2′), and the $H(w)$, defined by (6′), are functions of the same sort, and the equations (1), (2), (4), (5), (6) are all valid.*

We have just proved the sufficiency of the condition. In order to establish the necessity, we notice that $v(M)$, given by (3), is harmonic, and in particular that $\int_{S_r} \partial v/\partial r \, d\sigma = 0$. Moreover (2) is satisfied, whence

$$\int_{S_r} | \partial v/\partial r + v/2 | \, d\sigma$$

is bounded,[*] and (j) is established by § 1.

COROLLARY. If $v(M)$ is given by a formula (3) in terms of an $F_1(s)$ which is a bounded additive function of plurisegments, its discontinuities not of necessity being regular, then also

$$v(M) = \frac{1}{2\pi} \int_S \frac{dF(s)}{(1 + r^2 - 2r \cos \alpha)^{1/2}} = \frac{1}{2\pi} \int_S \frac{dF(w)}{(1 + r^2 - 2r \cos \alpha)^{1/2}}$$

where

$$F(w) = \int_S q(P, w) \, dF_1(s)$$

$q(P, w)$ being the circular density of ω on S at P, a function measurable in the Borel sense.[†]

3. We pass now to the integral equation (5). If it has a solution $F(w)$, the function $v(M)$, given by (3), will be harmonic within the sphere, of class (j) and will satisfy the boundary condition (6′). On the other hand, if $v(M)$, of class (j), is harmonic within the sphere it will be given by a formula (3), where $F(w)$ is determined by (2′); and if also it satisfies the boundary condition (6′), the $F(w)$ will satisfy (5).

Since the discontinuities of $H(w)$ are regular, it is sufficient to consider the segments s on the sphere. Write

$$F(s) = H(s) + \Phi(s).$$

Equation (5) becomes

$$\Phi(s) = \frac{1 - 2\lambda}{8\pi} \int_S dH(s') \int_\sigma \frac{d\sigma}{\sin(\alpha/2)} + \frac{1 - 2\lambda}{8\pi} \int_S d\Phi(s') \int_\sigma \frac{d\sigma}{\sin(\alpha/2)}$$

where the first term of the right hand member is to be regarded as a known function, and σ is the region bounded by s.

We shall show first that on the hypothesis that $H(s)$, $\Phi(s)$ are bounded additive functions of plurisegments, the function $\Phi(s)$ will be *absolutely continuous.* In fact, if we write

---

[*] Bray and Evans, *loc. cit.,* p. 168.

[†] Bray and Evans, *loc. cit.,* p. 168.

$$p_n(\alpha) = \frac{1}{\sin(\alpha/2)}, \qquad \frac{1}{\sin(\alpha/2)} < n$$
$$= n, \qquad \frac{1}{\sin(\alpha/2)} \geq n,$$

the function so defined will be not negative, continuous on the sphere and non-decreasing with n, and

$$\int_S dH(s') \int_\sigma p_n(\alpha)\,d\sigma = \int_\sigma d\sigma \int_S p_n(\alpha)\,dH(s').$$

Moreover

$$\left| \int_S dH(s') \int_\sigma p_n(\alpha)\,d\sigma \right| \leq \int_S dT(s') \int_\sigma p_n(\alpha)\,d\sigma,$$

where $T(s)$ is the total variation function for $H(s)$. But this is

$$\leq \int_\sigma dT(s') \int_\sigma \frac{d\sigma}{\sin(\alpha/2)}.$$

Hence the quantity

$$\int_S \frac{dH(s')}{\sin(\alpha/2)}$$

exists almost everywhere on S, and represents a summable function;[*] moreover:

$$\int_\sigma d\sigma \int_S \frac{dH(s')}{\sin(\alpha/2)} = \int_\sigma dH(s') \int_\sigma \frac{d\sigma}{\sin(\alpha/2)} \tag{7}$$

But the left hand member is an absolutely continuous function, and the point is proved.

If then we write

$$h(P) = \frac{1-2\lambda}{8\pi}\int_S \frac{dH(s')}{\sin(\alpha/2)}, \quad f(P) = \Phi'(\sigma;P) = \frac{d\Phi}{d\sigma}$$

equation (5) implies the following:

$$f(Q) = h(Q) + \frac{1-2\lambda}{8\pi}\int_S \frac{f(P)}{\sin(\alpha/2)}\,d\sigma_P \tag{8}$$

$$\Phi(s) = \int_\sigma f(P)\,d\sigma \tag{8'}$$

$$F(s) = \Phi(s) + H(s).$$

On the other hand, if (8) has a solution $f(P)$, summable in the Lebesgue sense on S, equation $(8')$ defines a function $F(s)$, bounded and additive, with regular discontinuities, which is a solution of (5).

---

[*] P. J. Daniell, "A General Form of Integral," *Annals of Mathematics*, Vol. 19 (1918), pp. 279-294, and continued in later papers; for the interchange of order of integration, G. C. Evans, "Fundamental Points of Potential Theory," *Rice Institute Pamphlet*, Vol. 7 (1920), pp. 252-329, see p. 258 and p. 262.

In passing, we note the particular value $\lambda = 1/2$, for which the only solution of (8) is $f(P) = 0$. For this value we have $F(s) = H(s)$ as the only solution of our problem. It is this case, as we shall see later, which corresponds to the value $\lambda = 0$ in the two dimensional problem; and thus it happens that in two dimensions the solution of the Neumann problem is given by a direct relation, while in three dimensions the "same" problem $(\lambda = 0)$ is answered in terms of the integral equation (4).

The kernel of (8)

$$K(Q,P) = K(P,Q) = 1/8\pi \cdot 1/\sin(\alpha/2)$$

becomes infinite when $P = Q$; but, as we have already seen, the function $\int_S K(Q,P)\, g(P)\, d\sigma_P$ is summable if $g(P)$ is summable, since this is merely a particular case of the result established earlier in this section. Hence the customary analysis applies,[*] and the solutions of (8) are identical with those of the following equation:

$$(9) \qquad f(Q) = h(Q) + \nu \int_S K(Q,P)h(P)\,d\sigma_P + \nu^2 \int_S K_1(Q,P)h(P)\,d\sigma_P \\ + \nu^3 \int_S K_2(Q,P)f(P)\,d\sigma_P,$$

unless $\nu = 1 - 2\lambda$ is such that ν^3 is a root of the twice iterated kernel

$$K_2(Q,P) = \int_S K(Q,X)\, K(X,Y)\, K(Y,P)\, d\sigma_X d\sigma_Y$$

which is easily shown to be a continuous function. The solution of equation (8) and therefore of equation (5) is accordingly unique if λ does not correspond to one of these exceptional values of ν. These values of ν are known to be the positive odd integers,[†] to which correspond, as values of λ, the negative integers and zero. But we may find them again, and at the same time completely solve our problem by means of a simple differential equation of the first order.

4. We return then to the equation (6). To say that (5) has one or more solutions implies that the equation (6) may be solved for $v(M)$ and that the resulting function $v(M)$ will be harmonic, in particular at the point $r = 0$. Conversely if $v(M)$ satisfies (6) it will be of class (j), and the corresponding $F(w)$ will therefore satisfy (5). If we write

---

[*] Goursat, *Cours d'Analyse*, Vol. 3, Paris (1915), p. 355 and p. 382.

[†] Goursat, *loc. cit.*, p. 513.

$$h(M) = h(r, P) = \frac{1}{4\pi} \int_S \frac{(1 - r^2)\, dH(s)}{(1 + r^2 - 2r\cos\alpha)^{3/2}}$$

the equation to be solved may be written in the form

$$r\partial v/\partial r + \lambda v = h(r, P), \tag{10}$$

where $h(r, P)$ is harmonic inside S, and may therefore be expanded in the form

$$h(r, P) = Y_0 + \sum_1^\infty Y_k(P) r^k, \tag{10'}$$

where $Y_k(P)$ is a surface or spherical harmonic of degree k, the series being convergent, uniformly in P, for any value of $r < 1$.

Equation (10) may be written in the form

$$\partial(r^\lambda v)/\partial r = Y_0 r^{\lambda - 1} + \sum_1^\infty Y_k(P) r^{k+\lambda-1}.$$

Hence if λ is not zero or a negative integer, we have

$$r^\lambda v = (1/\lambda) Y_0 r^\lambda + \sum_1^\infty [Y_k(P)/(k + \lambda)] r^{k+\lambda} + C(P)$$

and therefore

$$v = (1/\lambda) Y_0 + \sum_1^\infty [Y_k(P)/(k + \lambda)] r^k \tag{11}$$

since $C(P) r^{-\lambda}$ cannot be harmonic at the origin unless $C(P) = 0$.

If λ is a negative integer or zero, the expansion yields the following:

$$r^\lambda v = \sum_0^\infty{}' [Y_k(P)/(k + \lambda)] r^{k+\lambda} + Y_{-\lambda}(P) \log r + C(P),$$

where in the summation the term $k = -\lambda$ is lacking. Hence

$$v(M) = \sum_0^\infty{}' [Y_k(P)/(k + \lambda)] r^{k+\lambda} + Y_{-\lambda}(P) r^{-\lambda} \log r + C(P) r^{-\lambda}. \tag{11'}$$

But this will not be harmonic in the neighborhood of the origin unless $Y_{-\lambda}(P) = 0$. Conversely, if this condition is satisfied, $v(M)$ will be a solution, in which $C(P)$ is an arbitrary surface harmonic of degree $\mu = -\lambda$.

The condition $Y_\mu(P) = 0$ may be written in the form

$$\int_S Y_\mu(P) h(r, P)\, d\sigma_P = 0,$$

as is seen from (10'). But

$$\int_S Y_\mu(P) h(r, P)\, d\sigma_P = \frac{1}{4\pi} \int_S dH(s_{P'}) \int_S \frac{Y_\mu(P)(1-r^2)}{(1+r^2-2r\cos PP')^{3/2}}\, d\sigma_P\ .$$

$$= (r^\mu/4\pi) \int_S Y_\mu(P)\, dH(s_P).$$

Hence we have the following theorem.

THEOREM 2. *If λ is not a negative integer or zero there is one and only one function $v(M)$ harmonic inside the sphere and of class* (j), *such that*

$$(6') \qquad \lim_{r=1} \int_\omega \{\lambda v(r, P) + \partial v(r, P)/\partial r\}\, d\sigma = H(w),$$

$H(w)$ being an arbitrary, bounded, additive function of regular curves, with regular discontinuties.

If $\lambda = -\mu$ is a negative integer there is no solution of the class (j) *subject to* (6′) *unless $H(w)$ satisfies the $2\mu + 1$ independent conditions*

$$(12) \qquad \int_S Y_\mu(P)\, dH(s_P) = 0,$$

$Y_\mu(P)$ being the general surface harmonic of degree μ. If (12) *is satisfied the solution exists, is of class (j) and is determined except for the addition of an arbitrary solid harmonic $Y_\mu(P)\, r^\mu$* of degree μ. In this case $F(w)$ is *determined as a solution of* (5) *except for an additive term of the form $\int_\omega Y_\mu(P)\, d\sigma_P$.*

If $\lambda = 0$, the necessary condition is $H(S) = 0$, and the function $v(M)$ *is determined except for an additive constant. In this case $F(w)$ is determined, as a solution of* (5), *except for an arbitrary additive term of the form $C\omega$.*

5. From this point results may be obtained corresponding to those of Bray and Evans in the article already cited. It is unnecessary to repeat them in detail except to point out that

$$\lim_{r=1}\ (r\partial v/\partial r + \lambda v) = dH/d\sigma$$

wherever that derivative exists, that is, almost everywhere on S; and that therefore

$$\lim_{r=1} \partial v/\partial r, \quad \lim_{r=1} v$$

exist for almost all points of S, since the condition (j) implies the condition

(i) $$\int_{S_r} |\,v(M)\,|\,d\sigma \text{ is bounded.}$$

In fact these limits all exist whenever $\partial G/\partial\sigma$ exists. For if lim $\partial v/\partial r$ exists, then along that radius $\partial v/\partial r$ is bounded, and lim v exists.

Finally, *if we replace the condition* (j) *by the condition*

(jj) $$\int_{\omega_r} (\partial v/\partial r)\,d\sigma \text{ is absolutely continuous, uniformly in } r,$$

a condition which is satisfied, in particular, if $\partial v/\partial r$ is bounded within S, and the condition (6′) *by the condition*

(6″) $$\lim_{r=1}\ (\partial v/\partial r + \lambda v) = h(P),\quad h(P) \text{ summable on } S,$$

the Theorem 2 remains valid. In this case $H(s) = \int_\sigma h(P)\,d\sigma$.

Thus we have a solution of the Neumann problem itself, and of a class of problems related to it.

6. *Supplementary note on the problem in n-dimensions.* The case of three dimensions being typical, we content ourselves with a brief description of the formulae involved in the n-dimensional situation. The Poisson integral for the hypersphere S of radius R takes the form

(13) $$u(M) = u(r, Q) = \frac{1}{RA_n}\int_S \frac{R^2 - r^2}{\overline{MP}^n}\ dF(s),$$

where $A_n = 2\pi^{n/2}/\Gamma(n/2)$ is the $(n-1)$-dimensional "area" or measure of the hypersphere of unit radius. Its n-dimensional measure of "volume" is of course $2\pi^{n/2}/[n\Gamma(n/2)]$. In (13) we have written

(14) $$F(s) = \lim_{r=R}\ F(r, s_r) = \lim_{r=R}\int_{s_r} u(M)\,d\sigma_r,$$

the $d\sigma_r$ being $r^{n-1}\,d\sigma$, with $d\sigma$ as the element of area of the unit hypersphere; in fact the constant before the integral has been chosen so that

(14′) $$u(0) = F(S)/A_n(R) = F(S)/R^{n-1}A_n$$

For the corresponding potential of a single layer we write

(15) $$v(M) = \frac{2}{(n-2)A_nR}\int_S \frac{dF(s)}{\overline{MP}^{n-2}} = \frac{2}{(n-2)A_nR}\int_S \frac{dF(s)}{(R^2 + r^2 - 2Rr\cos\alpha)^{(n/2)-1}}.$$

We have

$$r\,\frac{\partial v}{\partial r} = -\frac{1}{A_n R}\int_s \frac{2r^2 - 2Rr\cos\alpha}{(R^2 + r^2 - 2Rr\cos\alpha)^{n/2}}\,dF(s),$$

and therefore,

$$r\partial v/\partial r + [(n-2)/2]v = u$$

$$r\partial v/\partial r + \lambda v = u + [\lambda - (n-2)/2]v = [(u - (n-2-2\lambda/2]v$$

$$= u - \frac{n-2-2\lambda}{(n-2)RA_n}\int_s \frac{dF(s)}{\overline{MP}^{n-2}}\,.$$

Hence we write

$$\lim_{r=R}\int_{S_r} r(\partial v/\partial r)\,d\sigma_r = G(s) = R\lim_{r=R}\int_{S_r}(\partial v/\partial r)\,d\sigma_r$$

$$\lim_{r=R}\int_{S_r}(r\partial v/\partial r + \lambda v)\,d\sigma_r = H(s),$$

where $G(s)$ and $H(s)$ are bounded additive functions of $(n-2)$-dimensional plurisegments on the hypersphere,[*] provided the condition (j) is valid:

$$\text{(j)}\qquad \int_{S_r} |\,\partial v/\partial r\,|\,d\sigma_r \textit{ is bounded, } r < R.$$

We have then the integral equations relating $H(s)$, $G(s)$ and $F(s)$, to determine $F(s)$:

$$(16)\qquad G(s) = F(s) - \frac{1}{RA_n}\int_S dF(s_P)\int_\sigma \frac{d\sigma}{\overline{QP}^{n-2}}$$

$$(17)\qquad H(s) = F(s) - \frac{n-2-2\lambda}{(n-2)RA_n}\int_S dF(s_P)\int_\sigma \frac{d\sigma}{\overline{QP}^{n-2}}$$

Here $G(s)$ of course vanishes, if $v(M)$ is given, and for particular values of $\lambda \neq 0$ there will also be necessary conditions on the $H(s)$.

In order to find these conditions we have merely to repeat the analysis of § 4. The characteristic values of λ will again be the negative integers $-\mu$ and 0; and if we write

$$h(r,Q) = \frac{1}{RA_n}\int_S \frac{(R^2 - r^2)\,dH(s)}{\overline{MP}^n} = \sum_0^\infty X_k(Q)\,r^k$$

the functions $X_k(Q)$ being orthogonal, the condition to be satisfied by $H(s)$ will be

$$\int_S Y_\mu(P)\,dH(s_P) = 0,$$

---

[*] Edmonson, *Poisson's Integral and Plurisegments on the Hypersphere*, in preparation.

where $Y_\mu(P)r^\mu$ is the general homogeneous polynomial of degree μ, which satisfies Laplace's equation.

The case $n = 2$ is not strictly regular, as is seen from the above formulae. But the problem corresponding to $\lambda = 0$ has already been considered in detail.[*] We write

$$v(M) = (1/\pi R) \int_S \log (1/\overline{MP})\, dF(s) + A, \textit{ with } F(S) = 0, \quad A = v(0),$$

$$u(M) = (1/2\pi R) \int_{S_1} [(R^2 - n^2)/\overline{MP^2}]\, dF(s),$$

and therefore

$$r\partial v/\partial r = u$$

$$r(\partial v/\partial r) + \lambda v = u + (\lambda/\pi R) \int_S \log (1/MP)\, dF(s) + \lambda A;$$

and the two equations corresponding to the integral equations (16), (17) are

$$G(s) = F(s)$$

$$\text{(18)} \qquad H(s) = F(s) + (\lambda/\pi R) \int_S dF(s_P) \int_S \log (1/QP)\, ds_Q + \lambda A s,$$

where, when $\lambda \neq 0$, we may still keep $F(S) = 0$ by taking $A = v(O) = H(S)/\lambda S$. The second of the equations (18) may therefore be written in the form

$$H(s) - [H(S)/\lambda S] = F(s) + (\lambda/\pi R) \int_S dF(s_P) \int_S \log (1/\overline{QP})\, ds_Q.$$

From this point on, the analysis offers nothing novel.

The conditions may be stated in a form invariant of conformal transformation. And if this is done the results will apply to general [simply connected] regions, and the special parameter values of λ will be unaltered. To this end we let $b(M_0, M)$ be the function conjugate to the Green's function $g(M_0, M)$ and write the condition (j) in the form

$$\int_{g=\text{const.}} |\, \partial v/\partial g \,|\, db = \int_{g=\text{const.}} |\, \partial v/\partial n \,|\, ds < N.$$

The boundary condition (6′) takes the form

$$\lim_{g=0} \int_{b'}^{b''} (v + \lambda \partial v/\partial g)\, db = H(b', b'') = H(s).$$

---

[*] Evans, *The Logarithmic Potential—Discontinuous Dirichlet and Neumann Problems*, New York (1927), Ch. IV.

Similarly in the condition (jj) the absolute continuity of the integral

$$\int_{b'}^{b} (\partial v/\partial g)\, db = \int_{(b')}^{(b'')} (\partial v/\partial n)\, ds$$

extended along curves $g = \text{const.}$, is uniform, $g > 0$; and the condition (6″) takes the form

$$\lim_{g=0} (v + \lambda \partial v/\partial g) = h(b),$$

$h(b)$ being summable with respect to b.

Under these conditions there is a unique solution of the respective boundary value problems, unless $\lambda = 0$ *or a negative integer.*[*]

THE RICE INSTITUTE,
HOUSTON, TEXAS.

---

[*] See the treatment for $\lambda = 0$ in Evans, *loc. cit.*, Ch. V.

On Taylor's Series Admitting the Circle of Convergence as a Singular Curve.[*]

By J. J. Gergen and D. V. Widder.[†]

1. *Introduction.* It is a familiar fact that certain Taylor's series

$$\sum_{n=0}^{\infty} a_n x^n \tag{1}$$

define functions which can not be continued analytically beyond their circles of convergence. The series of Weierstrass and Fredholm,

$$\sum_{n=0}^{\infty} x^{2^n}, \tag{2}$$

$$\sum_{n=0}^{\infty} x^{n^2}, \tag{3}$$

are classic examples. Both of these series are *lacunary;* that is, they are particular cases of (1) in which certain of the coefficients are zero. General classes of such series have been studied by a great number of authors including J. Hadamard, E. Borel, S. Mandelbrojt, G. Faber, and E. Fabry. We recall several of the results. It was shown by Hadamard that the series

$$\sum_{m=0}^{\infty} a_m x^m = \sum_{n=0}^{\infty} a_{\lambda_n} x^{\lambda_n}, \tag{4}$$

where

$$a_m = 0 \text{ when } m \neq \lambda_n, \quad a_{\lambda_n} \neq 0,$$

admits the circle of convergence as a cut if

$$(\lambda_{n+1} - \lambda_n)/\lambda_n$$

is greater than some positive constant for all n. The series (2) of Weierstrass is a case in point. More generally, it has been shown [‡] that the conclusion is the same if

$$(\lambda_{n+1} - \lambda_n)/\lambda^{\sigma}_n, \qquad 0 < \sigma \leq 1,$$

---

[*] Preliminary report presented to the American Mathematical Society, September 9, 1927. A preliminary account of the results of this paper was published in a note in *Comptes Rendus,* Vol. 185 (1927), pp. 829-831. Since the publication of that note, the authors have simplified several of the proofs.

[†] National Research Fellow.

[‡] For complete references see J. Hadamard and S. Mandelbrojt, "La Série de Taylor et son prolongement analytique," *Collection Scientia,* Gauthier-Villars, 1926.

is greater than a positive constant for all n. This was first proved for the case $\sigma = \frac{1}{2}$ by Borel. It is seen that this criterion applies to the series (3) of Fredholm.

In a recent paper Mandelbrojt[*] has developed a new type of criterion. He has shown that if ϕ is any irrational number, the series (4) admits its circle of convergence as a cut if $e^{2\pi i \lambda_n \phi}$ has a limit as n becomes infinite, and if a_{λ_n} has a limit different from zero. The proof of the theorem is based on a theorem of Cesàro. In the present paper the authors prove a similar theorem, replacing the condition that a_{λ_n} have a limit by the restriction that they all lie in a single quadrant of the complex plane. The method of proof is simplified, no use of Cesàro's theorem being necessary. By means of the criterion thus developed the authors are able to give a new proof of the theorem of Hadamard and its generalizations mentioned above. It has been the aim to make the proof elementary in character. It will be seen by an examination of the following sections that the proof is based on only two theorems from the theory of analytic continuation, two theorems that lie at the very foundation of that theory.

2. *A Lemma.* The proof of the theorem of the next section is made by use of the following

LEMMA. *The function defined by the series*

$$f(x) = \sum_{n=0}^{\infty} a_{\lambda_n} x^{\lambda_n}, \qquad \overline{\lim_{n=\infty}} \mid a_{\lambda_n} \mid^{1/\lambda_n} = 1, \tag{5}$$

has the point $x = 1$ as a singular point if the real part a_{λ_n}' of *a_{λ_n} satisfies the conditions $a_{\lambda_n}' > 0$, $\overline{\lim_{n=\infty}} \, a_{\lambda_n}'^{\,1/\lambda_n} = 1$.*

This Lemma is a generalization of the well-known fact that the function defined by (5) has the point $x = 1$ singular if the a_{λ_n} are all real and positive.[†] Indeed, this result enables us to see at once that the function

$$\phi(x) = \sum_{n=0}^{\infty} a_{\lambda_n}' x^{\lambda_n}$$

---

[*] *Comptes Rendus*, Vol. 184 (1927), pp. 1307-1309. It may be shown that the set of functions which satisfies the conditions of Mandelbrojt and the set which satisfies the conditions of the theorem of this paper form subclasses of the class for which $\lim (\lambda_{n+1} - \lambda_n) = \infty$. It is well known that a function of this latter class has its circle of convergence as a singular curve. [Gergen.]

[†] Compare a generalization by P. Dienes, "Essai sur les singularités des fonctions analytiques," *Journal de Mathématiques*, 6 série, t. 5 (1909), p. 338.

has the point $x = 1$ singular. That is, if β is any real number between zero and unity, the series

$$\sum_{n=0}^{\infty} \frac{\phi^{(n)}(\beta)}{n!} \, (x - \beta)^n$$

has the radius of convergence equal to $1 - \beta$. Hence for every positive ϵ there exist an infinite number of integers m for which

$$(6) \qquad \frac{\phi^{(m)}(\beta)}{m!} = \sum_{p=0}^{\infty} \frac{(m+p)!}{m!\,p!} \, a_{m+p}' \, \beta^p > \left(\frac{1-\epsilon}{1-\beta} \right)^m .$$

We wish to prove that $x = 1$ is a singular point of (5), or that

$$(7) \qquad \left| \frac{f^{(m)}(\beta)}{m!} \right| > \left(\frac{1-\epsilon}{1-\beta} \right)^m$$

for an infinite number of integers m. Set

$$a_{\lambda_n} = a_{\lambda_n}' + i \, a_{\lambda_n}'', \quad \psi(x) = \sum_{n=0}^{\infty} a_{\lambda_n}'' x^{\lambda_n}.$$

Then

$$f(x) = \phi(x) + i \, \psi(x),$$

and

$$f^{(m)}(\beta) = \phi^{(m)}(\beta) + i \, \psi^{(m)}(\beta).$$

Since β is real, the quantities $\phi^{(m)}(\beta)$ and $\psi^{(m)}(\beta)$ are real. Hence

$$| f^{(m)}(\beta) | \geq \phi^{(m)}(\beta),$$

and (7) becomes an immediate consequence of (6). The Lemma is thus completely proved.[*]

3. *The Theorem. If* $\{\lambda_n\}$ *is a sequence of positive integers such that* $\lim_{n=\infty} e^{2\pi i \lambda_n \phi}$ *exists for some irrational number* ϕ, *and if* $\{a_{\lambda_n}\}$, $a_{\lambda_n} \neq 0$, *is a sequence of numbers all lying in a single quadrant of the complex plane,*[†] *then the function defined by the series*

$$(8) \qquad \sum_{n=0}^{\infty} a_{\lambda_n} x^{\lambda_n}, \qquad \overline{\lim_{n=\infty}} \, | a_{\lambda_n} |^{1/\lambda_n} = 1,$$

admits the circle of convergence as a singular curve.

---

[*] Note that it is not true, in general, that all the singularities of $\sum a_n' x^n$ are singularities of $\sum a_n x^n$. Witness the example

$$\sum a_n x^n = x/(x^2+1) + i/(x^2+1) = 1/(x-i).$$

[†] We agree that a_{λ_n} lies in the kth quadrant ($k = 1, 2, 3, 4$) if

$$a_{\lambda_n} = r_{\lambda_n} \exp i\theta_{\lambda_n}, \quad (k-1)\pi/2 \leq \theta_{\lambda_n} < k\pi/2.$$

Without loss of generality we may suppose that all the a_{λ_n} lie in the first quadrant, for the case in which they lie in the kth quadrant may be reduced to this case by multiplying each term of the series (8) by $e^{-\pi i(k-1)/2}$. Obviously this has no effect on the position of the singularities of (8). Set

$$a_{\lambda_n} = a_{\lambda_n}{}' + i\, a_{\lambda_n}{}'', \quad a_{\lambda_n}{}' > 0, \quad a_{\lambda_n}{}'' \geq 0.$$

Then at least one of the following equations holds:

$$\overline{\lim_{n=\infty}}\; a_{\lambda_n}{}'^{\,1/\lambda_n} = 1, \qquad \overline{\lim_{n=\infty}}\; a_{\lambda_n}{}''^{\,1/\lambda_n} = 1.$$

For definiteness suppose the first to be true, and consider the series

$$\sum_{n=0}^{\infty} a_{\lambda_n}\, e^{2\pi i \lambda_n \phi p} x^{\lambda_n}, \tag{9}$$

where p is any positive integer. We show that $x = 1$ is a singular point of this series by showing it to be a singular point of the following series (differing from it only by a constant factor):

$$-i \sum_{n=0}^{\infty} a_{\lambda_n}\, e^{2\pi i \psi_n}\, x^{\lambda_n}, \tag{10}$$

where

$$\psi_n = \lambda_n p \phi + 1/8 - p\psi,$$

$$e^{2\pi i \psi} = \lim_{n=\infty} \; e^{2\pi i \lambda_n \phi}.$$

Then we have

$$\lim_{n=\infty} \; e^{2\pi i \psi_n} \; = \; e^{\pi i/4}.$$

If we decompose the coefficients of (9) into real and imaginary parts, we obtain

$$-i\, a_{\lambda_n} e^{2\pi i \psi_n} = -i(a'_{\lambda_n} \cos 2\pi\psi_n - a_{\lambda_n}{}'' \sin 2\pi\psi_n) + (a_{\lambda_n}{}'' \cos 2\pi\psi_n + a_{\lambda_n}{}' \sin 2\pi\psi_n)\,.$$

We can now show that the coefficients of (10) satisfy all the conditions of the Lemma (after rejecting, if necessary, a finite number of terms at the beginning of the series), since

$$\lim_{n=\infty} \sin 2\pi\psi_n = \lim_{n=\infty} \cos 2\pi\psi_n = 1/2^{1/2},$$

it follows that

$$c_{\lambda_n} = a_{\lambda_n}{}'' \cos 2\pi\psi_n + a_{\lambda_n}{}' \sin 2\pi\psi_n$$

is positive for all n sufficiently large $(n > m)$. Moreover,

$$\overline{\lim_{n=\infty}}\; c_{\lambda_n}{}^{1/\lambda_n} = 1, \quad n > m.$$

For,

$$\overline{\lim_{n=\infty}}\ (a_{\lambda_n}{}' \sin 2\pi\psi_n)^{1/\lambda_n} = \overline{\lim_{n=\infty}}\ a_{\lambda_n}{}'^{\,1/\lambda_n} = 1,$$

and

$$c_{\lambda_n} \geqq a_{\lambda_n}{}' \sin 2\pi\psi_n, \quad n > m.$$

This proves that

$$\overline{\lim_{n=\infty}}\ c_{\lambda_n}{}^{1/\lambda_n} \geqq 1.$$

That this upper limit can not be greater than unity is a consequence of the fact that c_{λ_n} is the real part of the general coefficient of (10), a series which clearly has unit radius of convergence:

$$\overline{\lim_{n=\infty}}\ | -i\, a_{\lambda_n} e^{2\pi i\psi_n} |^{1/\lambda_n} = \overline{\lim_{n=\infty}}\ | a_{\lambda_n} |^{1/\lambda_n} = 1.$$

By the Lemma the point $x = 1$ is a singular point of (10). That is, the points

$$x = e^{2\pi i p\phi}, \qquad (p = 1, 2, 3, \cdots),$$

are singular points of (8). Since this set of points is everywhere dense on the circle $|x| = 1$, it follows that the series (8) admits this circle as a singular curve. A similar proof applies if

$$\overline{\lim_{n=\infty}}\ a_{\lambda_n}{}''^{1/\lambda_n} = 1.$$

4. *Applications.* By use of the theorem of section 3 we are now able to prove the following result:[*]

If $\{\lambda_n\}$ is a sequence of positive integers such that the series

$$\sum_{n=0}^{\infty} 1/\lambda_n$$

converges, then the function defined by

$$(11) \qquad \sum_{n=0}^{\infty} a_{\lambda_n} x^{\lambda_n}, \qquad \overline{\lim_{n=\infty}}\ | a_{\lambda_n} |^{1/\lambda_n} = 1$$

has the circle $|x| = 1$ is a singular curve.

We begin by choosing a subset $\{\mu_n\}$ of the set $\{\lambda_n\}$ for which the unique limit (in contradistinction with the upper limit of the hypothesis), $\lim_{n=\infty} | a_{\mu_n}|^{1/\mu_n}$, is equal to unity, and for which all the a_{μ_n} lie in a single quadrant. Let ϕ be any irrational number. The set of points $e^{2\pi i\mu_n\phi}$ on the unit circle must contain an infinite number of distinct points, so that there is at least one limit point $e^{2\pi i\psi}$. Consequently it is possible to find a subset $\{\nu_n\}$ of the set $\{\mu_n\}$ such that

---

[*] Compare Hadamard and Mandelbrojt, *loc. cit.*, p. 71.

$$\lim_{n=\infty} e^{2\pi i\nu_n\phi} = e^{2\pi i\psi},$$

$$\lim_{n=\infty} |a_{\nu_n}|^{1/\nu_n} = 1. \tag{12}$$

Let the set $\{\nu_n'\}$ consist of those values of λ_n not equal to ν_n. If the set $\{\nu_n'\}$ contains only a finite number of elements, the series

$$\sum_{n=0}^{\infty} a_{\nu_n} x^{\nu_n} \tag{13}$$

differs from (11) by the polynomial

$$\sum a_{\nu_n'} x^{\nu_n'}.$$

But (13) satisfies all the conditions of the theorem of section 3, and consequently admits the circle $|x| = 1$ as a cut. And (13) must have the same singularities as (11), so that the theorem is proved.

On the other hand, if the set $\{\nu_n\}$ contains an infinite number of elements, we set up the entire function

$$g(z) = \prod_{n=0}^{\infty} (1 - z/\nu_n')^2. \tag{14}$$

The series

$$\sum_{n=0}^{\infty} 1/\nu_n'$$

converges, so that the infinite product (14) also converges. Moreover, if ϵ is an arbitrarily small positive constant, $g(z)$ satisfies the inequality

$$|g(z)| < e^{\epsilon|z|}$$

for all $|z|$ sufficiently large.* On this account we may apply a theorem of

---

* This is a result of the general theory of entire functions. For this special case it may be proved in the following manner: There exists a positive number A such that $|1-u| \leq e^{A|u|}$ for all complex numbers u. To every positive ϵ there corresponds an integer m such that

$$\sum_{n=m}^{\infty} 1/\nu_n' < \epsilon/4A.$$

Consequently,

$$\left|\prod_{n=m}^{\infty} (1 - z/\nu_n')^2\right| < e^{\epsilon|z|/2}.$$

But for $|z|$ sufficiently large

$$\left|\prod_{n=0}^{m-1} (1 - z/\nu_n')^2\right| < e^{\epsilon|z|/2}.$$

since the modulus of $(1 - z/\nu_n')^2$ increases less rapidly than $e^{\epsilon z/4m}$ as $|z|$ becomes infinite. The result is thus established.

G. Faber,[*] and be assured that the series

$$(15) \qquad \sum_{n=0}^{\infty} g(\lambda_n) x^{\lambda_n} = \sum_{n=0}^{\infty} g(\nu_n) x^{\nu_n}$$

is an element of a function which has no singularities in the extended plane except the point $x = 1$.

But

$$(16) \qquad \sum_{n=0}^{\infty} g(\lambda_n) a_{\lambda_n} x^{\lambda_n} = \sum_{n=0}^{\infty} g(\nu_n) a_{\nu_n} x^{\nu_n},$$

and, since $g(\nu_n)$ is real and positive, the coefficients all lie in a single quadrant. Moreover, it can be shown that the radius of convergence of (16) is unity. Indeed,

$$\overline{\lim_{n=\infty}}\, g(\nu_n)^{1/\nu_n} = 1$$

since the radius of convergence of (15) is unity. By virtue of (12) we have

$$\overline{\lim_{n=\infty}}\, | a_{\nu_n} g(\nu_n) |^{1/\nu_n} = \overline{\lim_{n=\infty}}\, | a_{\nu_n} |^{1/\nu_n} = 1.$$

Consequently, the theorem of section 3 is applicable to (16), and implies that the circle $|x| = 1$ is a cut for this series.

Finally, by an application of Hadamard's † theorem on the multiplication of singularities we see that (11) has all the singularities of (16), and the theorem is completely established.

As an immediate corollary to this theorem we prove the following result, mentioned in the introduction:

If $\{\lambda_n\}$ *is a sequence of positive integers such that*

$$(\lambda_{n+1} - \lambda_n)/\lambda_n^{\sigma}, \qquad 0 < \sigma \leq 1,$$

is greater than some positive constant for all values of n, then the series (11) *admits its circle of convergence as a cut.*

For, the inequality

$$(\lambda_{n+1} - \lambda_n)/\lambda_n^{\sigma} > M > 0$$

---

[*] G. Faber, "Über die Fortsetzbarkeit gewisser Taylorscher Reihen," *Mathematische Annalen*, Vol. 57 (1903), p. 369.

† J. Hadamard, "Théorème sur les séries entières," *Acta Mathematica*, Vol. 22 (1898), p. 55.

evidently implies the following inequalities:

$$\begin{aligned}
\lambda_{n+1} &> \lambda_n + n^\sigma M, \\
\lambda_{n+1} &> M\,[n^\sigma + (n-1)^\sigma + \cdots + 2^\sigma + 1], \\
\lambda_{n+1} &> (M/n^{1-\sigma})[n + (n-1) + \cdots + 2 + 1], \\
\lambda_{n+1} &> Mn(n+1)/2n^{1-\sigma}.
\end{aligned}$$

Consequently, we have

$$\sum_{n=0}^{\infty} 1/\lambda_{n+1} < (2/M) \sum_{n=0}^{\infty} 1/n^{1-\sigma},$$

and we are in the situation imposed by the hypotheses of the theorem of this section.

In conclusion, we note that the set $\lambda_n = n!$ provides an example of a set of integers satisfying the conditions of the theorem of section 3. For, take $\phi = e$. Then

$$n!\,e = I_n + R_n,$$

$$R_n = 1/(n+1) + 1/(n+1)(n+2) \cdots < 2/(n+1),$$

where I_n is an integer. Hence

$$e^{2\pi i n! e} = e^{2\pi i R_n},$$

$$\lim_{n=\infty} e^{2\pi i n! e} = 1.$$

THE RICE INSTITUTE,
June, 1927.

A Mathematical Theory of Depreciation and Replacement.

By C. F. Roos.*

1. *Introduction.* In a previous paper I have formulated a problem of depreciation in economics as a Lagrange problem in the calculus of variations and shown that the resulting problem is sufficiently general to include as special cases all of the existing depreciation theories, i. e. such theories as the straight line, the compound interest, the sinking fund, the unit cost plus and that of Hotelling.†

Inasmuch as a machine is seldom used until its useful life has ended, a more important problem is that of determining the time at which a machine in operation should be replaced by another machine having a different operating expense so that a maximum profit is obtained for some period of time extending from an initial time t_1 through a replacement time ω to some final time t_2 greater than ω. When formulated as a problem in the calculus of variations this replacement problem is a type of Lagrange problem with variable end-points and discontinuous integrand.‡

In this paper it is intended to formulate a replacement problem as a general problem in the calculus of variations and to solve in some detail a special example believed to be typical of an actual economic situation. In order to save space I will not attempt to give the whole economic setting of the problem since this can be obtained from other papers.§

2. *The general problem.* The value of a machine to its operator at a time t_1 is the sum of the anticipated rentals which it will yield from the time

---

* National Research Fellow in Mathematics.

† Roos, "The Problem of Depreciation in the Calculus of Variations," to be offered to the *Bulletin of the American Mathematical Society.* See, also, H. Hotelling, "A General Mathematical Theory of Depreciation," *Journal of American Statistical Association,* Sept. 1925.

‡ Bliss, *Lectures on the Calculus of Variations,* University of Chicago, Summer 1925, mimeographed by O. E. Brown, Northwestern University, Evanston, Ill. See, also, E. H. Clarke, *On the Minimum of the Sum of a Definite Integral and a Function of a Point,* Doctoral Dissertation, University of Chicago.

§ For a brief setting of the problem see footnote (†) above. For a more detailed economic account see Roos, "A Dynamical Theory of Economics," *Journal of Political Economy,* Vol. 35 (Oct. 1927).

t_1 to a time ω at which it is to be salvaged each multiplied by a discount factor to allow for interest plus the salvage value also discounted. If we let $\psi_1(u, u', p, p', t)$ be the cost of operating a machine producing $u(t)$ units of output per unit time which sell at a price $p(t)$; $D_1(u, u', p, p', t)$ be the rate of depreciation in market value of the machine, and K_1 be the cost of the machine at t_1, then, as I have shown,* the value $V_1(t_1)$ of this machine at $t = t_1$ will be

$$(1) \quad V_1(t_1) = \int_{t_1}^{\omega} [pu - \psi_1(u, u', p, p', t) - D_1(u, u', p, p', t)]\, E(t_1, t)\, dt + K_1 E(t_1, \omega).$$

Here and following $E(a, b)$ stands for $\exp\left[-\int_a^b \delta(\nu)\, d\nu\right]$, where $\delta(\nu)$ is the rate of increase of an invested sum s divided by s. The value $V_2(t_1)$ at the time t_1 of a machine which will replace this machine at the time ω will be

$$(2) \quad V_2(t_1) = \int_{t_2}^{\omega} [pu - \psi_2(u, u', p, p', t) - D_2(u, u', p, p', t)]\, E(t_1, t)\, dt + K_2 E(t_1, t_2)$$

where ψ_2, D_2 and K_2 are defined in a way analogous to ψ_1, D_1 and K_1, respectively. For convenience in notation let us write $Q_i = \psi_i + D_i$, $(i = 1, 2)$.

As a likely hypothesis we may suppose that the operator will endeavour to maximize the expression

$$(3) \qquad [V_1 + V_2 - K_1 E(t_1, \omega) - K_2 E(t_1, t_2)] = \int_{t_1}^{\omega} [pu - Q_1] E(t_1, t)\, dt + \int_{\omega}^{t_2} [pu - Q_2]\, E(t_1, t)\, dt$$

which is the sum of the values V_1 and V_2 at t_1 minus the respective cost values discounted to take care of interest. In reality K_1 is the sum necessary to replace the first machine by an identical machine at the time ω, and hence

$$(4) \qquad K_1(\omega) = K_1(t_2)\, E(\omega, t_2)\,.$$

The preceding hypothesis is, therefore, justified by the fact that if (4) holds, and, furthermore, if $K_1(t_2) \equiv K_2(t_2)$ and $Q_1(u, u', p, p', t) \equiv Q_2(u, u', p, p', t)$, the expression (3) reduces to the corresponding expression for the case of a single machine. In fact, for this special case the replacement problem is actually the problem of depreciation.

We may suppose the rate of production $u(t)$ and the selling price, $p(t)$, of the commodity produced to be related by a differential equation of demand,†

---

* Roos, "The Problem of Depreciation in the Calculus of Variations," *loc. cit.*

† Roos, "Dynamical Economics," *Proceedings of the National Academy of Sciences*, Vol. 13, No. 3, pp. 145-150 (March 1927.)

$$(5) \qquad \phi(u, u', p, p', t) = 0, \qquad (t_1 \leq t \leq t_2),$$

and to satisfy end conditions,

$$(6) \qquad \theta_\mu(t_1, u(t_1), p(t_1), t_2, u(t_2), p(t_2)) = 0, \quad (\mu = 1, \cdots, n \leq 6).$$

We may then state the replacement problem as that of: *Finding among the arcs $u(t)$, $p(t)$, satisfying an equation of demand* (5) *and end conditions* (6), *a set which maximizes an integral, $J = I_1 + I_2$, where*

$$I_1 = \int_{t_1}^{\omega} [pu - Q_1(u, u', p, p', t)]\, E(t_1, t)\, dt$$

and

$$I_2 = \int_{\omega}^{t_2} [pu - Q_2(u, u', p, p', t)]\, E(t_1, t)\, dt.$$

This problem differs from the depreciation problem in that the integrand of the integral to be maximized is discontinuous at some time ω when the change in machines is to be made.

3. *Conditions for a solution.* Let P_1 and P_2 be the end points of the curve which maximizes J, and let P_ω be the corner point of this maximizing curve. Then it is evident that along the segment P_1P_ω the Euler-Lagrange equations and other conditions of the calculus of variations for the Lagrange problem with variable end-points must hold, for, otherwise, it would be possible to introduce a variation of P_1P_ω which would give a larger value of the integral I_1 taken along it and consequently, a larger value to J, thus contradicting the assumption that P_1P_ω and $P_\omega P_2$ together give J a maximum value. A similar argument holds for the arc $P_\omega P_2$.

In writing these conditions it will be convenient to introduce the notations $y_1(t) = u(t)$, $y_2(t) = p(t)$ and

$$F_\sigma(y_1, y_1', y_2, y_2', t, t_1, \lambda_\sigma)$$

$$= [y_1y_2 - Q_\sigma(y_1, y_1', y_2, y_2', t)]\, E(t_1, t) + \lambda_\sigma(t)\phi(y_1, y_1', y_2, y_2', t);$$

$$H(t_\sigma) = (\sigma - 2)F_1(t_1) + (\sigma - 1)F_2(t_2) + \int_{t_1}^{\omega} [F_{1t_\sigma} + F_{1y_k(t_\sigma)}y_k'(t_\sigma)]\,dt$$

$$+ \int_{\omega}^{t_2} [F_{2t_\sigma} + F_{2y_k(t_\sigma)}y_k'(t_\sigma)]\,dt;$$

$$H_{y_i}'(t_\sigma) = (\sigma - 2)F_{1y_i}'(t_1) + (\sigma - 1)F_{2y_i}'(t_2)$$

$$+ \int_{t_1}^{\omega} F_{1y_i(t_\sigma)}\,dt + \int_{\omega}^{t_2} F_{2y_i(t_\sigma)}\,dt,$$

$$(\sigma = 1, 2;\ i = 1, 2);$$

where k is used as an umbral index with range 1, 2 and subscripts y_i, y_i', $(i = 1, 2)$, and t_σ, $(\sigma = 1, 2)$, denote differentiation.

Then the maximizing arc must satisfy the following conditions:

(a) the Euler-Lagrange equations

$$F_{1y_i'} = \int_{t_1}^{t} F_{1y_i} dt + C_{1i}, \qquad (t_1 \leqq t \leqq \omega), \quad \text{and}$$

$$F_{2y_i'} = \int_{t_2}^{t} F_{2y_i} dt + C_{2i}, \qquad (\omega \leqq t \leqq t_2);$$

(b) the transversality conditions which require that every determinant of order $n + 1$ of the matrix:

$$\left\| \begin{matrix} H(t_1) - y_k'(t_1) H_{y_k'}(t_1) & H_{y_i'}(t_1) & H(t_2) - y_k'(t_2) H_{y_k}'(t_2) & H_{y_i}'(t_2) \\ \theta_{\mu t_1} & \theta_{\mu y_i(t_1)} & \theta_{\mu t_2} & \theta_{\mu y_i(t_2)} \end{matrix} \right\|$$

vanish;

(c) certain second order conditions of the calculus of variations which we will not give here;

(d) at the corner point P_ω the following corner conditions

$$F_1(1) - F_2(2) - \dot{y}_k'(\omega) F_{1\dot{y}_k}'(1) + \bar{y}_k'(\omega) F_{2\bar{y}_k}'(2)$$
$$+ \int_{t_1}^{\omega} F_{1\omega} dt + \int_{\omega}^{t_2} F_{2\omega} dt = 0, \ k \text{ umbral as before,}$$

$$F_{1y_\sigma}'(1) - F_{2\bar{y}_\sigma}'(2) + \int_{t_1}^{\omega} F_{1\dot{y}_\sigma(\omega)} dt + \int_{\omega}^{t_2} F_{2\bar{y}_\sigma(\omega)} dt = 0, \quad (\sigma = 1, 2),$$

where $F_1(1)$ means $F_1(y_1, y_1', y_2, y_2', t, t_1, \lambda_1)$ with the arguments y_1, y_1', y_2, y_2', λ_1 replaced by the extremal set $\dot{y}_1$, $\dot{y}_1'$, $\dot{y}_2$, $\dot{y}_2'$, λ_1 for the arc P_1P_ω; $F_2(2)$ means $F_2(y_1, y_1', y_2, y_2', t, t_1, \lambda_2)$ with the arguments y_1, y_1', y_2, y_2', λ_2 replaced by the extremal set $\bar{y}_1$, $\bar{y}_1'$, $\bar{y}_2$, $\bar{y}_2'$, λ_2 for the arc $P_\omega P_2$, and $F_{1y_\sigma}'(1)$ etc. are defined in an analogous manner.*

The corner conditions (d) are, of course, only first necessary conditions. The condition that the second variation at the corner point be not positive is a further necessary condition for a maximum, but we will not attempt to derive this second order condition.†

---

* Roos, "A General Problem of Minimizing an Integral with Discontinuous Integrand," *Transactions of the American Mathematical Society*, not yet published. See, also, E. H. Clarke, *On the Minimum of the Sum of a Definite Integral and a Function of a Point, loc. cit.*

† Dresden, *Transactions of the American Matehmatical Society*, Vol. IX, p. 472 (1908), gives the second order corner conditions for the simplest problems of the calculus of variations.

The time $t = \omega$ at which the replacement should occur and the rate of production and price at that time can be determined from the three corner conditions (d). The end times t_1 and t_2 and the end values $y_1(t_1)$, $y_2(t_1)$, $y_1(t_2)$ and $y_2(t_2)$ can be determined from the n end conditions $\theta_\mu = 0$, $(\mu = 1, \cdots, n \leq 6)$ and the $6 - n$ transversality conditions (b).

4. *A replacement example.* Although the theory of the Lagrange problem in the calculus of variations is fairly well developed, there are known comparatively few problems for which the mathematical analysis is sufficiently simple to permit the actual determination of the maximizing or minimizing curves. For this reason it will be all the more interesting and instructive to consider a replacement example which seems to be sufficiently general to typify many economic situations and yet which is sufficiently simple to admit a ready solution.

In order to obtain an expression for the operating expense of a machine or other property we may as well follow the example set by Walras and define certain quantities called coefficients of production.[*] These quantities are defined as the quantities of the services of the factors of production, i. e. services of land, services of persons and services of capital that enter into the manufacture of a unit of a given commodity. As I have already pointed out, there seems to be justification for writing the coefficients of production for a commodity C as functions of the rate of production of C, the selling price of C and the first derivatives of these quantities.[†]

If there is one producer manufacturing an amount $u(t)$ of C in unit time and if, furthermore, the selling price of one unit of C is $p(t)$, the coefficients of production are functions, $f_\alpha(u, u', p, p', t)$, $(\alpha = 1, \cdots, m)$, where m is the number of services required to manufacture one unit of C and primes denote derivatives with respect to time. If we denote the prices of the m services required to produce one unit of C by $p_\alpha(t)$, then by the definition of the coefficients of production the cost of producing $u(t)$ units of C for the first or operating machine is

$$\psi_1(u, u', p, p', t) = \sum_{\alpha=1}^{m} u(t) f_{\alpha 1}(u, u', p, p', t) p_\alpha(t).$$

We may suppose the prices of the services $p_\alpha(t)$ to be known functions of the time.

As an approximation we may assume that the depreciation Δ_1 of the operating machine can be represented by $\Delta_1 = \xi_{11} u(t) + \xi_{12} + \xi_{13} u'(t) u'(t)$

---

[*] V. Pareto, *Manuel d'Economie Politique,* pp. 607.

[†] Roos, "A Dynamical Theory of Economics," *loc. cit.*

$+ \xi_{14}p'(t)p'(t) + \xi_{15}u'(t) + \xi_{16}p'(t)$, where $\xi_{11}, \cdots, \xi_{16}$ are known functions of the time or constants. We may further suppose that the coefficients of production f_{a1} depend only upon the rate of production, but, since there is an overhead expense when no units are produced, we would not be justified in assuming the f_{a1} to be linear functions of the rate of production. We may, however, suppose them to have the form $f_{a1} = a_{a1}u(t) + b_{a1} + c_{a1}/u(t)$, where a_{a1}, b_{a1} and c_{a1} are known functions of the time or constants.

We may, therefore, write the function Q_1, which is the cost of production including depreciation in market value of the machine, as

$$(7) \qquad Q_1 = f_{a1}p_{a1}(t) + \Delta_1 = A_1u^2 + B_1u + C_1 + D_1u'u' \\ + E_1p'p' + F_1u' + G_1p',$$

where

$$A_1 = a_{a1}p_a, \quad B_1 = \xi_{12} + b_{a1}p_a, \quad C_1 = \xi_{12} + c_{a1}p_a, \quad D_1 = \xi_{13}, \\ E_1 = \xi_{14}, \quad H_1 = \xi_{15}, \quad G_1 = \xi_{16}$$

and α is an umbral index with range $1, \cdots, m$ according to the convention that whenever a literal suffix appears twice in a term that term is to be summed for all values of the suffix.* The quantities $A_1, \cdots, G_1$ are, therefore, known functions of the time or constants.

By a similar argument we may write for the replacement machine

$$(8) \qquad Q_2 = f_{a2}p_{a2}(t) + \Delta_2 = A_2u^2 + B_2u + C_2 + D_2u'u' \\ + E_2p'p' + F_2u' + G_2p',$$

where $A_2, \cdots, G_2$ are known functions of the time, analogous, respectively, to $A_1, \cdots, G_1$ defining Q_1.

If we suppose the rate of demand to be a linear function of the price and the rate of change of price, and, furthermore, that as many units are sold as are produced, the demand equation of (5) becomes

$$(9) \qquad u - a(t)p(t) - h(t)p'(t) - b(t) = 0,$$

where a, b and h are known functions of the time or constants.
When the price p and the rate of production u are known at $t_1 = \tau_1$, the equations $\theta_\mu = 0$, $\mu = 1, 2, 3$, of (6) become

$$(10) \qquad \begin{aligned} \theta_1 &= t_1 - \tau_1 = 0 \\ \theta_2 &= u(\tau_1) - u_1 = 0 \\ \theta_3 &= p(\tau_1) - p_1 = 0. \end{aligned}$$

---

* A. S. Eddington, "The Mathematical Theory of Relativity," p. 50, 22.1.

The operator of the machine will desire to choose the rate of production $u(t)$ and the price $p(t)$, satisfying a differential equation of demand (9) and having end values satisfying the end conditions (10), so that they maximize the integral

$$(11)\quad J=\int_{t_1}^{\omega}[pu-A_1u^2-B_1u-C_1-D_1u'u'-E_1p'p'-H_1u'-G_1p']E(t_1,t)\,dt$$
$$+\int_{\omega}^{t_2}[pu-A_2u^2-B_2u-C_2-D_2u'u'-E_2p'p'-H_2u'-G_2p']E(t_1,t)\,dt.$$

We first consider the part of the arc which lies between $t=\tau_1$ and $t=\omega$, i. e. what would correspond to P_1P_ω of section (3). Since the function F_1 of section (3) is $F_1=[pu-A_1u^2-B_1u-C_1-D_1u'u'-E_1p'p'-H_1u'-G_1p'+\lambda_1(t)(u-ap-hp'-b)]E(t_1,t)$, we have*

$$\partial F_1/\partial p=[u-a\lambda_1]E(t_1,t)\,;\qquad \partial F_1/\partial p'=[2E_1p'-G_1-h\lambda_1]E(t_1,t)$$
$$\partial F_1/\partial u=[p-2A_1u-B_1+\lambda_1]E(t_1,t)\,;\qquad \partial F_1/\partial u'=[2D_1u'-H_1]E(t_1,t).$$

The Euler-Lagrange equations (a) in the form $\partial F_1/\partial p=(d/dx)\partial F_1/\partial p'$, etc.. obtained by differentiating (a) with respect to t, become on cancelling the factor $E(t_1,t)$

$$u-a\lambda_1=[2E_1p'-G_1-h\lambda_1]\delta+2E_1'p'+2E_1p''-G_1'-h'\lambda_1-h\lambda_1'$$
$$p-2A_1u-B_1+\lambda_1=[2D_1'u'-H_1]\delta+2D_1'u'+2D_1u''-H_1'.$$

It is possible to carry the problem through as it is, but since we do not want to lose the reader in a mass of detail, we will assume $A_1,\cdot\cdot\cdot,G_1,a,b,h$ and δ to be constant. We may then write the last equations as

$$(11A)\qquad u-a\lambda_1=(2E_1p'-G_1-h\lambda_1)\delta+2E_1p''-h\lambda_1'$$

$$(11B)\qquad p-2A_1u-B_1+\lambda_1=(2D_1u'-H_1)\delta+2D_1u''.$$

If we solve the second of these equations for λ_1 and substitute the resulting value of λ_1 and $\lambda_1'=d\lambda_1/dt$ in the first, we obtain on collecting terms

$$(12)\quad (a-h\delta)(B_1-H_1\delta)+G_1\delta+[1-(a-h\delta)2A_1]u$$
$$+[2A_1h-(a-h\delta)2D_1\delta]u'+[2D_1h\delta-(a-h\delta)D_1]u''$$
$$+D_1hu'''+(a-h\delta)p-(2\delta E_1-h)p'-2E_1p''=0.$$

The demand equation, $u=ap+b+hp'$, gives us by differentiation $u'=ap'+hp''$, $u''=ap''+hp'''$ and $u'''=ap'''+hd^4p/dt^4$. When we substitute these values of $u,\cdot\cdot\cdot,u'''$ in (12), we obtain

---

* We have chosen λ_1 as $\lambda_1E(t_1,t)$. Since t_1 is fixed this choice does not introduce any new complications and allows us to cancel the factor $E(t_1,t)$.

$$(13)\quad L_{14}d^4p/dt^4+L_{13}d^3p/dt^3+L_{12}d^2p/dt^2+L_{11}dp/dt+L_{10}p+p_{10}=0,$$

where

$$\begin{aligned}
L_{14}&=hD_1,\\
L_{13}&=3D_1h\delta-aD_1+hD_1,\\
L_{12}&=2A_1h^2-2ahD_1+2D_1h^2\delta^2+3ah\delta D_1-a^2D_1-2E_1,\\
L_{11}&=(2A_1h^2-2a^2D_1+2ah\delta D_1-2E_1)\delta,\\
L_{10}&=2a(1-A_1a)-h\delta(1-2aA_1),\\
p_{10}&=[1-(a-h\delta)2A_1]b+(a-h\delta)(B_1-H_1\delta)+G_1\delta.
\end{aligned}$$

Since (13) is a fourth order differential equation with constant coefficients, its general solution is

$$(14)\qquad p_1=p_{10}+K_{11}\,e^{m_{11}t}+K_{12}\,e^{m_{12}t}+K_{13}\,e^{m_{13}t}+K_{14}\,e^{m_{14}t},$$

where $m_{11},\cdots,m_{14}$ are the four roots of the quartic $L_{14}m^4+L_{13}m^3+L_{12}m^2+L_{11}m+L_{10}=0$ and $K_{11},\cdots,K_{14}$ are arbitrary constants.

Similarly, if we assume that $A_2,\cdots,G_2$ and δ, for the replacement machine, are constant for the period $\omega\leq t\leq t_2$, we can obtain

$$(15)\qquad p_2=p_{20}+K_{21}\,e^{m_{21}t}+K_{22}\,e^{m_{22}t}+K_{23}\,e^{m_{23}t}+K_{24}\,e^{m_{24}t},$$

with $p_{20},\cdots,m_{24}$ defined by replacing the set $A_1,\cdots,G_1$ in $p_{10},\cdots,m_{14}$ for the operating machine by the corresponding set $A_2,\cdots,G_2$ for the replacement machine.

If we substitute the value of p_1 given by (14) in the equation of demand (9), we obtain on collecting terms

$$(16)\quad u_1=ap_{10}+b+K_{11}(a+hm_{11})e^{m_{11}t}+\cdots+K_{14}(a+hm_{14})e^{m_{14}t}.$$

By a similar procedure with p_2 we obtain

$$(17)\quad u_2=ap_{20}+b+K_{21}(a+hm_{21})e^{m_{21}t}+\cdots+K_{24}(a+hm_{24})e^{m_{24}t}.$$

In order to complete the solution of the problem we must determine the constants $K_{11},\cdots,K_{14}$, $K_{21},\cdots,K_{24}$, the time t_1, the time ω at which the replacement should occur and the time t_2 at which the replacement machine should be scrapped. To determine these eleven quantities we must make use of the end conditions (10), the transversality conditions (b) and the corner conditions (d). We proceed to set up the matrix (b) for the three end conditions (10) and the function F defined as $F=F_1$ on $t_1\leq t\leq\omega$ and $F=F_2$ on $\omega\leq t\leq t_1$. We easily obtain the rectangular matrix with four rows and six columns

$$\begin{Vmatrix} S_1 & \partial F_1(t_1)/\partial p_1' & \partial F_1(t_1)/\partial u_1' & S_2 & \partial F_2(t_2)/\partial p_2' & \partial F_2(t_2)/\partial u_2' \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{Vmatrix}$$

where

$$S_\sigma = (-1)^\sigma [F(t_\sigma) - p_\sigma'(t_\sigma)\, \partial F_\sigma(t_\sigma)/\partial p_\sigma' - u_\sigma'(t_\sigma) \partial F_\sigma(t_\sigma)/\partial u_\sigma'] + I_\sigma \delta \qquad (\sigma = 1, 2)$$

and $\partial F_\sigma(t_\sigma)/\partial p_\sigma'$ means $\partial F_\sigma/\partial p'$ with p_σ, p_σ', u_σ, u_σ' at t_σ substituted for p, p', u, u'. The value of λ_σ to be used in $\partial F_\sigma/\partial p_\sigma'$ is the value of λ_σ given by (11B) and an analogous equation for λ_2. In (11B) p, u, u', u'' must be replaced by p_1, u_1, u_1', u_1'', and similarly for λ_2.

In order that every determinant of order four of this matrix vanish, it is necessary and sufficient that

$$(18) \qquad S_2 = 0, \quad \partial F_2(t_2)/\partial p_2' = 0, \quad \partial F_2(t_2)/\partial u_2' = 0.$$

We have thus three transversality conditions.

From the corner conditions (d) we obtain the three additional conditions

$$(18) \qquad \begin{aligned} & F_1(\omega) - F_2(\omega) - p_1'(\omega)\partial F_1(\omega)/\partial p_1' - u_1'(\omega)\partial F_1(\omega)/\partial u_1' \\ & \qquad + p_2'(\omega)\partial F_2(\omega)/\partial p_2' + u_2'(\omega)\partial F_2(\omega)/\partial u_2' = 0, \\ & \partial F_1(\omega)/\partial u_1' - \partial f_2(\omega)/\partial u_2' \equiv 2D_1 u_1'(\omega) - 2D_2 u_2'(\omega) = 0, \\ & \partial F_1(\omega)/\partial p_1' - \partial F_2(\omega)/\partial p_2' \equiv 2E_1 p_1'(\omega) - 2E_2 p_2'(\omega) \\ & \qquad - [\lambda_1(\omega) - \lambda_2(\omega)] = 0, \end{aligned}$$

where $\partial F_1(\omega)/\partial p_1'$ is defined as $\partial F_1(t_1)/\partial p_1'$ with t_1 replaced by ω.

Since we have required our maximizing arc to be continuous at $t = \omega$, the following equations must be satisfied

$$(19) \qquad p_1(\omega) - p_2(\omega) = 0, \quad u_1(\omega) - u_2(\omega) = 0.$$

For the determination of the eleven quantities t_1, ω, t_2, $K_{11}, \cdots, K_{14}$, $K_{21}, \cdots, K_{24}$ we have, therefore, eleven equations consisting of the three end conditions (10), the three transversality conditions (17), the three corner conditions (18) and the two continuity conditions (19). As one may readily verify, the equations of the highest degree in t_1, ω, t_2, $K_{11}, \cdots, K_{24}$ will be the transversality equation $S_2 = 0$, and the first corner condition of (18). These equations will be quadratics in $K_{11}, \cdots, K_{24}$ and will contain t_1, ω and t_2 as exponents of e. There will in general, therefore, be more than one solution for the eleven quantities $t_1, \cdots, K_{24}$. To determine which solution maximizes J we need sufficiency conditions for this problem. In an actual case we would probably be able to fix t_1 and t_2, and this would simplify matters considerably. The condition $S_2 = 0$ would then drop out, and the only quadratic would be the first corner condition. We could then discard all values of ω which did not lie between t_1 and t_2. The discussion of this phase of the problem is, however, beyond the purpose of this paper.

Let us observe that if we fix all end values and take $\delta \equiv D_1 \equiv \cdots \equiv G_1 \equiv D_2 \equiv \cdots \equiv G_2 \equiv 0$ and $A_1 \equiv A_2$, $B_1 \equiv B_2$ and $C_1 \equiv C_2$, our transversality and corner conditions vanish identically and the differential equation (13) reduces to the differential equation obtained by G. C. Evans in a paper which introduced the new dynamical theory of economics.*

5. *Some further generalizations.* The analogue of the Hotelling hypothesis, that the operator of a machine will do everything in his power to maximize the present value $V_1(t_1)$ of his machine, would require for the replacement problem that the operator endeavor to maximize the sum, $V_1(t_1) + V_2(t_1)$ of section 2. This replacement problem does not seem to be a special case of a very general Mayer problem discussed by Bliss, because of the way in which the corner value ω appears.† It is, however, a special case of the general problem to which the conditions (a), (b), (c) and (d) of section 3 belong, for, we can obtain the Euler-Lagrange, transversality and corner conditions by replacing F_1 and F_2 in (a), (b) and (d) of section 3 by

$$G_1 \equiv F_1 + (K_1 + K_2)E(t_1, t)/\omega - t_1 \quad \text{and} \quad G_2 \equiv F_2, \text{ respectively.}$$

I have already discussed a related depreciation problem for the case of several competing machines all of which continue in operation for the period $t_1 \leq t \leq t_2$, and I do not believe that the introduction of the interest factor would unduly complicate matters.‡ The replacement problem for several competing machines is, however, much more difficult because of the fact that all replacements need not occur at the same time ω, i. e. one machine may be replaced at a time ω_1, another at a time ω_2, and so on.

The replacement problem for the case of n cooperating machines, which we may suppose to be replaced at the n time $\omega_1, \omega_2, \cdots, \omega_n$, ought not, however, to impose any serious mathematical complications. For example, if all of the n cooperating machines produced a like commodity, we would need to maximize an expression of the form

---

* G. C. Evans, "The Dynamics of Monopoly," *American Mathematical Monthly*, Vol. 31, No. 2 (1924).

† Bliss, "The Problem of Mayer with Variable End-points," *Transactions of the American Mathematical Society*, Vol. 19 (1918).

‡ Roos, "A Mathematical Theory of Competition," *American Journal of Mathematics*, Vol. 47, No. 3 (July 1925).

$$J = \int_{t_1}^{\omega_1} f_{11}dt + \int_{\omega_1}^{t_2} f_{12}dt + \int_{t_1}^{\omega_2} f_{21}dt + \int_{\omega_2}^{t_2} f_{22}dt + \cdots + \int_{t_1}^{\omega_n} f_{n1}dt + \int_{\omega_n}^{t_2} f_{n2}dt,$$

where the $f_{i\sigma}$ $(i = 1, \cdots, n;\ \sigma = 1, 2)$, are functions of the rates of production $u_1(t_1), \cdots, u_n(t)$ of the n cooperating machines and of the price $p(t)$, satisfying a single equation of demand $\phi(u_1, u_1', u_2, u_2', \cdots, u_n, u_n', p, p', t) = 0$. The problem is, however, too important to be disposed of in this cursory manner.

It is possible that this cooperation problem might yield the key by which it would be possible to consider a competition problem for a variable number of competitors. Certainly, the cooperation problem for a variable number of competitors can be set up by the methods of the preceding paragraph, for, it is the special case of the general cooperation problem of the preceding paragraph when $f_{i1} \equiv 0$.

There is another important possible direction of extension of this paper. As I have already pointed out in another paper, there are instances when a more general equation of demand

$$\phi(u, u', p, p', t) = \int_{t_1}^{t} \theta(u, u', p, p', t, s)ds$$

should replace the differential equation (5) of demand.[*] This leads to a problem somewhat similar to the problem of maximizing a functional proposed by Hahn.[†]

---

[*] Roos, "A Dynamical Theory of Economic Equilibrium," *Proceedings of the National Academy of Sciences*, Vol. 13 (1927), pp. 280-285.

[†] H. Hahn, "Ueber die Lagrange'sche Multiplicatorenmethode," *Sitzungsberichte der Akademie Wien*, Vol. 131 (1922), pp. 531-550. See, also, citation (*) above.

Book N........
LIBRARY

A Theorem on Correspondences on Algebraic Curves.[*]

BY S. LEFSCHETZ.

1. The object of the present paper is to prove the following

THEOREM. *The Abelian functions attached to a general algebraic curve of genus p, C_p, are without complex multiplication.*

The equivalent geometric property is: *C_p carries only valence correspondences.*

This basic proposition was first stated by Hurwitz[†] and several proofs are found in the literature,[‡] but so far as we know, they are all open to objection if only on the ground of being essentially too complicated. Two proofs are given here, one purely function theoretic, the other geometric,[§] both very straightforward. The method followed consists in showing that: (*a*) If the Abelian functions of every C_p have a complex multiplication, then there exists a fixed complex multiplication common to them all, i. e. independent of C_p. (*b*) There exist C_p's without complex multiplication.

2. Let $\omega = \| \omega_{j\mu} \|$, $(j = 1, 2, \cdots p; \mu = 1, 2, \cdots, 2p)$ be the period matrix of a set of p independent integrals of the first kind $u_1, \cdots, u_p$ attached to C. As Riemann has shown there exists an associated alternating matrix c of order $2p$, with rational terms, such that:[||]

I. $\omega c \omega' = 0$.

II. $\omega c \bar{\omega}'$ is a definite Hermitian matrix.

---

[*] For a résumé of their theory and complete references see the forthcoming Report of the National Research Council: *Selected Topics in Algebraic Geometry*, Chs. 7, 16.

[†] *Mathematische Annalen*, Vol. 28 (1887), pp. 561-585.

[‡] See Severi, *Mathematische Annalen*, Vol. 74 (1913), pp. 515-544; Enriques-Chisini, *Teoria Geometrica delle Equazioni*, Vol. 3, p. 493.

[§] This geometric proof is based on a theorem of Severi's recently completed by Zariski, *American Journal of Mathematics*, Vol. 50 (1928). At the end of his paper Zariski gives another proof of the theorem here considered similar to ours. His two proofs and ours have been obtained independently. In a Note which just appeared in the *Rendiconti dei Lincei*, Ser. 6, Vol. 6 (1927), p. 435, Severi returned to his theorem and completed it.

[||] We use substantially the matrix notations of Bôcher's *Higher Algebra*. The matrix $\| a_{ij} \|$ is designated by a, its transverse by a', the matrix of the conjugates of its terms by $\bar{a}$.

These conditions characterize $\boldsymbol{\omega}$ as a *Riemann matrix* in the sense of Scorza.*

Now any algebraic correspondence T on C results in a relation,

$$\alpha\boldsymbol{\omega} = \boldsymbol{\omega}\boldsymbol{a} \tag{1}$$

(α square of order p, $\boldsymbol{a}$ square of order $2p$ and with terms rational) defining a multiplication of the Abelian functions attached to $\boldsymbol{\omega}$ (more briefly: multiplication of $\boldsymbol{\omega}$). As shown by Scorza (1) leads to a relation

$$\boldsymbol{\omega}\boldsymbol{d}\boldsymbol{\omega}' = 0 \tag{2}$$

analogous to I except that $\boldsymbol{d}$ need not be alternating.

Explicitly (2) implies that

$$\sum_{\mu,\nu} d_{\mu\nu}\omega_{j\mu}\omega_{k\nu} = 0, \quad (j, k = 1, 2, \cdots, p). \tag{3}$$

From this follows that if u, v are arbitrary integrals of the first kind with Ω_μ, Ω_ν' for μth periods, then

$$\sum d_{\mu\nu}\Omega_\mu\Omega_\nu' = 0 \tag{4}$$

Conversely of course if (4) holds whatever u, v so does (3) and also (2), so that (2) and (4) are equivalent.

When $\boldsymbol{c}$ and $\boldsymbol{d}$ are linearly dependent, the multiplication (1) is *ordinary* and T is a *valence* correspondence, otherwise the multiplication is *complex* and T is a *singular* correspondence. Thus valence correspondences are characterized by the fact that they impose no new relation on $\boldsymbol{\omega}$ other than I.

The question arises however if the mysterious relations existing between the periods for $p \geqq 4$ † imply a complex multiplication, or a singular T. The answer according to our theorem is negative.

3. We recall that if we choose a set of retrosections as the fundam circuits on C_p, and the related normal integrals as the independent s integrals, $\boldsymbol{\omega}$ assumes the form

$$\| \delta_{jk};\ \tau_{jk} \|, \quad (j, k = 1, 2, \cdots p)$$

where $\boldsymbol{\delta}$ is the unit matrix of order p. In place of I and II we have now

I′. $\boldsymbol{\tau}$ is symmetrical.

---

* A summary of his work and application to correspondences on a curve with complete references will be found in the Report, Chs. 15, 16, 17.

† When $p = 4$ there is a unique relation discovered by Schottky; beyond that we are completely in the dark.

II′. Let $\tau_{jk} = \tau'_{jk} + i\,\tau''_{jk}$. Then the quadratic form $\sum \tau''_{jk} x_j x_k$ is definite.

The periods τ_{jk}, are analytical functions of $3p - 3$ parameters for $p > 1$, of *one* for $p = 1$.[*]

The existence of a complex multiplication implies the existence of relations

$$(5) \qquad Q_h(\tau_{11}, \tau_{12}, \cdots) = 0; \; h = 1, 2, \cdots, p(p-1),$$

where the Q's are quadratic polynomials in the τ's with integer coefficients.

4. Consider the τ's as coördinates of a point in a function space S_n, $n = p(p+1)/2$. In that space there exists a certain analytical q-spread M_q, $q \leqq 3p - 3$ representing the ∞^q sets of functionally independent normal periods of a generic C_p. For $p = 1$ we have $q = 1$.

Since the coefficients of the Q's in (4) are integers the set of all such systems is denumerable. Let $\{W_i\}, i = 1, 2, \cdots$ be the infinite set of the corresponding spreads in some order. If our theorem is false some W_i passes through every point of M_q. In that case I say that *at least one W carries M_q itself.* Let us assume that this is false. Let $A(\tau_{jk}{}^0)$ be an arbitrary point of M_q. Its neighborhood E_q on M_q is an analytical element represented by parametric equations

$$\tau_{jk} - \tau_{jk}{}^0 = \phi_{jk}(t_1, \cdots, t_q), \; |t_h| < \alpha$$

where the τ's are power series zero at $t_1 = \cdots = t_q = 0$, convergent and with a Jacobian matrix of rank q within the range considered. The intersections of E_q with the W's are a denumerable set $\sum$ of analytical spreads of less than q dimensions. Since the sections of E_q by a set of parallel hyperplanes of S_n constitute a non denumerable set of non intersecting analytical $q - 1$ spreads on E_q, one of them at least will be different from every one of those of $\sum$. Hence E_q carries a $q - 1$ spread behaving exactly like M_q with respect to the set $\{W_i\}$. Proceeding step by step we come to a situation such as that assumed at the start but with $q = 1$. Assume then $q = 1$. If no W_i carries M_1, or a definite analytical element E_1 of M_1, it intersects it in a finite number of points. Hence the intersections of the W's with E_1 constitute a denumerable set of points on E_1. Every point of E_1 belongs to the set—which is absurd since E_1 is a two dimensional real analytical element and the set of all its points has the power of the continuum. It follows that M_q is carried by at least one W. In other words, *either our theorem holds or else there is a fixed complex multiplication for the period matrix* ω *of every* C_p.*

---

[*] See for example Picard, *Cours d'Analyse*, Vol. 2, 3rd ed., p. 578.

The task that we are facing is then to show that for every p *there is at least one curve of that genus whose* ω *is without complex multiplication or whose correspondences all have valence.*

We shall show that there exists for every p at least two systems of curves having the requisite property, one of them composed of arbitrary hyperelliptic curves.

5. *Proof that arbitrary hyperelliptic curves carry only valance correspondences.* Consider the general hyperelliptic curve C_p:

$$y^2 = \prod_{i=1}^{2p+2} (x - a_i). \tag{6}$$

We have to show that when the a's are arbitrary there is no complex multiplication attached to the period matrix ω. This is true for $p = 1, 2$, since the corresponding period matrices are arbitrary Riemann matrices of their genus. We shall therefore assume $p > 2$.

Let K be the two-sheeted Riemann surface of C_p, L_μ a loop drawn on K from $x = a$ to the branch point a_μ in the first sheet and back to the point $x = a$ in the second. We assume that when we turn around the point in the positive direction the loops follow one another around $x = a$ in the order in which they are numbered.

The system of linear cycles

$$\begin{aligned}\gamma_{2\mu-1} &= L_{2\mu} - L_{2\mu-1}\\ \gamma_{2\mu} &= -L_{2\mu} + L_{2\mu+1} - L_{2\mu+2} + \cdots + L_{2p+1}\end{aligned}$$

constitutes a set of retrosections. An equivalent system is as follows: We draw non-intersecting cuts $a_{2\mu-1}a_{2\mu}$. Then $\gamma_{2\mu-1}$ is deformable into a circuit surrounding the cut just named in the first sheet, $\gamma_{2\mu}$ into a circuit crossing the cuts of order μ and $p + 1$ and no others. The circuits can furthermore be so drawn that $\gamma_{2\mu-1}$ and $\gamma_{2\mu}$ cross one another but meet no other γ.†

---

* A highly sophisticated way of looking at the matter is as follows: It can readily be shown that of two analytical elements E_q, E_r, either one carries the other, or else their intersection has relatively to one of them, say E_q zero Lebesgue measure,—the measure being taken with respect to E_q and its $2q$ dimensional real volume elements. Hence our theorem is equivalent to the statement that the W's intersect E_q in a set of measure zero, or, in the usual language of real variables to the following: *Almost all algebraic curves carry only valence correspondences.*

† See for example Appell et Goursat, *Théorie des fonctions algébriques*, p. 124.

6. Let then u, v be two integrals of the first kind attached to C, Ω_μ, Ω_μ', their periods relatively to γ_μ. Since the γ's are retrosections we have

$$(7) \qquad \sum (\Omega_{2\mu-1}\Omega'_{2\mu} - \Omega_{2\mu}\Omega'_{2\mu-1}) = 0$$

and we must show that any bilinear relation with integer coefficients,

$$(8) \qquad \sum c_{\mu\nu}\,\Omega_\mu\Omega'_\nu = 0,$$

satisfied whatever u, v, necessarily reduces to the preceeding. In so doing we may assume $p > 2$, and the theorem proved for $p - 1$, then proceed by induction.

The method to be followed will consist in making $a_{2\mu-1} \rightarrow a_{2\mu}$, $\mu \leqq p$, and finding the limits of the Ω's for u, v suitably chosen. From (8) we shall thus deduce a limiting relation for a curve of genus $p - 1$ from which our theorem will readily follow.

7. It will be convenient to write $a_{2\mu-1} = z$, $a_{2\mu} = z'$. We choose

$$u = \int [P(x)(x - z')/y]\,dx, \quad v = \int [Q(x)(x - z')/y]\,dx,$$

P and Q linearly independent polynomials of degree $p - 2$. We are going to assume that $z' \rightarrow z$ in such manner that $z - z'$ remains real and positive. In other words z' approaches z along a segment parallel to the real axis and to the right of the point.

8. Let us introduce the auxiliary C_{p-1}:

$$y_1^2 = (x - a_1) \cdot\cdot\cdot (x - a_{2\mu-2})(x - a_{2\mu+1}) \cdot\cdot\cdot (x - a_{2p+2}).$$

We have in

$$u_1 = \int (P/y_1)\,dx, \quad v_1 = \int (Q/y_1)\,dx$$

two perfectly arbitrary integrals of the first kind attached to C_{p-1}.

Let K_1 be the Riemann surface of C_{p-1} derived from K by suppressing the branch points $a_{2\mu-1}$, $a_{2\mu}$.

The circuits γ_ρ, $\rho \neq 2\mu - 1$ or 2μ, constitute a set of restrosections for K_1. Let $\Omega_{1\rho}$, $\Omega'_{1\rho}$ be the corresponding periods of u_1, v_1. When $z' \rightarrow z_1$ it is at once seen that $\Omega_\rho \rightarrow \Omega_{1\rho}$, $\Omega_\rho' \rightarrow \Omega'_{1\rho}$, $\rho \neq 2\mu - 1$ or 2μ. Furthermore $\Omega_{2\mu-1}$ and $\Omega'_{2\mu-1}$ both $\rightarrow 0$ for in the limit they are equal to u_1, v_1, integrated along a circuit surrounding an ordinary point of K_1.

9. There remains then merely to find the limits of $\Omega_{2\rho}$, $\Omega'_{2\mu}$. Let us confine z' to a circle of center z and radius ϵ as small as we please and in any case such that all branch points other than z and z' are exterior to it. We have then

$$\Omega_{2\mu} = 2\int_{z'}^{z+\epsilon} du + 2\int_{z+\epsilon}^{a_{2p+1}} du = 2I_1 + 2I_2 .$$

Here I_1 and I_2 designate the two definite integrals, the paths of integration being a straight line for I_1, some finite regular arc exterior to the above circle for I_2. The latter is a holomorphic function of z' about the point z therefore

$$\lim_{z'\to z} \int_{z+\epsilon}^{a_{2p+1}} du = \int_{z+\epsilon}^{a_{2p+1}} du_1 .$$

Within and on a certain circle $|x - z| = \eta$, where $\eta > \epsilon$ and is to be fixed in the sequel, $P(x)/y_1$ is holomorphic hence within that same circle, $|P(x)/y_1| < M$, a fixed number, and

$$\lim_{z'\to z} \int_{z+\epsilon}^{a_{2p+1}} du = \int_{z}^{a_{2p+1}} du_1 + \zeta,\ |\zeta| < M$$

the path of integration being some finite regular arc joining the end points and avoiding the critical points. Also $\xi = x - z$ is real on the path of integration of I_1 with $0 < z' - z \leqq \xi \leqq \epsilon$. Hence on that path

$$[(x-z')/(x-z)]^{1/2} = [1-(z'-z)/\xi]^{1/2} \text{ is real and } \leqq 1.$$

Therefore on the path of integration the integrand of I_1 is in absolute value $< M$, and $|I_1| < M(z+\epsilon-z') < 2M\epsilon$.
It follows

$$\left| \lim_{z'\to z} \Omega_{2\mu} - 2\int_{z}^{a_{2p+1}} du_1 \right| < 6\,M\epsilon$$

and since ϵ is arbitrarily small $\quad \Omega_{2\mu} \to 2\int_{z}^{a_{2p+1}} du_1$.

We can set by definition $\quad u_1 = \int_{a_{2p+1}}^{\infty} du_1$. Then $\Omega_{2\mu} \to -2u_1(z)$.

With a similar situation for v_1 we shall have $\Omega'_{2\mu} \to -2v_1(z)$.

10. The relation (8) will then yield in the limit a relation

$$(9) \qquad Au_1(z)v_1(z) + Bu_1(z) + Cv_1(z) + \sum c_{\rho\sigma}\Omega_{1\rho}\Omega'_{1\sigma} = 0$$

where A, B, C are independent of z and ρ, σ run from 1 to $2p$ omitting $2\mu - 1$

and 2μ. If z follows the closed circuit γ_ν, $\nu \neq 2\mu - 1, 2\mu$, the increment of the left side must be zero. Hence

$$A(\Omega_{1\nu} v_1(z) + \Omega'_{1\nu} u_1(z)) = \text{const.}$$

Since the periods Ω_1 are not all zero, and since u_1, v_1 are like P, Q, linearly independent, it follows that $A = 0$. For the same reason then $B = C = 0$. Therefore (9) reduces to

$$\sum c_{\rho\sigma} \Omega_{1\rho} \Omega'_{1\sigma} = 0; \quad \rho, \sigma \neq 2\mu - 1 \text{ or } 2\mu.$$

But C_{p-1} is an arbitrary hyperelliptic curve of genus $p - 1 \geqq 2$, and u_1, v_1, are arbitrary integrals of the first kind attached to it. Our theorem being assumed true for $p - 1$ it follows that

$$c_{2\nu-1,\, 2\nu} = -c_{2\nu,\, 2\nu-1} = c, \text{ a fixed integer, } \nu \neq \mu$$

while all other coefficients $c_{\rho\sigma}$ are zero, $\rho, \sigma \neq 2p - 1$ or 2μ.

Since $p > 2$ there are at least two positive integers $\rho, \sigma \leqq p$ and $\neq \mu$. Bring into coincidence $a_{2\rho-1}$ and $a_{2\rho}$ but no other critical points. This time we shall find

$$c_{2\mu-1,2\mu} = -c_{2\mu,2\mu-1} = c$$
$$c_{2\mu-1,\sigma} = c_{\sigma,2\mu-1} = c_{2\mu,\sigma} = c_{\sigma,2\mu} = 0.$$

Since σ is any integer $\leqq p$ and $\neq \mu$, it follows ultimately that $c_{2\mu-1,\, 2\mu} = -c_{2\mu,\, 2\mu-1} = c$, whatever $\mu \leqq p$, all other coefficients $c_{\rho\sigma}$ being zero. In other words (8) merely reduces to (7) which proves our theorem.

11. *Another type of curves carrying only valence correspondences.* Its existence is established by means of the following proposition established by Zariski in the paper already quoted: *On the general curve C of a simple linear system** *drawn on a regular algebraic surface*† *all correspondences have a valence.*

Let x, y, z be homogeneous plane coordinates and consider the linear system $|C|$ of all curves

$$\sum_{q=0}^{\mu} \sum_{s=0}^{\nu} a_{qs}\, x^q y^s z^{m-q-s} = 0, \qquad (\mu + \nu = m). \tag{10}$$

---

* A linear system $|C|$ of curves on the surface F is *simple* whenever the curves C that pass through an arbitrary point A of F are not constrained to pass through other points of F variable with A.

† The surface F is *regular* whenever its linear connection index is zero or, equivalent condition, when it possesses no simple Picard integrals of the first kind. If every circuit on F can be deformed into a point it is certainly regular. It follows that the plane is regular—which is all that we need here.

We shall show that μ, ν can be so chosen that the generic C be of genus p, and that the system be simple. By Zariski's theorem it will then have the requisite property.

12. The generic curve C has just two multiple points with distinct tangents at $(1, 0, 0)$ and $(0, 1, 0)$ with respective multiplicities ν, μ. Its genus is therefore

$$p = \binom{m-1}{2} - \binom{\mu}{2} - \binom{\nu}{2} = (\mu - 1)(\nu - 1)$$

It is evident that given p we can always find μ, ν and therefore a $| C |$ whose curves are of genus p.*

The *dimension* of $| C |$ is one less than the number of distinct coefficients a_{qs} in (10), or $r = (\mu + 1)(\nu + 1) - 1$. The *degree* of $| C |$, or number of variable intersections of two arbitrary C's is

$$d = (\mu + \nu)^2 - \mu^2 - \nu^2 = 2\mu\nu.$$

I say that $| C |$ is simple. For in the contrary case every C through an arbitrary point A of the plane would be constrained to pass through $k - 1 > 0$ other points variable with A. But there are at least two C's going through $r - 1$ arbitrary points. Therefore any two C's would intersect in no less than $(r - 1)k$ variable points and $d \geqq (r - 1)k$. This gives

$$\begin{gathered} 2\mu\nu \geqq k(\mu + 1)(\nu + 1) - 2k \\ (k - 2)\mu\nu + k(\mu + \nu - 1) \leqq 0 \end{gathered}$$

which is manifestly impossible since $k \geqq 2$, $\mu \geqq 1$, $\nu \geqq 1$. Thus $| C |$ is simple and a generic C carries only valence correspondences.

13. *Conclusion.* We have reduced our basic theorem to the more special problem is finding some curves of genus p that carry only valence correspondences. We have two types of curves of the required nature. For the first—generic hyperelliptic curves—the proper behavior has been established by a strictly transcendental method. For the second the same result has been achieved by a geometrical procedure based in part on a theorem due to Zariski. We have therefore given two essentially distinct proofs of the theorem.

PRINCETON UNIVERSITY.

---

* Snyder has established the existence of curves of any genus p on a non-degenerate quadric, *Bulletin of the American Mathematical Society*, Vol. 15 (1908-09), pp. 1-4. If the quadric is mapped perspectively on a plane his curves give rise to the linear system here considered.

Concerning the Structure of a Continuous Curve.[*]

By Gordon T. Whyburn.

1. *Introductory. Notation and Definitions. Cyclic Elements.* A continuous curve M is said to be cyclicly connected [†] provided every two points of M lie together on some simple closed curve which is a subset of M. A cyclicly connected continuous curve C which is a subset of a continuous curve M is said to be a maximal cyclic curve of M if and only if C is not a proper subset of any other cyclicly connected continuous curve which is also a subset of M. The point P of a continuous curve M is an endpoint of M if and only if no simple continuous arc of M has P as one of its interior points.[‡] The point P of a *continuum* M is an endpoint of M provided that if N is any subcontinuum of M containing P, then P is not a limit point of any connected subset of $M - N$.[§] In this paper the ordinary notation will be used for the sum, product, and difference of point sets. If a symbol X is used to denote a point set, the symbol $\overline{X}$ will be used to denote the set X plus all the limit points of the set X. It is to be understood that the point sets considered in this paper lie in a Euclidean plane.[||] Also the hypothesis that "M is a continuous curve" is understood to imply that the point set M is bounded.

A subset E of a continuous curve M will be called a cyclic element of M

---

[*] Presented to the American Mathematical Society December 31, 1926. The results in § 5 were presented to the Society June 18, 1927.

[†] G. T. Whyburn, "Cyclicly Connected Continuous Curves," *Proceedings of the National Academy of Sciences*, Vol. 13 (1927), pp. 31-38.

[‡] This definition is equivalent to the one given by R. L. Wilder in his paper "Concerning Continuous Curves," *Fundamenta Mathematicae*, Vol. 7 (1925), p. 358. For a proof of this fact see my paper "Concerning Continua in the Plane," *Transactions of the American Mathematical Society*, Vol. 29 (1927), No. 2, pp. 369-400, Theorem 12, p. 385.

[§] See G. T. Whyburn, *loc. cit.*, p. 382. See also H. M. Gehman, *Transactions of the American Mathematical Society*, Vol. 30 (1928), pp. 63-84.

[||] With the aid of some results established recently by Dr. W. L. Ayres on the cyclic elements of a continuous curve in n dimensions it follows that practically all the theorems in the present paper hold true in n dimensions. I have found that all except numbers 24-27 and 29 so hold. In most cases the arguments as given extend to n dimensions without modification, although in a few instances a slightly different proof is necessary.

provided that E is either (a) a maximal cyclic curve of M, (b) a cut point* of M, or (c) an endpoint of M. The cyclic elements of a continuous curve will be referred to by classes (a), (b), and (c) as here defined. That every continuous curve is the sum of its cyclic elements follows from the results, in particular Theorem 7, of my paper "Cyclicly Connected Continuous Curves," cited above. On account of the frequent references to that paper, it will be referred to hereafter as C. C. The following conclusions are easily deduced from the definition of the cyclic elements of a continuous curve M: (1) no two cyclic elements of M have more than one point in common, (2) the common part of every pair of cyclic elements of M is either vacuous or is itself a cyclic element of class (b), and (3) if N, K, and H respectively denote the sum of all the cyclic elements of classes (a), (b), and (c) respectively, then (I) $N \cdot K$ is countable, and (II) $N \cdot H = K \cdot H = 0$. By an *internal point* of a maximal cyclic curve C of a continuous curve M will be meant a point of C which is not a cut point of M. The notation $I(C)$ will be used to designate the set of all the internal points of maximal cyclic curve C of a continuous curve. It is readily seen from (I) above that the set of all the non-internal points of a maximal cyclic curve must be countable. Also it is easy to see that no two maximal cyclic curves of a continuous curve can have an internal point of either in common without being identical.

By an acyclic continuous curve is meant a continuous curve which contains no simple closed curve. In this paper it will be shown in §§ 2 and 3 that if we regard a continuous curve as being composed of its cyclic elements instead of points, then *every continuous curve* is acyclic with respect to these elements. Many of the characteristic properties of acyclic continuous curves are established for any continuous curve which is regarded as being composed of its cyclic elements. In particular it is shown that every continuum of cyclic elements is a continuous curve, and that every connected set of cyclic elements is arcwise connected in the ordinary sense. In § 4 the notions of the *nodes* and *endelements* of a continuous curve are introduced and characterized in various ways, and some of the familiar propositions concerning the endpoints of acyclic continuous curves are shown to hold for the endelements of any continuous curve. In §§ 5 and 6 some theorems will be established relating to the structure of a continuous curve from the ordinary point of view.

---

* The point P of a connected set M is a cut point of M provided that the set of points $M - P$ is not connected. Cf. R. L. Moore, "Concerning the Cut Points of Continuous Curves and of other Closed and Connected Point Sets," *Proceedings of the National Academy of Sciences*, Vol. 9 (1923), pp. 101-106.

2. *Simple Cyclic Chains.*

Definition. A point set X will be called a ***simple cyclic chain*** of M between two cyclic elements E_1 and E_2 of M provided that X is connected and contains E_1 and E_2 and is the sum of the elements of some collection of the cyclic elements of M, and furthermore no proper connected subset of X contains E_1 and E_2 and is the sum of the elements of such a collection.

In my paper "Some Properties of Continuing curves,"* I showed that if A and B are any two points of a continuous curve M, K denotes the set of all those points of M which separate† A and B in M, and t is any arc in M from A to B, then (1) $K+A+B$ is a closed set of points, and (2) each maximal segment S of $t-(K+A+B)$ determines a unique maximal cyclic curve of M which contains S.

THEOREM 1. *In order that the point set X should be a simple cyclic chain of a continuous curve M between two cyclic elements E_1 and E_2 of M it is necessary and sufficient that if t is any arc in M joining a point A of E_1 and a point B of E_2, and K denotes the set of all those points of M which separate A and B in M, then X is the set of points E_1+E_2+K+ all those maximal cyclic curves of M determined by the maximal segments of $t-(K+A+B)$.*

Proof. The condition is sufficient. Let X denote the set of points E_1+E_2+K+ all those maximal cyclic curves of M determined by the maximal segments of $t-(K+A+B)$ as described in the statement of this theorem. Then X is a simple cyclic chain of M from E_1 to E_2. Evidently X contains E_1 and E_2 and is the sum of the elements of a collection of cyclic elements of M. That X is connected follows immediately from the fact that X contains t and every cyclic element of M belonging to X has at least one point in common with t. No proper subset of X has these properties. For suppose, on the contrary, that some proper subset Y of X does have these properties. Then there exists at least one cyclic element C of M which is a subset of X but not of Y. Since each point of K separates A and B in M, and since Y is connected and contains both A and B, then Y must contain every point of K. Therefore C must be a maximal cyclic curve of M determined by some maximal segment S of $t-(K+A+B)$.

---

[*] *Bulletin of the American Mathematical Society*, Vol. 33 (1927), pp. 305-308.

[†] The point X of a connected set M is said to separate in M the two points A and B of M provided that $M-X$ is the sum of two mutually separated point sets M_a and M_b containing A and B respectively.

Let E and F be the endpoints of S. Since E and F belong to $K + A + B$, therefore they belong also to Y. Now Y can contain no internal point of C, for if it did it would necessarily contain all of C. Hence $Y \cdot C$ is countable. Since $Y \cdot C$ is countable and contains at least two points, it is not connected and hence can be expressed as the sum of two mutually separated point sets H_1 and H_2. For each point P of Y which does not belong to C, let T_p denote the maximal connected subset of $M - C$ which contains P. By theorem 2 of C. C., for each P, the set T_p has exactly one limit point Q which belongs to C. It is clear that for each P, the point Q must belong to Y and hence must belong either to H_1 or to H_2. Let G_1 be the set of all the points P of $Y - Y \cdot C$ such that the corresponding point Q belongs to the set H_1, and let G_2 be the set of all those points of $Y - Y \cdot C$ such that the corresponding point Q belongs to H_2. Then clearly $Y = H_1 + H_2 + G_1 + G_2$. Now no point of H_1 is a limit point of G_2. For suppose, on the contrary, that some point Z of H_1 is a limit point of G_2. Then since for no point P of G_2 is Z a limit point of the set T_p, it follows that there exists in G_2 an infinite sequence of points $P_1, P_2, P_3, \cdots$, having Z as its sequential limit point and such that the sets $T_{p_1}, T_{p_2}, T_{p_3}, \cdots$, are all distinct and have no points in common. But for each i, T_{p_i} has a limit point Q_i which belongs to H . Now for each point Q of the set $Q_1 + Q_2 + Q_3 + \cdots$, there are not more than a finite number of sets T_{p_i} which have Q for a limit point. For if the contrary were true, then since Z is the sequential limit point of the sequence $P_1, P_2, P_3, \cdots$, and since for any given number $\epsilon > 0$, not more than a finite number of the sets T_p are of diameter $> \epsilon$, it would follow that Q, a point of H_2, is a limit point of the set $P_1 + P_2 + P_3 + \cdots$, and therefore is identical with Z, a point of H_1, which is impossible. It follows, then, that the set of points $Q_1 + Q_2 + Q_3 + \cdots$ contains infinitely many distinct points, and since Z is a limit point of the sequence $P_1, P_2, P_3, \cdots$, it readily follows that Z is also a limit point of the set $Q_1 + Q_2 + Q_3 + \cdots$. But this is impossible, for Z belongs to H_1 and $Q_1 + Q_2 + Q_3 + \cdots$ belongs to H_2, and H_1 and H_2 are mutually separated sets. Thus the supposition that H_1 contains a limit point of G_2 leads to a contradiction. Now clearly no point of G_2 can be a limit point of H_1, for H_1 belongs to C, and G_2 belongs to $M - C$. Therefore H_1 and G_2 are mutually separated point sets. A similar argument shows that H_2 and G_1 are mutually separated sets. Then since H_1 and H_2 and also G_1 and G_2 are mutually separated, it follows that the sets $(G_1 + H_1)$ and $(G_2 + H_2)$ are mutually separated. But the sum of these two sets is Y, and Y, by hypothesis, is connected. Thus the supposition that the set Y exists having the properties of the set X leads to a contradiction, and it follows that X is a simple cyclic chain of M from E_1 to E_2.

The condition is also necessary. For let Y be any simple cyclic chain of M between two cyclic elements E_1 and E_2 of M. Let X be the set of points $E_1 + E_2 + K +$ all those maximal cyclic curves of M determined by the maximal segments of $t - (K + A + B)$, where t is any arc in M from a point A of E_1 to a point B of E_2, and K is the set of all those points of M which separate A and B in M. It was shown above that X is a simple cyclic chain of M from E_1 to E_2. I shall show that Y must be identical with X. Suppose this is not so. Then X contains a cyclic element C of M which is not a subset of Y. Then by an argument identical with that given in the first part of the proof of this theorem, beginning with the sentence "Since each point of K separates A and B in M, . . . ," it is shown that this supposition leads to a contradiction. Hence Y is identical with X, and the theorem is proved.

The method of proof used under Theorem 1 yields immediately the following theorem.

THEOREM 2. *Let X be a simple cyclic chain between two cyclic elements* E_1 and E_2 *of a continuous curve M. Let C be any cyclic element of class* (a) *belonging to X and identical with neither E_1 nor E_2. Let E and F be the endpoints of the segment of t which determines C (see Theorem* 1). *Then $C - (E + F)$ separates E_1 and E_2 in M, i. e., $M - [C - (E + F)]$ is the sum of two mutually separated sets containing E_1 and E_2 respectively. Likewise $I(C)$ separates E_1 and E_2 in M.*

THEOREM 3. *If E_1 and E_2 are any two cyclic elements of a continuous curve M, then there exists in M one and only one simple cyclic chain from E_1 to E_2.*

Proof. Let t be any arc in M from a point A of E_1 to a point B of E_2. Let X be defined as in Theorem 1. Then by Theorem 1, X is a simple cyclic chain of M from E_1 to E_2. That X is the only simple cyclic chain of M from E_1 to E_2 follows at once from Theorem 1.

THEOREM 4. *A simple cyclic chain is a continuous curve.*

Proof. Let X be a simple cyclic chain of M between two cyclic elements E_1 and E_2 of a continuous curve M. Let A and B denote points of E_1 and E_2 respectively and t any arc in M from A to B. Then by Theorem 1, X is the set of points $E_1 + E_2 + K +$ all those maximal cyclic curves of M determined by the maximal segments of $t - (K + A + B)$, where K is the set of all those points of M which separate A and B in M. By definition X is connected. And since, for any positive number ϵ, at most a finite number of the cyclic elements of M are of diameter $> \epsilon$, since t belongs to X, and each cyclic element of M which belongs to X contains a point of t, it readily follows that X is closed. Hence X is a continuum. The continuum X is also regular (connected im kleinen). For let P be a point of X and C any circle with P as center. Now if P is on t, there exists a circle C_1 with P as center and such that all those points of t lying within C_1 belong to some subarc of t lying within C and containing P. There exists a circle C_2 concentric with C_1 and such that every cyclic element of M which has a point within C_2 either lies wholly within C_1 or else it contains P. If N is any cyclic element lying wholly within C_1 and belonging to X, then since N has a point in common with t, it follows that N lies together with P in a connected subset of M lying within C. And since only a finite number of cyclic elements contain P and also some point without C, and each cyclic element is a continuous curve, it follows that there exists a circle C_3 concentric with and within C_2 and such that every point of X interior to C_3 lies together with P in a connected subset of M lying within C. Hence X is regular at P if P is on t. Now if P is nto on t, then P belongs to some maximal cyclic curve N of M, and is not a limit point of $X - N$. Hence in this case also X is regular at P. Thus we have shown that X is a continuous curve.

DEFINITION. If X is a simple cyclic chain between two cyclic elements E_1 and E_2 of a continuous curve M, the elements E_1 and E_2 will be called the *endelements of M.*

It may happen, in case E_1 or E_2 or both is a cut *point* of M, that some cyclic element of M different from E_1 belongs to X and contains E_1, and similarly for E_2. This will be the case when E_1 is not a limit point of the set K of points which separate in M a point A of E, and a point B of E_2. In this case, $E_1(E_2)$ is an endpoint of a maximal segment of $t - (K + A + B)$, where t is any arc in M from A to B, and clearly not more than one cyclic element of M different from $E_1(E_2)$ can belong to X and contain $E_1(E_2)$. It will be seen later, when the term endelement is defined for continuous curves in general, that any element of X which contains either E_1 or E_2 is also an endelement of X under the generalized definition. However, in this paper wherever we speak of an "endelement of a simple cyclic chain," we mean an endelement in the sense as above defined. All the cyclic elements of M which belong to X and contain neither E_1 nor E_2 will be called the *interior* elements of the chain X.

THEOREM 5. *If X is a simple cyclic chain between two cyclic elements E_1 and E_2 of a continuous curve M, and C is any interior element of X, then $X - C = X_1 + X_2$, where X_1 and X_2 are mutually separated point sets, and (1) X_1 and X_2 are connected and contain points of E_1 and E_2 respectively, and (2) $X_1 + C$ and $X_2 + C$ are simple cyclic chains from E_1 to C and from E_2 to C respectively.*

Proof. Since C contains neither E_1 nor E_2, therefore, by Theorem 1, C is either a point of K or a maximal cyclic curve of M which contains at least one point of K, where K denotes the set of all those points of M which separate in M a point A of E_1 (an internal point of E_1 in case E_1 is an element of class (a)) and a point B of E_2. In either case it is easy to see that $X - C$ is not connected and that it can be expressed as the sum of two mutually seperated sets X_1 and X_2 containing points A and B of E_1 and E_2 respectively.

(1.) The sets X_1 and X_2 are connected. For suppose one of them, say X_1, is not connected. Then let Y denote the maximal connected subset of X_1 containing A. Now since, by Theorem 4, X is a continuous curve, it follows that $\overline{X}_2$ and $\overline{Y}$ are each the sum of the elements of some collection of cyclic elements of M, and that C contains a limit point of Y. Then since $X_2 + C$ is connected, $Y + X_2 + C$ is connected and contains E_1 and E_2 and is the sum of the elements of some collection of cyclic elements of M. But X is a simple cyclic chain of M from E_1 *to* E_2, and $Y + X_2 + C$ is a proper subset of X. This is contrary to the definition of a simple cyclic chain. It follows that X_1 is connected, and a similar proof shows that X_2 is connected.

(2.) The set of points $X_1 + C$ is a simple cyclic chain from E_1 to C. For $X_1 + C$ is connected and contains both E_1 and C and is the sum of the elements of some collection of cyclic elements of M. Furthermore $X_1 + C$ is irreducible with respect to these properties, for any assumption to the contrary leads immediately to the conclusion that X is reducible with respect to similar properties between E_1 and E_2, contrary to the fact that X is a simple cyclic chain from E_1 to E_2. Therefore $X_1 + C$ is a simple cyclic chain from E_1 to C, and a similar proof shows that $X_2 + C$ is a simple cyclic chain from E_2 to C. This completes the proof of the theorem.

THEOREM 6. *A simple cyclic chain X of a continuous curve M is not disconnected by the omission of either one or both of its endelements, E_1*

and E_2 nor by the omission of either one or both of the cyclic elements of M (if such exist) which belong to X and contain E_1 or E_2.

Proof. Let X_1 denote the set of points $X - E_1$. Then by almost the same argument as given in part (1) of the proof of Theorem 5, it is shown that X_1 (if it exists, of course) is connected. Similarly it is shown that $X - E_2$ is connected. Now let H denote the set of points $X - (E_1 + E_2)$. Suppose, contrary to this theorem, that H is not connected. Then since, by Theorem 4, X is a continuous curve, it follows that some maximal connected subset Y of H has at least one limit point in each of the sets E_1 and E_2. Then $E_1 + E_2 + Y$ is a connected point set which contains E_1 and E_2 and is the sum of the elements of some collection of the cyclic elements of M. But $E_1 + E_2 + Y$ is a proper subset of X, and by hypothesis X is a simple cyclic chain from E_1 to E_2. Thus the supposition that H is not connected leads to a contradiction.

Now if in the argument just given, we replace E_1 and E_2 by E_1^0 and E_2^0, where E_1^0 and E_2^0 respectively denote the cyclic elements of M (if any such exist) which lie in X and contain E_1 and E_2 respectively, we arrive at the conclusion that $X - E_1^0$, $X - E_2^0$, and $X - (E_1^0 + E_2^0)$ are connected point sets. This completes the proof of the theorem.

The close relationship between a simple cyclic chain between two cyclic elements E_1 and E_2 of a continuous curve M and a simple continuous arc is at once apparent from the foregoing propositions and discussion. The definition given above for a simple cyclic chain parallels Lennes'* definition of a simple continuous arc. However, the analogy between the two sets is not complete, due to the fact that two cyclic elements of a continuous curve may have a point in common without being identical. In the case of an arc, it is true † that a necessary and sufficient condition that a bounded continuum be a simple continuous arc between two of its points A and B is that it be disconnected by the omission of any one of its points different from A and from B. Theorem 5 shows that a simple cyclic chain is disconnected by the omission of any one of its interior elements. It is not sufficient, however, in order that a bounded continuum which is the sum of the elements of a collection of cyclic elements of a continuous curve M should be a simple cyclic chain between two of its elements E_1 and E_2 that it should be disconnected by the omission of any one of its cyclic elements different from E_1 and E_2. For let E_1 and E_2 be two circles each without the other except for P and which are tangent to each other at the point P. Let C_1 and C_2 be circles lying within E_1 and E_2 respectively except for the point P and which are also tangent at P. Let M denote the continuous curve $E_1 + E_2 + C_1 + C_2$. The cyclic elements of M are E_1, E_2, C_1, C_2, and P, and the omission of *any* cyclic element of M disconnects M. But M is not a simple cyclic chain from E_1 to E_2.

---

* N. J. Lennes, "Curves in Non-Metrical Analysis Situs with an Application in the Calculus of Variations," *American Journal of Mathematics*, Vol. 33 (1911), p. 308.

† Cf. R. L. Moore, "Concerning Simple Continuous Curves," *Transactions of the American Mathematical Society*, Vo. 21 (1920), pp. 333-347.

3. *Some Acyclic Properties of Continuous Curves in General.* In this section it is purposed to show that many of the characteristic properties of acyclic continuous curves are properties of any continuous curve which is regarded as being composed of its cyclic elements. It is well known that a simple closed curve may be defined as a point set which is the sum of two simple continuous arcs having in common only their two endpoints. In strict analogy, let us define a *closed curve of cyclic elements* of a continuous curve M as a point set which is the sum of two *different* simple cyclic chains of M which have in common only their two endelements. Then Theorem 3 shows that there does not exist any closed curve of cyclic elements of a continuous curve. Thus we get the following interesting theorem.

THEOREM 7. *Every continuous curve is acyclic with respect to its cyclic elements.*

Thus it is apparent that so long as we are dealing with a single continuous curve, a study of the *closed curves* of its cyclic elements is vacuous. However, a consideration of such curves in the study of two or more continuous curves having points in common will probably lead to interesting conclusions. It is not the purpose of this paper to proceed any further in this direction than merely to mention the possibility of further research.

DEFINITION. A subset H of a continuous curve M will be said to be *cyclic chainwise connected* provided every two cyclic elements of M which belong to H can be joined by a simple cyclic chain of M which is a subset of H.

THEOREM 8. *If the sum H of the elements of a collection of cyclic elements of a continuous curve M is connected, then H is both arcwise connected and cyclic chainwise connected. Furthermore H contains every simple continuous arc of M which joins two points of H.*

Proof. Let H be any connected point set which is the sum of the elements of some collection of the cyclic elements of M, and let A and B be any two points of H. Let E_1 and E_2 be cyclic elements of M containing A and B respectively and belonging to H, and let K be the set of all those points of M which separate A and B in M. By Theorem 3, there exists in M a simple cyclic chain X from E_1 to E_2. Then X must be a subset of H. For suppose it is not. Then it contains a cyclic element C which is not a subset of H. Since H is connected and contains A and B, then it must contain every point of K; and therefore, by Theorem 1, C is a maximal cyclic curve of M which is determined by some maximal segment of $t-(K+A+B)$, where t is any arc in M from A to B. Now no internal point of C can belong to H. For if it did, then it would belong to some cyclic element of M belonging to H, and this element must be identical with C, for no two cyclic elements can have an internal point of either in common without being identical. Hence $H \cdot I(C)=0$. But this is contrary to Theorem 2, which states that $I(C)$ separates E_1 and E_2 in M, whereas H is a connected subset of $M-I(C)$ containing both E_1 and E_2. Thus the supposition that X is not a subset of H leads to a contradiction. Then since H contains X and, by Theorem 1, X contains t, therefore H contains an arc t from A to B. Hence H is arcwise connected.

That H is cyclic chainwise connected follows by the same argument as just given, where we let E_1 and E_2 be any two cyclic elements of M belonging to H. The same argument shows that H contains every arc in M which joins any two points of H, for A and B were any two points of H and t any arc in M from A to B, and it was shown that H contains t.

It is interesting to note that the following theorem of C. M. Cleveland's[*]: *Every connected subset of the set of all the cut points of a continuous curve is arcwise connected,* is a corollary to Theorem 8. This is evidently true because each cut point of a continuous curve is, by definition, a cyclic element of that curve.

In my paper *Concerning certain types of continuous curves*[†] I showed that a necessary and sufficient condition that a bounded continuum M should be an acyclic continuous curve is that every connected subset of M should be uniformly connected im kleinen. I shall now prove the following related theorem.

---

[*] Cf. an abstract of his paper "Concerning Cut Points of a Continuous Curve" in the *Bulletin of the American Mathematical Society,* Vol. 32 (1926), p. 420.

[†] *Proceedings of the National Academy of Sciences,* Vol. 12 (1926), pp. 761-767, Theorem 2.

THEOREM 9. *If the sum H of the elements of a collection of the cyclic elements of a continuous curve M is connected, then H is uniformly connected im kleinen.*

Proof. Suppose, contrary to this theorem, that H is not uniformly connected im kleinen. Then there exists a positive number ϵ such that H contains two infinite sequences of points $X_1, X_2, X_3, \cdots$, and $Y_1, Y_2, Y_3, \cdots$, having the property that for each positive integer n, the distance from X_n to Y_n is less than $1/n$, but such that for no integer n can X_n be joined to Y_n by a connected subset of H of diameter less than ϵ. But since M is uniformly connected im kleinen, there exists an integer k such that X_k and Y_k can be joined in M by an arc X_kY_k of diameter less than ϵ. But by Theorem 8, the arc X_kY_k, since it joins two points of H, must belong to H. Thus the supposition that H is not uniformly connected im kleinen leads to a contradiction, and the theorem is proved.

THEOREM 10. *If the sum H of the elements of a collection of cyclic elements of a continuous curve is a continuum, then H is a continuous curve.*

Theorem 10 is a corollary to Theorem 9.

THEOREM 11. *If the sum H of a collection of the cyclic elements of a continuous curve M is connected, then each point of $\overline{H} - H$ is a cyclic element of M and $\overline{H}$ is a continuous curve.*

Proof. Let P be any point of $\overline{H} - H$. Then P is either a cut point or an endpoint of M. For suppose it is neither. Then by Theorem 7 of C. C. it follows that P is an internal point of some maximal cyclic curve C of M. Now H contains no point of $I(C)$, for if it did it would contain all of $I(C)$ and therefore would contain P. Let N denote the maximal connected subset of $M - I(C)$ which contains H. Then N is a continuous curve having exactly one point Q in common with C. For since H is connected and $C - I(C)$ is countable, therefore H contains a point X in common with $M - C$. Let N_x denote the maximal connected subset of $M - C$ containing X. Then by Theorem 2 of C. C., C contains exactly one limit point Q of N_x, and clearly Q is a cut point of M. Let G denote the collection of all the maximal connected subsets of $M - C$ which have Q as a limit point, and let N_0 denote the sum of all the point sets of the collection G. Since by a theorem of W. L. Ayres',* at most a finite number of the elements of G are of diameter

---

* "Concerning Continuous Curves and Correspondences," *Annals of Mathematics*, Vol. 28 (1927), Theorem 1. The special case of this theorem here used holds in n dimensions.

greater than any preassigned positive number, it follows at once that $N_0 + Q$ is closed and, in fact, is a continuous curve. Clearly $N_0 + Q$ is a subset of N. It must be identical with N. For suppose N contains a point Y not belonging to $N_0 + Q$. It is easy to see that every point of N belongs to a cyclic element of M which is contained in N. Hence, by Theorem 8, N contains an arc XY. On the arc XY, in the order from Y to X, it is readily seen that Q must be the first point belonging to $N_0 + Q$. By the definition of the set N_0, it follows that the segment YQ of the arc XY must contain at least one point of C; but since it can contain only a countable number of points of C, it follows that it contains some segment S whose endpoints are on C but which contains no point of C. This is contrary to Theorem 2 of C. C., which states that no connected subset of $M - C$ has more than one limit point in C. Thus the supposition that N is not identical with $N_0 + Q$ leads to a contradiction. Hence N is a continuous curve having in common with C only the point Q. But H is a subset of N, and by supposition P, an internal point of C, is a limit point of H. Clearly this is impossible. Thus the supposition that P is neither an endpoint nor a cut point of M leads to a contradiction.

Now since every point of $\bar{H} - H$ is either an endpoint or a cut point of M, and since every such point is, by definition, a cyclic element of M, it follows that there exists a collection of cyclic elements of M whose sum is $\bar{H}$. Therefore, since $\bar{H}$ is a continuum, by Theorem 10, it is a continuous curve. This completes the proof of our theorem.

4. *Nodes and Endelements.* A cyclic element E of a continuous curve M will be called a *node* of M provided E neither is itself a cut point of M nor contains more than one cut point of M. Following a definition of an *endpoint* of a continuous curve given in my thesis,* let us define an *endelement* of a continuous curve as follows. The cyclic element E of a continuous curve M will be called an *endelement* of M provided that if X is any simple cyclic chain in M having E as one of its endelements, then no connected subset of $M - X$ has a limit point in E. It is clear from these definitions that every *endpoint* of M is both a *node* and an *endelement* of M and, indeed, that every endelement of M is also a node of M. However, not all nodes of M are necessarily endelements of M. For let $M = C + I$, where I is the interval $(-1, 1)$ and C is the circle $x^2 + (y - 1)^2 = 1$. Then C is a node of M; but

---

* G. T. Whyburn, "Concerning Continua in the Plane," *loc. cit.*, p. 386. An endpoint of a continuous curve M is there defined as a point P of M such that if t is any arc of M having P as one of its extremities, then P is not a limit point of any connected subset of $M - t$.

it is not an endelement of M, because C contains a limit point of the segment $(-1, 0)$, a connected subset of M minus the simple cyclic chain consisting of the interval $(1, 0)$ plus C.

THEOREM 12. *In order that a cyclic element of a continuous curve M should be a node of M it is necessary and sufficient that it should be either an endpoint of M or a maximal cyclic curve C of M such that $M - I(C)$ is connected (or vacuous).*

Proof. The condition is necessary. For let C be a node of M which is not an endpoint of M. Then since by definition C is not a cut point of M, therefore it is a maximal cyclic curve of M. Now unless C is identical with M, in which case our theorem is obvious, then C contains at least one cut point P of M. By the definition of a node it follows that P is the only cut point of M which belongs to C. Hence P is a limit point of every maximal connected subset of $M - C$, and therefore the sum H of all the maximal connected subsets of $M - C$ plus P is connected. But $H = M - C = M - P - (C - P)$, and $C - P = I(C)$. Therefore $H + P = M - I(C)$, and hence $M - I(C)$ is connected.

The condition is also sufficient. For let C be any cyclic element of M which is either an endpoint of M or a maximal cyclic curve of M such that $M - I(C)$ is connected. Now if C is an endpoint of M, obviously it must be a node. So let us suppose C is a maximal cyclic curve of M. Now it was shown in the proof of Theorem 11 that if C is any maximal cyclic curve of M, then every maximal connected subset of $M - I(C)$ is a continuous curve having one and only one point in common with C. And since, in this case, $M - I(C)$ is connected, therefore it is a continuous curve having exactly one point P in common with C. Hence $I(C) = C - P$, and P is the only cut point of M which belongs to C. Therefore C satisfies all the requirements of the definition of a node of M, and the theorem is proved.

THEOREM 13. *In order that a cyclic element E of a continuous curve M should be a node of M it is necessary and sufficient that no simple cyclic chain in M should have E as one of its interior elements.*

Proof. The condition is necessary. For suppose, on the contrary, that a node E of M is an interior element of some simple cyclic chain X in M between two cyclic elements E_1 and E_2 of M. Now by the definition of an interior element of X, E contains neither E_1 nor E_2. Hence there exist points A and B in E_1 and E_2 respectively which do not belong to E. Let t be an arc in M from A to B, and let K denote the set of all those points of M which separate A and B in M. By Theorem 1, X is the set of points $E_1 + E_2 + K +$ all those maximal cyclic curves of M determined by the maximal segments of $t-(K+A+B)$. Since E is a node, it cannot belong to K, and it does not belong to either E_1 or E_2. Hence E is a maximal cyclic curve of M determined by some maximal segment S of $t-(K+A+B)$. Let F and G denote the endpoints of S. Since F and G belong to E, therefore A, B, F, and G are all distinct points. But then F and G are points of K and hence cut points of M. This is contrary to the fact that a node can contain at most one cut point of M. Thus the supposition that the condition of this theorem is not necessary leads to a contradiction.

The condition is also sufficient. For let E be a cyclic element of M which is not an interior element of any simple cyclic chain in M. Then E is not a cut point of M. For suppose it is. Then E separates in M some two points A and B of M. Then by Theorem 1, E is an element of the simple cyclic chain X which joins a cyclic element E_1 containing A and an element E_2 containing B. And as E contains neither E_1 nor E_2, therefore it is an interior element of X, contrary to hypothesis. Now since E is not a cut point of M, it must be either an endpoint of M or a maximal cyclic curve of M. In the former case our theorem is obvious. So let us suppose that E is a maximal cyclic curve of M. Then $M-I(E)$ is connected. For suppose it is not. Then there exist two distinct maximal connected subsets N_1 and N_2 of $M-I(E)$. It was shown in the proof of Theorem 11 that N_1 and N_2 are continuous curves containing points P_1 and P_2 respectively in common with E. Let A and B be points of N_1-P_1 and N_2-P_2 respectively. Let E_1 and E_2 be cyclic elements of M which contain A and B respectively. By Theorem 3, M contains a simple cyclic chain X from E_1 to E_2. Let t be an arc in M from A to B and K the set of all those points of M which separate A and B in M. Then P_1 and P_2 belong to K, and the interval $P_1 P_2$ of t belongs to E. Hence by Theorem 1 it follows that E is an element of X, and since E contains neither A nor B, it is therefore an interior element of X. Thus the supposition that $M-I(E)$ is not connected leads to a contradiction. Hence $M-I(E)$ is connected and, by Theorem 12, E is a node of M.

THEOREM 14. *If the sum H of the elements of a collection G of cyclic elements of a continuous curve M is a continuum, and H itself has more than one cyclic element, then H contains at least two nodes of itself.*

Proof. Every cyclic element of H is a cyclic element also of M. For every maximal cyclic curve of H is a maximal cyclic curve also of M, every cut point of H is a cut point also of M, and every endpoint of H is a cut point or an endpoint of M.

Let us suppose, contrary to this theorem, that H does not contain as many as two nodes of itself. Then since H, by hypothesis, has more than one cyclic element, it follows by Theorem 12 and the definition of a node that H contains a cyclic element C which is either a cut point of H or a maximal cyclic curve of H such that $H - I(C)$ is not connected. Let E denote C or $I(C)$ according as C is a cut point or a maximal cyclic curve of H. Then there exist at least two distinct maximal connected subsets H_1 and H_2 of $H - E$. Since, by supposition, H has not more than one node, then either $H_1 + C$ or $H_2 + C$, say $H_1 + C$, contains no node of H. Now either H_1 has no point whatever in common with C or else (see proof of Theorem 11), H_1 is a continuous curve containing exactly one point P in common with C. In the former case, C is a cut point of H and a limit point of H_1. In this case let $P = C$. In either case H_1 contains a countable set of points $P_1, P_2, P_3, \cdots$,—all distinct from P, and such that every point of H_1 is a limit point of this set. For each positive integer i, let $C(P_i)$ denote a cyclic element of H which either is identical with P_i or contains P_i, and let E_i denote $C(P_i)$ or $I[C(P_i)]$ according as $C(P_i)$ is a cut point or a maximal cyclic curve of H. Since H_1 contains no node of H, then by Theorem 12 and the definition of cyclic element it follows that for every positive integer i, $H - E_i$ is not connected. It is readily seen that for no integer $i > 0$ does E_i contain the point P. For every integer $i > 0$, let H_{ip} denote the maximal connected subset of $H - E_i$ which contains P. There exists an integer n_1 which is the least integer such that P_{n_1} belongs to $H - [H_{1p} + C(P_1)]$. Now E_{n_1} contains no point whatever in common with $H_{1p} + C(P_1)$. Therefore, since $H_{1p} + C(P_1)$ is connected, it is a subset of H_{n_1p}. There exists an integer n_2 which is the least integer such that P_{n_2} belongs to $H - [H_{n_1p} + C(P_{n_1})]$ Just as above it follows that $H_{n_1p} + C(P_{n_1})$ is a subset of H_{n_2p}. There exists a least integer n_3 such that P_{n_3} belongs to $H - [H_{n_2p} + C(P_{n_2})]$, and so on. This process may be continued indefinitely, giving an infinite sequence of distinct points P_{n_1}, P_{n_2}, P_{n_3}, $\cdots$, and an infinite sequence of connected sets H_{n_1p}, H_{n_2p}, $\cdots$, such that for every positive integer i, $H_{n_ip} + C(P_{n_i})$ is a subset of $H_{n_{i+1}p}$.

The set of points $P_{n_1} + P_{n_2} + P_{n_3} + \cdots$ has at least one limit point Q which belongs to H_1. It is easily seen from the method of selection of the points P_{n_i} that Q belongs to none of the sets H_{n_ip}. The point Q is not a cut point of H. For suppose it is. Then $H - Q = N_1 + N_2$ where N_1 and N_2 are mutually separated sets, one of which, say N_1, contains $\sum H_{n_ip}$. But N_2 must contain some point, say P_k, of the sequence $P_1, P_2, P_3, \cdots$. Clearly this is impossible, because the above process of selection of the points P_{n_1}, P_{n_2}, $\cdots$ would, after at most $k+1$ steps, throw the point P_k into one of the sets H_{n_ip}, contrary to what has just been shown. Hence Q is not a cut point of H. Therefore, by a theorem of the author's,[*] Q is either an endpoint of H or else it is an internal point of some maximal cyclic curve F of H. The point Q is not an endpoint of H, because by supposition H_1 contains no node of H. But neither is P an internal point of any maximal cyclic curve F of H, because Q is a limit point of $\sum H_{n_ip}$ and $\sum H_{n_ip}$ is connected and a subset of $H - I(F)$, and therefore no internal point of F is a limit point of $\sum H_{n_ip}$. Thus the supposition that H does not contain two nodes of itself leads to a contradiction, and the theorem is proved.

Mazurkiewicz has shown [†] that if M is an acyclic continuous curve and t is any arc of M, then M contains a maximal arc which contains t, *i. e.*, M contains an arc T containing t and which is not a proper subset of any other arc of M. This theorem is evidently not true for any continuous curve which contains a simple closed curve. I shall now prove the following theorem which gives a similar result for *any continuous curve,* provided we use simple cyclic chains instead of simple continuous arcs.

THEOREM 15. *If X is any simple cyclic chain of a continuous curve M, then M contains a maximal simple cyclic chain which contains X,* i. e., *a simple cyclic chain which contains X and is not a proper subset of any other simple cyclic chain of M.*

Proof. Let E_1 and E_2 denote the endelements of X. Now if E_1 is not a node of M, then, by Theorem 12, it is either a cut point of M or a maximal cyclic curve of M such that $M - I(E_1)$ is not connected. In either case, it follows with the aid of Theorem 6 that there exists a maximal connected subset H_1 of $M - X$ which has exactly one limit point P_1 in E_1, where P_1 is not a limit point of $X - E_1$ if E_1 is an element of class (a). Likewise if E_2 is not a node, then there exists a maximal connected subset H_2 of $M - X$ which has exactly one limit point P_2 in E_2, where P_2 is not a limit point of $X - E_2$ if E_2 is an element of class (a). Now if either of the continuous curves $H_1 + P_1$ and $H_2 + P_2$ has only one cyclic element, it is clear that that element must be a node of M. And if either of these curves contains more than one cyclic element, then by Theorem 14 it follows that that curve contains at least two nodes of itself; and since at least one of these nodes does not contain the point $P_i (i = 1, 2)$, it is readily seen that in each case at least one of these nodes must be a node also of M. Hence in any case if E_1 and E_2 themselves are not nodes of M, then $H_1 + P_1$ and $H_2 + P_2$ contain nodes N_1 and N_2 respectively of M. If either E_1, or E_2, or both, is a node of M, then let $N_1 = E_1$, $N_2 = E_2$, or both. By Theorem 3, M contains one simple cyclic chain T from N_1 to N_2. With the aid of Theorem 1 it is easy to see that T must contain X, and since N_1 and N_2 are nodes of M, therefore, by Theorem 13, they are the endelements of every simple cyclic chain of M which contains them. Hence by Theorem 3 it follows that T is identical with every simple cyclic chain of M which contains T. Therefore T is a maximal simple cyclic chain in M containing X.

---

[*] "Concerning Continua in the Plane," *loc. cit.*, Theorem 22.

[†] S. Mazurkiewicz, "Un théorème sur les lignes de Jordan," *Fundamenta Mathematicae*, Vol. 2 (1921), p. 129.

R. L. Wilder has shown [*] that a necessary and sufficient condition that a point of an acyclic continuous curve M should be an endpoint of M is that it should be a non-cut point of M. This theorem obviously does not hold for any continuous curve which contains a simple closed curve. However, I shall now show that it does hold for any continuous curve provided we use cyclic elements instead of points.

THEOREM 16. *In order that the cyclic element E of a continuous curve M should be an endelement of M it is necessary and sufficient that it should be a non-cut element of M.*

Proof. The condition is necessary. For suppose, on the contrary, that an endelement E of M is a cut element of M. Then since E is a node of M, it readily follows that there exist two maximal connected subsets H and N of $M - E$ and a point P of E which is a limit point of each of the sets H and N. Let x be a point of H and F a cyclic element of M containing x. By Theorem 3, there exists in M a simple cyclic chain X from F to E. Since, by Theorem 6, $X - E$ is connected, therefore it is a subset of H. Hence N contains no point in common with X. But the point P of E is a limit point of N, which is impossible, since E is an endelement of M. Thus the supposition that the condition of this theorem is not necessary leads to a contradiction.

The condition is also sufficient. For let E be an element of M such that $M - E$ is connected, and suppose, contrary to this theorem, that E is not an endelement of M. Then there exists in M a simple cyclic chain X from E to some other element F of M and a maximal connected subset N of $M - X$ having a limit point P in E. Now since $M - E$ is connected, by Theorem 2

---

[*] "Concerning Continuous Curves," *loc. cit.*, Theorem 7.

of C. C., P is the only limit point of $M - E$ which belongs to E. There exists in X an arc AP, where A is some point of $X - E$. By a theorem of R. L. Wilder's,[*] there exists an arc BP such that $BP - P$ belongs to N. By a theorem of R. L. Moore's,[†] $M - E$ contains an arc AB. The sum of the arcs AB, AP, and BP contains a simple closed curve J which contains a segment of each of the arcs AP and BP. But then by Theorem 1, J must belong to X, and BP belongs to N and therefore contains no point of X. Thus the supposition that E is not an endelement of M leads to a contradiction.

Theorem 16 gives immediately the following theorem:

THEOREM 17. *In order that a cyclic element E of a continuous curve M containing more than one cyclic element should be an endelement of M it is necessary and sufficient that it should be either an endpoint of M or a maximal cyclic curve C of M which contains exactly one cut point P of M and such that $M - P = (C - P) + S$, where $C - P$ and S are mutually separated and connected point sets.*

It has been shown by the author [‡] and also by Menger [‡] that the sum of all the endpoints of a continuous curve is a totally disconnected point set. It is not necessarily true, however, that the sum of all the endelements of a continuous curve contains no connected set containing more than one endelement of M. For let $M = C_1 + C_2$, where C_1 and C_2 are circles tangent to each other at a point P. Then both C_1 and C_2 are endelements of M and their sum is connected. The following two theorems are of interest in this connection.

THEOREM 18. *In order that two different endelements E_1 and E_2 of a continuous curve M should have a common point it is necessary and sufficient that $E_1 + E_2 = M$.*

Proof. The condition is sufficient. For E_1 and E_2 are closed point sets, and if their sum is M, a connected point set, evidently they must have a common point.

The condition is also necessary. For suppose the endelements E_1 and E_2 of a continuous curve M have a common point P, and suppose, contrary to this theorem, that $E_1 + E_2 \neq M$. Then there exists a maximal connected subset N of $M - (E_1 + E_2)$. But since E_1 and E_2 have a common point, it is clear that if X denotes the simple cyclic chain in M from E_1 to E_2, then $E_1 + E_2 = X$. Hence $M - (E_1 + E_2) = M - X$. But X contains at least one limit point P of N. The point P must belong either to E_1 or to E_2. But this is impossible since both E_1 and E_2 are endelements of M, and hence, by definition, neither contains a limit point of any connected subset of $M - X$.

---

[*] "Concerning Continuous Curves," *loc. cit.*, Theorem 1.

[†] "Concerning Continuous Curves in the Plane," *Mathematische Zeitschrift*, Vol. 15 (1922), pp. 254-260, Theorem 1.

[‡] G. T. Whyburn, "Concerning Continua in the Plane," *loc. cit.*, Theorem 21; K. Menger, "Grundzüge einer Theorie der Kurven," *Mathematische Annalen*, Vol. 95 (1925), pp. 272-306.

THEOREM 19. *If the continuous curve M has more than three cyclic elements, or has two mutually exclusive elements, then the sum of all the endelements of M contains no connected point set which contains more than one endelement of M.*

Proof. If M has more than three cyclic elements, then no two endelements of M can have a common point. For if two endelements E_1 and E_2 of M have a common point P, then by Theorem 18, $E_1 + E_2 = M$, and M has only the three cyclic elements E_1, E_2, and P. Furthermore, if M has two mutually exclusive cyclic elements, then again no two endelements of M can have a common point. For if two such elements E_1 and E_2 had a common point P, then by Theorem 18, $E_1 + E_2 = M$, and M has only the three cyclic elements E_1, E_2, and P, every two of which have the point P in common. Hence, in either case, no two endelements of M can have a common point.

Now suppose, contrary to this theorem, that the sum of all the endelements of M contains a connected set H which contains as many as two endelements E_1 and E_2 of M. Let N denote the sum of all the endelements of M which contain at least one point of H. Then by Theorem 8, N contains a simple cyclic chain X of M from E_1 to E_2. But since E_1 and E_2 have no common point, it follows that there exists an element E of M which is an interior element of X. But E belongs to N and hence is an endelement of M. This is contradictory to Theorem 13 and the fact that every endelement of M is also a node of M. Thus the supposition that Theorem 19 is not true leads to a contradiction.

5. *Some Theorems on Continuous Curves.*

THEOREM 20. *Let P be a cut point of a continuous curve M which is an irregular** *point of some subcontinuum N_0 of M. Then P belongs to some simple closed curve in M.*

---

* The point P of a point set N is an irregular point of N provided N is not connected im kleinen at P.

Proof. Since for each positive number ϵ, not more than a finite number of maximal connected subsets of $M - P$ are of diameter greater than ϵ, it readily follows that there exists a maximal connected subset H of $M - P$ such that P is an irregular point of the continuum $N_0 \cdot (H + P)$. Let N denote the continuum $N_0 \cdot (H + P)$. Now P is not a cut point of the continuous curve $H + P$. Therefore, by a theorem of the author's,[*] P either is an endpoint of $H + P$ or belongs to some simple closed curve in $H + P$. Let us suppose that P is an endpoint of $H + P$. Then since, as proved by H. M. Gehman,[†] for the case of a continuous curve, our definitions for the term *endpoint* as applied to continuous curves and for the same term as applied to continua in general are equivalent, it follows that P is an endpoint of the *continuum* $H + P$ and hence is an endpoint of the sub-continuum N of $H + P$. Now P obviously is accessible from some complementary domain of N, namely, from the one which contains some maximal connected subset of $M - (H + P)$. Therefore, by a theorem of mine,[‡] it follows that P is a regular point of N, contrary to hypothesis. Thus the supposition that P is an endpoint of $H + P$ leads to a contradiction, and it follows that P must belong to some simple curve in $H + P$. This completes the proof.

THEOREM 21. *If G denotes the set of all those cut points $[P]$ of a continuous curve M such that P is an irregular point of some subcontinuum of M, then G is countable.*

Theorem 21 is an immediate consequence of Theorem 20 and of the theorem of the author's [§] which states that the set of all the cut points of a continuous curve M which belong to the sum of all the simple closed curves contained in M is countable.

I have recently succeeded in establishing the conclusion of Theorem 21 for *any plane continuum M*. Since the proof of this much more general proposition is of considerable length and is foreign to the material presented in this paper, it will be reserved for a later paper.

THEOREM 22. *In order that the point P of a continuous curve M should belong to no simple closed curve in M it is necessary and sufficient,*

---

[*] G. T. Whyburn, "Concerning continua in the plane," *loc. cit.*, Theorem 22.

[†] "Concerning End Points of Continuous Curves and Other Continua," *Transactions of the American Mathematical Society*, Vol. 30 (1928), pp. 63-84.

[‡] *Loc. cit.*, Theorem 30.

[§] *Loc. cit.*, Theorem 29.

that it should be an endpoint of every continuous curve obtained by adding it to some maximal connected subset of $M - P$.

Proof. The condition is necessary. For suppose P belongs to no simple closed curve in M, and let N be any continuous curve obtained by adding P to some maximal connected subset H of $M - P$. Then P belongs to no simple closed curve in N. And since $H = N - P$ is connected, therefore P is not a cut point of N. But every continuous curve is the sum of its cut points, endpoints, and simple closed curves.[*] Therefore P is an endpoint of N.

The condition is also sufficient. For suppose P is an endpoint of every continuous curve obtained by adding it to some maximal connected subset of $M - P$, but suppose, contrary to this theorem, that P belongs to some simple closed curve J in M. Then since $J - P$ is connected, it must belong to some maximal connected subset H of $M - P$, and clearly P is not an endpoint of $H + P$, contrary to hypothesis. Thus the supposition that P belongs to some simple closed curve in M leads to a contradiction.

THEOREM 23. *Let P be any endpoint of a continuous curve M. Then there exists in M an infinite sequence of points $X_1, X_2, X_3, \cdots$, having P as its sequential limit point and such that if A is any point whatever of $M - P$, then there exists an integer k_a such that if i is any integer greater than k_a, then X_i separates A from P in M.*

Proof. Let A_0 be any point of $M - P$, t_0 any arc in M from A_0 to P, and K_0 the set of all those points in M which separate A_0 and P in M. Then P is a limit point of K_0. For if not, there exists [†] a maximal cyclic curve of M which contains P, contrary to the fact that P is an endpoint of M. Hence there exists a sequence of points $X_1, X_2, X_3, \cdots$, all belonging to K_0, and having P as its sequential limit point. Now let A be any other point of $M - P$, t any arc in M from A to P, and K the set of all those points of M which separate A from P in M. On t, in the order from A to P, let E be the first point belonging to t_0. Since P is an endpoint of M, E must be different from P. There exists an integer k_a such that if i is any integer greater than k_a, then X_i belongs to the segment EP of t_0. Let j be any integer greater than k_a. Then X_j must belong to K. For $M - X_j = M_{A_0} + M_P$, by hypothesis, where M_{A_0} and M_P are mutually separated sets containing A_0 and P respectively. Now the arc AEA_0 does not contain X_j, for X_j belongs to the segment

---

[*] G. T. Whyburn, *loc. cit.*, Theorem 22.

[†] G. T. Whyburn, "Some Properties of Continuous Curves," *loc. cit.*, Theorem 2.

EP of t_0. Therefore the arc AEA_0, and hence also the point A, belongs to M_{A_0}. Hence X_j separates A from P in M and therefore belongs to K. This completes the proof of the theorem.

Theorem 23 is a generalization of H. M. Gehman's[*] theorem that every endpoint of a continuous curve is a limit point of cut points of that curve.

THEOREM 24. *In order that the continuous curve M should fail to separate the plane it is necessary and sufficient that every maximal cyclic curve of M should be a simple closed curve plus its interior.*

Proof. The condition is necessary. For suppose M does not separate the plane. Let C be any maximal cyclic curve of M, and let J be the boundary of the unbounded complementary domain of C. By Theorem 10 of C. C., J is a simple closed curve. Since M does not separate the plane, clearly every point within J belongs to M and also to C. That J plus its interior must be identical with C follows immediately from the fact that J is the boundary of the unbounded complementary of C.

The condition is also sufficient. For suppose every maximal cyclic curve of a continuous M is a simple closed curve plus its interior, and suppose, contrary to this theorem, that M separates the plane. Then there exists at least one bounded complementary domain R of M. Let J denote the boundary of the unbounded complementary domain of the boundary of R. By a theorem of R. L. Moore's,[†] J is a simple closed curve whose interior I contains R. By Theorem 3 of C. C., M contains a maximal cyclic curve C which contains J. By hypothesis there exists a simple closed curve K such that if D is the interior of K, then $K + D = C$. But C contains J and hence must also contain I, the interior of J. But I contains R, a complementary domain of M. Thus the supposition that M separates the plane leads to a contradiction.

THEOREM 25. *In order that a maximal cyclic curve C of a continuous curve M should be a simple closed curve it is necessary and sufficient that C should be the common boundary of two complementary domains of M.*

Proof. That the condition is sufficient is a consequence of a theorem of R. L. Moore's.[‡] I shall show that it is necessary. Let C be a simple closed curve in M which is a maximal cyclic curve of M. If the interior I of C is a

---

[*] H. M. Gehman, *loc. cit.* It is interesting to note that the condition in Theorem 23 characterizes an endpoint of a continuous curve, whereas that in Gehman's theorem does not.

[†] R. L. Moore, "Concerning Continuous Curves in the Plane," *Mathematische Zeitschrift*, Vol. 15 (1922), pp. 254-260, Theorem 4.

[‡] *Loc. cit.*

complementary domain of M, then let $I = R$. And if I is not complementary to M, then there exists in I a maximal connected subset H_0 of $M - C$ which has exactly one limit point P in C. Let H be the sum of all the maximal connected subsets of $M - C$ which lie in I and have P for a limit point. Let B be a point of $C - P$ and A_0 a point of H. There exists an arc A_0B such that $A_0B - B$ belongs to I. In the order from B to A_0 on the arc A_0B, let A be the first point belonging to the continuum $H + P$. Then A belongs to H, and there exists an interval AX of the arc AB such that $AX - A$ contains no point whatever of M. Now let R denote the complementary domain of M which contains the point X. Then every point of C belongs to the boundary of R. For suppose some point Y of C does not belong to the boundary of R. Then by a theorem of R. L. Moore's,[*] there exists in M a simple closed curve J which either encloses R and has Y in its exterior or encloses Y and has R in its exterior. And since, by hypothesis, C is a maximal cyclic curve of M, it is readily seen that J cannot enclose Y. Hence J encloses R and has Y in its exterior. Now since C is a maximal cyclic curve of M, it follows that J can have at most one point Q in common with C. Then since $(J + Q) - Q$ is a connected subset of $M - C$, which evidently is within C, therefore it belongs to some maximal connected subset N of $M - C$. Now N cannot belong to H, for since X belongs to the segment AB of the arc AB, which contains no point of H, evidently X, and hence also R, must belong to the unbounded complementary domain of $H + P$. Now N has exactly one limit point Q in C, and Q is different from P. But since the segment AX of the arc AB belongs to R and contains no point whatever of M, the connected point set $R + H + P$ contains no point of N but contains a limit point of the exterior of C. Therefore R belongs to the unbounded complementary domain of $N + Q$. This is contrary to the fact that R is within J, a simple closed curve belonging to $N + Q$. Thus the supposition that C is not a part of the boundary of R leads to a contradiction.

Now by a suitable inversion of the plane, the exterior of C can be thrown into the interior of the image of C. And by the same argument as given above it is shown that there exists in E, the exterior of C, a complementary domain D of M whose boundary contains C. Therefore C is the common part of the boundaries of the two complementary domains R and of M.

THEOREM 26. *The boundary of every complementary domain of a maximal cyclic curve of a continuous curve M belongs to the boundary of some single complementary domain of M.*

---

[*] *Loc. cit.*

Proof. By Theorem 10 of C. C., the boundary of every complementary domain of a maximal cyclic curve of M is a simple closed curve, and by the same method as used in the proof of Theorem 25 it is shown that there exists a complementary domain of M whose boundary contains this simple closed curve.

THEOREM 27. *If the continuous curve M is not the boundary of any domain, then in order that every proper subcontinuum of M should be the boundary of a domain it is necessary and sufficient that M should be the sum of three arcs joining two points A and B of M and having no other points common to any two of them.*

Proof. The condition is evidently sufficient. I shall show that it is necessary. In my thesis* I proved the following theorem. THEOREM *A. In order that the continuous curve M should be the boundary of a connected domain it is necessary and sufficient that if J is any simple closed curve in M, then* (1) *M is a subset either of $J + I$ or of $J + E$, where I and E respectively denote the interior and exterior of J, and* (2) *if A and B are any two points of J, then $M - (A + B)$* is not connected. Now M must satisfy condition (1) of Theorem A. For suppose, on the contrary, that M contains a simple closed curve J such that I contains a point X of M and E contains a point Y of M. Then M contains an arc XY, and the arc XY contains a proper subarc t containing points of both I and E. But the continuum $t + J$ is a proper subcontinuum of M and hence, by hypothesis, is the boundary of some domain. This is contrary to Theorem A. It follows that M satisfies condition (1) of Theorem A.

Now since, by hypothesis, M is not the boundary of any domain and since M satisfies condition (1) of Theorem A, it follows that M cannot satisfy condition (2) of Theorem A. Hence there exists a simple closed curve J in M and two points X and Y of J such that $M - (X + Y)$ is connected. Let A_0 be a point on one of the segments of J from X to Y, and B_0 a point on the other. Then by a theorem of R. L. Moore's,† $M - (X + Y)$ contains an arc t from A_0 to B_0. There exist points A and B on t which belong to the segments XA_0Y and XB_0Y respectively of J and such that the segment AB of t lies wholly either within J or without J. Let Z be a point on the segment AB of t. Then the three arcs AXB, AYB, and AZB all belong to J and every two of them have in common exactly the points A and B. T $AXB + AYB + AZB = M$ follows immediately now, using the fact that satisfies condition (1) of Theorem A. This completes the proof of Theorem [illegible].

---

* "Concerning Continua in the Plane," *loc cit.*, Theorem 6.

† "Concerning Continuous Curves in the Plane," *loc. cit.*, Theorem 1.

6. *Properties of a Continuous Curve as determined by its Maximal Cyclic Curves.* The nature of a continuous curve is determined to a large extent by the nature of its maximal cyclic curves. A considerable number of propositions are true about a continuous curve M provided the same propositions are true for all or for some of the maximal cyclic curves of M and vice versa. In this section some propositions of this sort will be given, and in some cases merely stated without proof, since the proofs present no especial difficulty.

THEOREM 28. *In order that every subcontinuum of a continuous curve M should be a continuous curve it is necessary and sufficient that every subcontinuum of each maximal cyclic curve of M should be a continuous curve.*

THEOREM 29. *In order that a continuous curve M should separate the plane it is necessary and sufficient that some maximal cyclic curve of M should separate the plane.* (See Theorem 24.)

THEOREM 30. *If H is any connected subset of a continuous curve M and C is any maximal cyclic curve of M, then $H \cdot C$ is either vacuous or connected.*

Proof. Suppose, on the contrary, that $H \cdot C = H_1 + H_2$, where H_1 and H_2 are mutually separated point sets. Now for each point X of H which belongs to $M—C$, the maximal connected subset M_x of $M—C$ which contains X has exactly one limit point P_x which belongs to C, and P_x must belong either to H_1 or to H_2. Let K_1 denote the set of all those points X of H in $M - C$ such that the corresponding point P_x belongs to H_1, and let K_2 be the set of all such points such that P_x belongs to H_2. Then clearly $H = H_1 + H_2 + K_1 + K_2$.

Now H_1 contains no limit point of K_2. For suppose it does contain a limit point P of K_2. Then since no subset of K_2 every point of which belongs to some single maximal connected subset of $M - C$ can have a limit point in H_1, it follows that K_2 contains an infinite subset D which has P for its sequential limit point and such that no two points of D lie together in the same maximal connected subset of $M - C$. Let G denote the collection of maximal connected subsets of $M - C$ each of which contains a point of D. Since P is a limit point of D, then P belongs to the limiting set of the collection G. Since P is not a limit point of H_2, there exists a number ϵ such that every point of H_2 is at a distance greater than ϵ from P. But now each element of G has a limit point in H_2, and therefore the limiting set of G has a point whose distance from P is $\geqq \epsilon$. It follows that infinitely many elements of G must be of diameter $> \epsilon/2$. This is contrary to a theorem of

W. L. Ayres.[*] Thus the supposition that H_1 contains a limit point of K_2 leads to a contradiction.

Now since H_1 belongs to C and K_2 belongs to $M - C$, it is clear that no point of K_2 can be a limit point of H_1. Hence H_1 and K_2 are mutually separated point sets. A similar argument shows that H_2 and K_1 are mutually separated. Hence, as H_1 and H_2, by supposition, are mutually separated, and clearly K_1 and K_2 are mutually separated, it follows that $(H_1 + K_1)$ and $(H_2 + K_2)$ are mutually separated point sets. But the sum of these two sets is H, and H, by hypothesis, is connected. Thus the supposition that $H \cdot C$ is neither vacuous nor connected leads to a contradiction, and the theorem is established.

THEOREM 31. *If the subcontinuum C of a continuous curve M has no cut point, then in order that C should be a maximal cyclic curce of M it is necessary and sufficient that if H is any connected subset of M, then $H \cdot C$ is either vacuous or connected.*

Proof. Theorem 30 shows that the condition is necessary. To show that it is sufficient, let us first show that C is a continuous curve. Suppose the contrary is true. Then there exists a point P of C and a positive number ϵ such that no matter what number δ we choose, there are points of C whose distance from P is less than δ but which lie in no connected subset of C containing P and of diameter less than ϵ. Now since M is connected im kleinen at P, there exists a positive number δ_ϵ such that every point of M whose distance from P is $< \delta_\epsilon$ can be joined to P by an arc[†] of M of diameter less than ϵ. Let X be a point of C whose distance from P is $< \delta_\epsilon$ and which cannot be joined to P by any connected subset of C of diameter less than ϵ. There exists in M an arc XP of diameter less than ϵ. The arc XP is not a subset of C. Hence there exists an interior point O of XP not belonging to C. On the arcs OX and OP of XP, in the order from O to X and from O to P respectively, let A and B denote the first points belonging to C. Then the arc AOB of XP is a connected subset of M having in common with C only the point set $A + B$, which clearly is not connected. This is contrary to hypothesis, and it follows that C is a continuous curve.

Since C has no cut point, then by Theorem 1 of C. C., C is cyclicly connected. Now suppose, contrary to this theorem, that C is not a maximal cyclic curve of M. Then by Theorem 2 of C. C., there exists a connected subset N of $M - C$ which has at least two limit points A and B which belong to C. This is impossible, because the connected subset $N + A + B$ of M has only the point set $A + B$ in common with C, and $A + B$ is not connected. Thus the supposition that C is not a maximal cyclic curve of M leads to a contradiction.

---

[*] W. L. Ayres, *loc. cit.* The special case of Ayres' theorem here used holds in n dimensions.

[†] Cf. J. R. Kline, "Concerning Approachability of Simple Closed and Open Curves," *Transactions of the American Mathematical Society*, Vol. 21 (1920), footnote to p. 453.

It is interesting to note that with the aid of Theorem 30 the proof of Theorem 1 can be considerably shortened.

THEOREM 32. *If H is a connected subset of a continuous curve M such that the subset of H contained in each maximal cyclic curve of M is arcwise connected, then H itself is arcwise connected.*

Proof. Let A and B be any two points of H, let t be any arc in M from A to B, and let K denote the set of all those points of M which separate A and B in M. Clearly K belongs to both T and H. By a theorem of the author's,[*] $K + A + B$ is a closed set of points. Hence $t - (K + A + B)$ is the sum of a countable number of non-overlapping segments S_1, S_2, S_3, $\cdots$ For each positive integer i, let A_i and B_i denote the endpoints of S_i. For each i, S_i determines [†] a maximal cyclic curve C_i of M which contains S_i. Now for each i, A_i and B_i belong to $K + A + B$, and hence belong also to H and to C_i. Then since by hypothesis $H \cdot C_i$ is arcwise connected and contains both A_i and B_i, therefore it contains an arc t_i from A_i to B_i Now let

$$T = K + A + B + \sum_{i = 1, 2, 3, \ldots} t_i .$$

Then T is a subset of H, and since for no two integers k and j can t_k and t_j have an interior point of either in common, and since for no positive number ϵ does there exist more than a finite number of maximal cyclic curves of M of diameter greater than ϵ, it is easy to see that T is a simple continuous arc from A to B. Therefore H is arcwise connected.

THEOREM 33. *In order that every connected subset of a continuous curve M should be arcwise connected it is necessary and sufficient that every connected subset of each maximal cyclic curve of M should be arcwise connected.*

Theorem 33 follows at once from Theorems 32 and 31.

THEOREM 34. *If every maximal cyclic curve of a continuous curve M is a simple closed curve, then every connected subset of M is arcwise connected.*

Theorem 34 is a corollary to Theorem 33.

It is to be noted that Theorem 34 includes the following important theorem of R. L. Wilder's [*]: every connected subset of the boundary of a complementary domain of a continuous curve is arcwise connected. It is readily seen with the aid of Theorem 25 that every maximal cyclic curve of the boundary B of a complementary domain of a continuous curve is a simple closed curve. Hence, by Theorem 34, every connected subset of B is arcwise connected.

---

[*] G. T. Whyburn, "Some Properties of Continuous Curves," *loc. cit.*, Theorem 1.

[†] *Loc. cit.*, Theorem 2.

In connection with Theorems 25 and 34, the following proposition is interesting because of the light it throws upon the nature of a continuous curve every maximal cyclic curve of which is a simple closed curve.

THEOREM 35. *If every maximal cyclic curve of a continuous curve M is a simple closed curve, then M is the sum of its endpoints plus the boundaries of all of its complementary domains.*

Proof. Suppose, on the contrary, that M contains a point P which neither is an endpoint of M nor belongs to the boundary of any complementary domain of M. Since P is not an endpoint of M, by the definition of an endpoint, there exists an arc APB in M having P as an interior point. Let d be the smaller of the two numbers d_1 and d_2, where d_1 is ½ the distance from A to P and d_2 is ½ the distance from B to P. Then since P belongs to the boundary of no complementary domain of M, by a theorem of mine,[†] there exists in M a simple closed curve J which encloses P and is of diameter less than d. Clearly J does not enclose either A or B. In the order from P to A and from P to B respectively on APB, let X and Y denote the first points belonging to J. Then J is not a maximal cyclic curve of M, because the segment XPY of the arc APB belongs to $M - J$ and has two limit points in J. This is contrary to Theorem 2 of C. C. Hence J is not a maximal cyclic curve of M. But J belongs to some maximal cyclic curve C of M. Therefore C is not a simple closed curve, contrary to hypothesis. Thus the supposition that Theorem 35 is false leads to a contradiction.

In conclusion I wish to point out that the above theory of the structure of a continuous curve as being composed of its cyclic elements appears much more perfect if we restrict the types of continuous curves considered to any one of the following three types: (1) continuous curves no two cyclic elements of which have a common point, (2) continuous curves which are the boundary of some domain, or (3) continuous curves which do not separate the plane (see Theorem 24).

UNIVERSITY OF TEXAS,
AUSTIN, TEXAS.

---

[*] R. L. Wilder, "Concerning Continuous Curves," *loc. cit.*, Theorem 20.

[†] "Concerning Continua in the Plane," *loc. cit.*, Theorem 7.

Compound Statements on Four Classes.

By Frances E. Baker.

1. *Introduction.* W. K. Clifford[*] first enumerated the compound statements on four classes, applying an intuitive method which allowed no check. Cayley[†] carried a check as far as three-fold statements. Up to this stage, the point of view of simple combinations was sufficient, but further progress demanded the use of a theorem of groups, viz., that if an object is invariant under a subgroup of order N, and the total group has order M, M/N is the number of conjugates. In the present paper this check is carried out completely, showing that in several cases Clifford was in error.

The order M of the group is here $384 = 4! \times 2^4$, representing the number of permutations of the four classes multiplied by the group of logical operations whose primitive elements are negations. Certain theorems about possible subgroups are proved.

A special notation proves convenient in which $[\alpha]$ means logical multiplication, or denial of the class A; and the usual signs, as $(\alpha\beta)$, mean interchange of classes. The rules for transposition of the square and round brackets are relatively simple. In the case of the eight-fold classes, the material to be worked upon is obtained by a special enumeration which gives duplicates but is inclusive.

2. *Notation.* To facilitate the counting of distances, and to serve as a basis for the subsequent group technique, a new notation has been introduced, namely:

The identical element $I \sim ABCD = 0$ (no $ABCD$ exists)
$\alpha \sim aBCD = 0$ (no $aBCD$ exists)
$\beta \sim AbCD = 0$ (no $AbCD$ exists)
$\gamma \sim ABcD = 0$ (no $ABcD$ exists)
$\delta \sim ABCd = 0$ (no $ABCd$ exists)

---

[*] Clifford, W. K., *Memoirs of the Literary and Philosophical Society of Manchester*, Vol. 16, No. 7, pp. 88-101. Also, *Mathematical Papers*, by W. K. Clifford, pp. 1-16; and *Lectures and Essays*, by W. K. Clifford, Vol. 2, pp. 89-105. For the origin of the problem, see also: Jevons, W. Stanley, *Proceedings of the Manchester Philosophical Society*, Vol. 6, pp. 65-68, and *Memoirs*, Third Series, Vol. 5, pp. 119-130; *The Principles of Science*, Vol. 1, pp. 154-164. [New Edition, pp. 134-143.]

[†] Cayley, A., *Memoirs of the Literary and Philosophical Society of Manchester*, Vol. 16, p. 113; *Collected Papers*, Vol. 10, p. 243.

Consequently:

$$\begin{aligned} \alpha\beta &\sim abCD = 0 \\ \alpha\beta\gamma &\sim abcD = 0 \\ \alpha\beta\gamma\delta &\sim abcd = 0 \text{ etc.} \end{aligned}$$

Economy of letters is an advantage gained. For example, $BCD = 0$ in the old notation representing the two-fold statement $ABCD = 0$, $aBCD = 0$, may now be expressed simply as I, α. Similarly, $AC = 0$, standing for $ABCD = 0$, $AbCD = 0$, $ABCd = 0$, $AbCd = 0$, becomes I, β, δ, $\beta\delta$. It may be noted that statements denying entirely a particular class, such as the eight-fold statement $I\ \alpha\ \beta\ \gamma\ \alpha\beta\ \alpha\gamma\ \beta\gamma\ \alpha\beta\gamma$, are inconsistent according to Jevons,[*] but are included by Clifford.

If we now consider I as the origin, the 4 proximates become $\alpha\ \beta\ \gamma\ \delta$; the 6 mediates become $\alpha\beta\ \alpha\gamma\ \alpha\delta\ \beta\gamma\ \beta\delta\ \gamma\delta$; the 4 ultimates become $\alpha\beta\gamma\ \alpha\beta\delta\ \alpha\gamma\delta\ \beta\gamma\delta$; and the obverse to I becomes $\alpha\beta\gamma\delta$. Hence there are altogether **16 marks**:

$$I\ \alpha\ \beta\ \gamma\ \delta\ \alpha\beta\ \alpha\gamma\ \alpha\delta\ \beta\gamma\ \beta\delta\ \gamma\delta\ \alpha\beta\gamma\ \alpha\beta\delta\ \alpha\gamma\delta\ \beta\gamma\delta\ \alpha\beta\gamma\delta.$$

This set of 16 objects is invariant as a whole under:

(1). Multiplication by any member of itself. Let [ ] be the symbol for multiplication, so that $[\alpha\beta\delta]$ means multiply by $\alpha\beta\delta$. There are 16 multipliers, in which $\alpha\ \beta\ \gamma\ \delta$ are commutative. Multiplication is taken modulo 2 with respect to exponents. Thus $[\alpha^\pi\beta^\kappa\gamma^\rho\delta^\sigma]$ represents a general multiplier, where $\pi, \kappa, \rho, \sigma = (0, 1)$.

Ex. $$[\alpha^0\beta^0\gamma^1\delta^1][\alpha^0\beta^1\gamma^1\delta^0] = [\gamma\delta][\beta\gamma] = I\cdot\beta^1\cdot I\cdot\delta^1 = [\beta\delta].$$

The permutations of the set $I\ \alpha\ \beta\ \gamma \cdots \alpha\beta\gamma\delta$ effected by [ ] all change I.

(2). 24 substitutions, viz., the $G^4{}_{24}$. The permutations of the set $I\ \alpha\ \beta\ \gamma \cdots \alpha\beta\gamma\delta$ effected by ( ) do not change I.

3. *The Order of the Group of Permutations.* To prove that the order of the group of permutations of the set $I\ \alpha\ \beta\ \gamma \cdots \alpha\beta\gamma\delta$ is 16×24:

Every operation is reducible either to [] () or to () [] or to a succession of [] () [] () $\cdots$. As a general element of the set take $E = \{\alpha^p\beta^q\gamma^r\delta^s\}$ and as elements of multiplication and substitution, respectively, take $M_1 = [\alpha^\pi\beta^\kappa\gamma^\rho\delta^\sigma]$ and $S_1 = (\alpha\beta)$ where $p, q, r, s, \pi, \kappa, \rho, \sigma = (0, 1)$. Let {} [] () mean first multiply into the operand, then permute; and let {} () [] mean first permute, then multiply into the result.

---

[*] *Loc. cit.*

Examples:

(1).
$$\left\{\begin{matrix}\alpha^p\beta^q\gamma^r\delta^s\\ E\end{matrix}\right\}\underset{M_1}{[\alpha^\pi\beta^\kappa\gamma^\rho\delta^\sigma]}\ \underset{S_1}{(\alpha\beta)} = \left\{\begin{matrix}\alpha^{p+\pi}\beta^{q+\kappa}\gamma^{r+\rho}\delta^{s+\sigma}\\ E_1\end{matrix}\right\}\underset{S_1}{(\alpha\beta)}$$
$$= \left\{\begin{matrix}\alpha^{q+\kappa}\beta^{p+\pi}\gamma^{r+\rho}\delta^{s+\sigma}\\ E_2\end{matrix}\right\}$$

Since we can write:
$$E_2 = \left\{\begin{matrix}\alpha^p\beta^q\gamma^r\delta^s\\ E\end{matrix}\right\}\underset{S_1}{(\alpha\beta)}\ \underset{M_2}{[\alpha^\pi\beta^\kappa\gamma^\rho\delta^\sigma]}$$

it is seen that $EM_1S_1 = ES_1M_2$. Treating the multiplication symbols as substitution symbols:

$$\begin{aligned} ES_1^{-1}M_1S_1 &= ES_1^{-1}S_1M_2\\ \therefore\ S_1^{-1}M_1S_1 &= M_2\\ \text{i. e. } (\alpha\beta)^{-1}M_1(\alpha\beta) &= M_2\end{aligned}$$

(2). Likewise

$$\begin{aligned} [M_1](\alpha\beta)(\alpha\gamma) &= (\alpha\beta)[M_2](\alpha\gamma)\\ &= (\alpha\beta)(\alpha\gamma)[M_3]\\ \text{where } [M_2] &= (\alpha\beta)^{-1}[M_1](\alpha\beta)\\ [M_3] &= (\alpha\gamma)^{-1}(\alpha\beta)^{-1}[M_1](\alpha\beta)(\alpha\gamma)\\ &= (\alpha\beta\gamma)^{-1}[M_1](\alpha\beta\gamma)\end{aligned}$$

(3). Similarly

$$\begin{aligned} [M_1](\alpha\beta)(\alpha\gamma)(\alpha\delta) &= (\alpha\beta)(\alpha\gamma)(\alpha\delta)[M_2]\\ [M_1](\alpha\beta\gamma\delta) &= (\alpha\beta\gamma\delta)[M_2]\\ \text{where } [M_2] &= (\alpha\beta\gamma\delta)^{-1}[M_1](\alpha\beta\gamma\delta).\end{aligned}$$

It is evident that when the substitution is moved from the right of the multiplier to the left, the new multiplier is the transform of the old by the substitution. Proceeding in the same way, we find upon reversing the order of multiplier and substitution that:

(4).
$$\begin{aligned} (\alpha\beta)[M_1] &= [M_2](\alpha\beta)\\ [M_2] &= \{(\alpha\beta)^{-1}\}^{-1}[M_2](\alpha\beta)^{-1}\end{aligned}$$

(5).
$$\begin{aligned} (\alpha\beta)(\alpha\gamma)[M_1] &= [M_2](\alpha\beta)(\alpha\gamma)\\ [M_2] &= \{(\alpha\beta\gamma)^{-1}\}^{-1}[M_1](\alpha\beta\gamma)^{-1}\end{aligned}$$

(6).
$$\begin{aligned} (\alpha\beta)(\alpha\gamma)(\alpha\delta)[M_1] &= [M_2](\alpha\beta)(\alpha\gamma)(\alpha\delta)\\ [M_2] &= \{(\alpha\beta\gamma\delta)^{-1}\}^{-1}[M_1](\alpha\beta\gamma\delta)^{-1}.\end{aligned}$$

Obviously, when the substitution is moved from the left of the multiplier to the right, the new multiplier is the transform of the old by the inverse of the substitution.

It is to be observed that similar multipliers and substitutions permute; also multipliers and substitutions neither of which contain any marks of the other.

Ex.

$$\begin{aligned}[\alpha\beta](\alpha\beta) &= (\alpha\beta)[\alpha\beta] \\ (\alpha\beta\gamma\delta)[\alpha\beta\gamma\delta] &= [\alpha\beta\gamma\delta](\alpha\beta\gamma\delta) \\ (\alpha\beta)[\gamma\delta] &= [\gamma\delta](\alpha\beta)\end{aligned}$$

Since every substitution can be expressed as the product of transpositions, the laws for the right-hand and left-hand transposition of the substitution symbol hold for all substitutions. Consequently the group of permutations contains only operations reducible to the form [] (). Therefore its order is 16 × 24, since these operations are distinct.

4. *Discussion of Subgroups.* As the multiplication laws show that the 16 multipliers do not affect the 24 substitutions, the possible orders of subgroups of the $G^4{}_{16\times24}$ are those of the orders of the subgroups of the $G^4{}_{24}$, which may be multiplied by 2^k where $k = 0, 1, 2, 3, 4$. The subgroups are therefore of orders: 1, 2, 4, 6, 8, 16, 24, 32, 48, 64, 96, 192, 384.

G_{12} does not occur, for a two-valued function of α, β, γ, δ, necessarily contains $(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)(\beta-\gamma)(\beta-\delta)(\gamma-\delta)$, which of course does not occur in this mere collection.

C_4 does not occur, because if C_4 leaves a collection invariant then if α occurs, α, β, γ, δ occur; if $\alpha\beta\gamma$ occurs, $\beta\gamma\delta$, $\gamma\delta\alpha$, $\delta\alpha\beta$, occur; if $\alpha\beta$ occurs, $\beta\gamma$, $\gamma\delta$, $\delta\alpha$ are present. At most only $\alpha\gamma$, $\beta\delta$ are absent, and these are invariant under G_8.

5. *Determination and Description of Types.* In the main, Clifford's work has been followed, the types overlooked by him in the six- and seven-fold statements having been found by trial. However, the number of mistakes in the eight-fold statements makes necessary for their detection an independent device, which gives some duplicates but includes all the possible types.

(1). The determination of the eight-fold set is accomplished in the following manner:— If one letter, say δ, not occurring oftener than any other be abstracted from an eight-fold type, the residue may be called a base. It represents a set of eight points of a cube, and may be the whole set $I\ \alpha\ \beta\ \gamma\ \alpha\beta\ \alpha\gamma\ \beta\gamma\ \alpha\beta\gamma$ or may have some duplicates and lack the same number. Not more than two can be alike, otherwise in the original set two would have agreed. The types of duplication and omission which are distinct may be classified by geometrical intuition on the cube, and verified by counting the number of conjugates and finding the total.

Ex. If two are omitted and two duplicated, the duplicated may be an edge, a face diagonal or a cube diagonal, and so for the omitted, but they can have no common members.

We get such symbols as $[I, \alpha](\gamma, \alpha\gamma)$, read "Double I, α, omit γ, $\alpha\gamma$." These are opposite edges of the same face. The symbol is invariant under $[\alpha]$, $[\gamma](12)$, i. e. under multiplication by α and under multiplication by γ followed by interchange of parts. Since the whole group on three classes is of order $2^3 \cdot 3! = 48$, and this subgroup is of order 4, there are 12 conjugates to it. Proceeding in this way until we have in all ${}_8C_2 \cdot {}_6C_2/2 = 210$ conjugates, we have a check. Similarly are treated the cases in which one is doubled, one omitted; three are doubled, three omitted; four are doubled, four omitted.

The set of possible bases is then set out, 32 in all. There is 1 base when none are doubled; there are 3 when one is doubled; 12 when two are doubled; 10 when three are doubled; 6 when four are doubled. Each base is now filled with δ's in all possible ways so that the δ does not occur oftener than any other letter. Repetitions which are ineffective because of a group invariance of the base are omitted. Duplications will arise if the number of δ's is equal to that of the γ's, for instance. Actually, out of 108 types found by this method, 74 are distinct.

(2). Throughout, the types have been named with reference to their sets of distances as a preliminary distinction. Taking the distance in a type of every mark from its successive neighbors, the number of times each distance occurs is noted. The five-fold statement $I\ \alpha\beta\gamma\delta\ \alpha\ \beta\gamma\ \alpha\delta$, for example, has distances 4122 : 322 : 31 : 4. The fact that 1 occurs twice, 2 occurs 4 times, 3 twice, 4 twice, is expressed as 2422. Likewise the eight-fold statement $I\ \alpha\ \beta\ \delta\ \alpha\beta\ \alpha\gamma\ \beta\delta\ \gamma\delta$ with distances 1112222 : 221133 : 21313 : 3311 : 224 : 42 : 2 is called 9, 11, 62.

Appearance of more than one type with the same set of distances necessitates further differentiation. As the above eight-fold type contains 3 α's, 3 β's, 2 γ's, 3 δ's, it may be designated a member of the 3323 set, indicated in the tables under "letters." Moreover, the three marks containing α form a triangle with distances 112; the three containing β form a triangle with distances 112; and the three containing δ form a triangle with distances 112. Because two other kinds of triangle are also possible, namely those with distances 123, and 222, the three kinds may be conveniently distinguished as numbers 1, 2 and 3. Since the marks containing γ are at distance 2, the above type 9, 11, 62 bears a further classification, 1121, under "basis."

When an eight-fold type exhibits a letter occurring four times, the four marks represent four corners of a cube. These sets are classified in the following order:

1.	I	α	β	γ	111222
2.	I	α	β	$\alpha\beta$	112211
3.	I	α	β	$\alpha\beta\gamma$	113222
4.	I	α	$\alpha\beta$	$\alpha\beta\gamma$	123121
5.	I	$\alpha\beta$	$\beta\gamma$	$\alpha\gamma$	222222
6.	I	α	$\beta\gamma$	$\alpha\beta\gamma$	123321

$I\ \beta\ \gamma\ \alpha\beta\ \alpha\delta\ \beta\delta\ \alpha\beta\gamma\ \alpha\gamma\delta$ can now be described thus:

Letters	*Distance*	*Basis*
4433	6, 11, 10, 1	6432

It allows a 2 group, with generators I, $[\alpha\beta](\alpha\beta)(\gamma\delta)$ and 192 conjugates.

(3). Should an n-fold type display a letter in more than $E\{n/2\}$ marks, it is impossible to determine the kinds of triangles and quadrilaterals until the type has been reduced to a minimum form in which for two- and three-fold types no letter appears more than once; for four- and five-fold types no letter appears more than twice; for six- and seven-fold types no letter appears more than three times; for eight-fold types no letter appears more than four times. Obviously this reduction can be accomplished by multiplication by the offending letter or letters. If the multiplier be not a member of the type, the identity will be lost. In the tables, the types are set out in minimum form.

Note that, in the tables, (all) means all the permutations on the letters, whereas [all] means multiplication by all the members, of the type form listed.

6. *Errors in Clifford's Work.* This table compares the present results with those of Clifford.

Type	*Clifford*	*Total*	*Corrected*	*Total*	*No. of Conjugates*
One-fold	1	1	1	1	$16 = {}_{16}C_1$
Two-fold	4	4	4	4	$120 = {}_{16}C_2$
Three-fold	6	6	6	6	$560 = {}_{16}C_3$
Four-fold					
Pure	12		12		
1 pair obverses	5		5		
2 pair obverses	2		2		
		19		19	$1820 = {}_{16}C_4$
Five-fold					
Pure	12		12		
1 pair obverses	12		12		
2 pair obverses	3		3		
		27		27	$4368 = {}_{16}C_5$
Six-fold					
Pure	12		14		
1 pair obverses	22		22		
2 pair obverses	11		12		
3 pair obverses	2		2		
		47		50	$8008 = {}_{16}C_6$
Seven-fold					
Pure	8		8		
1 pair obverses	24		23		
2 pair obverses	19		20		
3 pair obverses	4		5		
		55		56	$11,440 = {}_{16}C_7$
Total $<$ eight-fold		159		163	26,332
Complementary		159		163	26,332
Eight-fold					
Pure	7		7		
1 pair obverses	14		17		
2 pair obverses	41		33		
3 pair obverses	12		13		
4 pair obverses	4		4		
		78		74	$12,870 = {}_{16}C_8$
Grand Total		396		400	$65,534 = 2^{2^4}$

The last column of statements forms the final check of the work.

7. *Tables of Statements.* There is of course only one type of simple statement, which may be represented as I, having a group of order 24, since it is susceptible to all interchanges, with 16 conjugates.

Four Classes; Two-fold Statements.

Letters	Distance	Type	Generators	Group Order	Conjugates
1111	0001	$I\ \alpha\beta\gamma\delta$	$(\alpha\beta)$; $(\alpha\gamma)$ $(\alpha\delta)$; $[\alpha\beta\gamma\delta]$	48	8
1110	0010	$I\ \alpha\beta\gamma$	$(\alpha\beta)$; $(\alpha\gamma)$; $[\alpha\beta\gamma]$	12	32
1100	0100	$I\ \alpha\beta$	$(\alpha\beta)$; $(\gamma\delta)$; $[\alpha\beta]$	8	48
1000	1000	$I\ \alpha$	$(\beta\gamma)$; $(\beta\delta)$; $[\alpha]$	12	32
					120

Four Classes; Three-fold Statements

Letters	Distance	Type	Generators	Group Order	Conjugates
1111	0120	$\alpha\ \beta\ \gamma\delta$	$(\alpha\beta)$; $(\gamma\delta)$	4	96
	0201	$I\ \alpha\beta\ \gamma\delta$	$(\alpha\beta)$; $(\alpha\gamma\beta\delta)$	8	48
	1011	$I\ \alpha\ \beta\gamma\delta$	$(\beta\gamma)$; $(\beta\delta)$	6	64
1110	0300	$\alpha\ \beta\ \gamma$	$(\alpha\beta)$; $(\alpha\gamma)$	6	64
	1110	$I\ \alpha\ \beta\gamma$	$(\beta\gamma)$	2	192
1100	2100	$I\ \alpha\ \beta$	$(\alpha\beta)$;$(\gamma\delta)$	4	96
					560

Four Classes; Four-fold Statements.

Letters	Distance	Basis	Type	Generators	Group Order	Conj gate
2222	0240		$I\ \alpha\beta\ \alpha\gamma\delta\ \beta\gamma\delta$	$(\alpha\beta)$; $(\gamma\delta)$; [all]	16	24
	0402		$I\ \alpha\beta\ \gamma\delta\ \alpha\beta\gamma\delta$	$(\alpha\beta)$; $(\alpha\beta\gamma\delta)$; [all]	32	12
	2022		$I\ \alpha\ \beta\gamma\delta\ \alpha\beta\gamma\delta$	$(\beta\gamma)$; $(\beta\delta)$; [all]	24	16
2221	0330		$I\ \alpha\beta\ \beta\gamma\ \alpha\gamma\delta$	$(\alpha\gamma)$; $[\alpha\beta]\ (\alpha\beta)$	6	64
	1221		$I\ \alpha\ \beta\gamma\ \alpha\beta\gamma\delta$	$(\beta\gamma)$	2	192
2220	0600		$I\ \alpha\beta\ \alpha\gamma\ \beta\gamma$	$(\alpha\beta)$; $(\alpha\gamma)$; [all]	24	16
	2220		$I\ \alpha\ \beta\gamma\ \alpha\beta\gamma$	$(\beta\gamma)$; [all]	8	48
2211	0501		$I\ \alpha\beta\ \alpha\delta\ \beta\gamma$	$(\alpha\beta)$; $(\gamma\delta)$; $[\alpha\beta](\alpha\beta)$	4	96
	1230		$I\ \alpha\ \beta\gamma\ \alpha\beta\delta$	$[\alpha](\gamma\delta)$	2	192
	2121	33	$I\ \alpha\ \beta\ \alpha\beta\gamma\delta$	$(\alpha\beta)$; $(\gamma\delta)$	4	96
	2121	11	$I\ \gamma\ \alpha\beta\ \alpha\beta\delta$	$(\alpha\beta)$; $[\alpha\beta](\gamma\delta)$	4	96
2210	2310		$I\ \alpha\ \beta\ \alpha\beta\gamma$	$(\alpha\beta)$	2	192
2200	4200		$I\ \alpha\ \beta\ \alpha\beta$	$(\alpha\beta)$; $(\gamma\delta)$; [all]	16	24
2111	1320		$I\ \delta\ \alpha\beta\ \alpha\gamma$	$(\beta\gamma)$	2	192
	2211		$I\ \alpha\ \beta\ \alpha\gamma\delta$	$(\gamma\delta)$	2	192
2110	3210		$I\ \alpha\ \beta\ \alpha\gamma$	$[\alpha](\beta\gamma)$	2	192
1111	0600		$\alpha\ \beta\ \gamma\ \delta$	$(\alpha\beta)$; $(\alpha\gamma)$; $(\alpha\delta)$	24	16
	2220		$I\ \alpha\ \delta\ \beta\gamma$	$(\alpha\delta)$; $(\beta\gamma)$	4	96
1110	3300		$I\ \alpha\ \beta\ \gamma$	$(\alpha\beta)$; $(\alpha\gamma)$	6	64
						1820

Four Classes; Five-fold Statements.

etters	Distance	Basis	Type					Generators	Group Order	Conjugates
22	0640		α	β	γ	δ	$\alpha\beta\gamma\delta$	$(\alpha\beta);(\alpha\gamma);(\alpha\delta)$	24	16
	0802		I	$\alpha\beta$	$\alpha\gamma$	$\beta\delta$	$\gamma\delta$	$(\alpha\delta);(\alpha\beta)(\gamma\delta)$	8	48
	1450		I	α	$\beta\gamma$	$\gamma\delta$	$\alpha\beta\gamma$	$(\beta\gamma)$	2	192
	1531		α	γ	δ	$\beta\delta$	$\alpha\beta\gamma$	$(\alpha\gamma)$	2	192
	2341	1122	α	β	$\alpha\gamma$	$\beta\delta$	$\gamma\delta$	$(\alpha\beta)(\gamma\delta)$	2	192
	2341	3232	I	α	γ	$\beta\delta$	$\alpha\beta\gamma\delta$	$(\alpha\gamma);(\beta\delta)$	4	96
	2422		I	α	β	$\alpha\gamma\delta$	$\beta\gamma\delta$	$(\alpha\beta);(\gamma\delta)$	4	96
	3232		I	α	$\alpha\beta$	$\gamma\delta$	$\beta\gamma\delta$	$(\gamma\delta)$	2	192
21	0901		α	β	γ	δ	$\alpha\beta\gamma$	$(\alpha\beta);(\alpha\gamma)$	6	64
	1630		I	δ	$\alpha\beta$	$\alpha\gamma$	$\beta\gamma$	$(\alpha\beta);(\alpha\gamma)$	6	64
	2440	223	I	α	β	$\gamma\delta$	$\alpha\beta\gamma$	$(\alpha\beta)$	2	192
	2440	112	α	β	δ	$\alpha\gamma$	$\beta\gamma$	$(\alpha\beta)$	2	192
	2521		I	β	$\alpha\gamma$	$\alpha\delta$	$\beta\gamma$		1	384
	3331	333	I	α	β	γ	$\alpha\beta\gamma\delta$	$(\alpha\beta);(\alpha\gamma)$	6	64
	3331	111	α	β	γ	$\alpha\delta$	$\beta\gamma$	$(\beta\gamma)$	2	192
	3331	211	I	α	δ	$\beta\gamma$	$\alpha\beta\gamma$	$(\beta\gamma)$	2	192
	3331	213	I	α	β	$\beta\gamma$	$\alpha\gamma\delta$		1	384
20	3610		I	α	β	γ	$\alpha\beta\gamma$	$(\alpha\beta);(\alpha\gamma)$	6	64
	4420		I	α	β	$\alpha\gamma$	$\beta\gamma$	$(\alpha\beta)$	2	192
11	2620		α	β	γ	δ	$\alpha\beta$	$(\alpha\beta);(\gamma\delta)$	4	96
	3430		I	β	δ	$\alpha\beta$	$\alpha\gamma$		1	384
	3511		I	α	β	γ	$\alpha\beta\delta$	$(\alpha\beta)$	2	192
	4321	11	I	α	β	$\alpha\delta$	$\beta\gamma$	$(\alpha\beta)(\gamma\delta)$	2	192
	4321	11	I	α	β	$\alpha\beta$	$\gamma\delta$	$(\alpha\beta);(\gamma\delta)$	4	96
210	5410		I	α	β	γ	$\alpha\beta$	$(\alpha\beta)$	2	192
111	4420		I	α	β	γ	$\alpha\delta$	$(\beta\gamma)$	2	192
111	4600		I	α	β	γ	δ	$(\alpha\beta);(\alpha\gamma);(\alpha\delta)$	24	16
										4368

Four Classes; Six-fold Statements.

etters	Distance	Basis	Type						Generators	Group Order	Conjugates
3333	0, 12, 03		I	$\alpha\gamma$	$\alpha\delta$	$\beta\gamma$	$\beta\delta$	$\alpha\beta\gamma\delta$	$(\alpha\beta);(\gamma\delta);[\alpha\beta\gamma\delta];[\beta\gamma](\beta\gamma)$	48	8
	2661		I	α	$\beta\gamma$	$\gamma\delta$	$\alpha\beta\delta$	$\alpha\beta\gamma\delta$	$(\beta\delta);[\alpha\beta\gamma\delta](\alpha\gamma)$	4	96
	4443		I	α	$\alpha\delta$	$\beta\gamma$	$\beta\gamma\delta$	$\alpha\beta\gamma\delta$	$(\beta\gamma);[\beta\gamma](\alpha\delta);[\alpha\beta\gamma\delta]$	8	48
332	1941		α	β	γ	δ	$\alpha\beta\gamma$	$\alpha\beta\gamma\delta$	$(\alpha\beta);(\alpha\gamma)$	6	64
	2670		I	α	$\beta\gamma$	$\beta\delta$	$\alpha\beta\gamma$	$\alpha\gamma\delta$	$[\alpha\beta\gamma];[\beta\gamma](\beta\gamma)$	4	96
	2832		α	γ	δ	$\beta\gamma$	$\alpha\beta\gamma$	$\alpha\beta\delta$	$[\alpha\beta](\alpha\beta)$	2	192
	3561		I	α	γ	$\beta\delta$	$\alpha\beta\gamma$	$\alpha\beta\gamma\delta$	$(\alpha\gamma)$	2	192
	4452	1213	I	β	δ	$\alpha\gamma$	$\alpha\beta\gamma$	$\alpha\beta\gamma\delta$	$(\alpha\gamma);[\alpha\beta\gamma]$	4	96
	4452	2221	I	α	γ	$\alpha\beta$	$\beta\gamma\delta$	$\alpha\beta\gamma\delta$	$[\alpha](\beta\gamma)$	2	192
3331	4641	222	I	α	β	γ	$\alpha\beta\gamma$	$\alpha\beta\gamma\delta$	$(\alpha\beta);(\alpha\gamma)$	6	64
	4641	232	I	α	γ	$\alpha\beta$	$\beta\gamma$	$\alpha\beta\gamma\delta$	$(\alpha\gamma)$	2	192
3330	6630		I	α	γ	$\alpha\beta$	$\beta\gamma$	$\alpha\beta\gamma$	$(\alpha\gamma);[\alpha](\beta\gamma);[\alpha\beta\gamma]$	12	32
3322	0, 13, 02		α	β	γ	δ	$\alpha\beta\gamma$	$\alpha\beta\delta$	$(\alpha\beta);(\gamma\delta);[\alpha\beta]$	8	48
	2760	3233	I	β	$\alpha\beta$	$\alpha\gamma$	$\alpha\delta$	$\beta\gamma\delta$	$(\gamma\delta);[\alpha\beta](\alpha\beta)$	4	96

Four Classes; Six-fold Statements—(*Continued*).

Letters	Distance	Basis	Type						Generators	Group Order	Conjugates
	2760	2222	α	β	γ	$\alpha\delta$	$\beta\delta$	$\alpha\beta\gamma$	$(\alpha\beta);[\alpha\beta]$	4	96
	3651	3232	I	β	γ	$\alpha\beta$	$\alpha\delta$	$\alpha\beta\gamma\delta$		1	384
	3651	1212	α	γ	δ	$\alpha\beta$	$\beta\gamma$	$\alpha\beta\delta$		1	384
	4542	2222	I	α	β	$\alpha\beta$	$\gamma\delta$	$\alpha\beta\gamma\delta$	$(\alpha\beta);(\gamma\delta);[\alpha\beta]$	8	48
	4542	2222	I	α	β	$\alpha\delta$	$\beta\gamma$	$\alpha\beta\gamma\delta$	$(\alpha\beta)(\gamma\delta)$	2	192
	4542	2213·	I	α	β	δ	$\alpha\beta\gamma$	$\alpha\beta\gamma\delta$	$(\alpha\beta)$	2	192
	4542	2111	I	γ	$\alpha\beta$	$\alpha\gamma$	$\beta\delta$	$\alpha\beta\delta$	$[\alpha\beta](\gamma\delta)$	2	192
	4542	1122	I	γ	δ	$\alpha\beta$	$\alpha\beta\gamma$	$\alpha\beta\delta$	$(\alpha\beta);(\gamma\delta);[\alpha\beta]$	8	48
3321	3921		I	α	β	γ	$\alpha\beta\gamma$	$\alpha\beta\delta$	$(\alpha\beta)$	2	192
	4650		I	α	β	$\alpha\gamma$	$\beta\gamma$	$\alpha\beta\delta$	$(\alpha\beta)$	2	192
	5541	223	I	α	β	γ	$\alpha\beta$	$\alpha\beta\gamma\delta$	$(\alpha\beta)$	2	192
	5541	211	I	β	γ	$\alpha\beta$	$\alpha\gamma$	$\alpha\beta\delta$		1	384
3320	6720		I	α	β	$\alpha\beta$	$\alpha\gamma$	$\beta\gamma$	$(\alpha\beta);[\alpha\beta]$	4	96
3311	6621		I	α	β	δ	$\alpha\beta$	$\alpha\beta\gamma$	$(\alpha\beta);[\alpha\beta](\gamma\delta)$	4	96
3222	2931		I	δ	$\alpha\beta$	$\alpha\gamma$	$\alpha\delta$	$\beta\gamma$	$(\beta\gamma)$	2	192
	3660	2232	I	α	δ	$\alpha\beta$	$\beta\gamma$	$\alpha\gamma\delta$	$[\alpha](\beta\delta)$	2	192
	3660	3111	β	γ	δ	$\alpha\beta$	$\alpha\gamma$	$\alpha\delta$	$(\beta\gamma);(\beta\delta);[\alpha]$	12	32
	3822		I	α	β	δ	$\alpha\beta\gamma$	$\alpha\gamma\delta$	$(\beta\delta)$	2	192
	4551	2233	I	α	γ	δ	$\alpha\beta$	$\alpha\beta\gamma\delta$	$(\gamma\delta)$	2	192
	4551	2211	I	β	γ	$\alpha\gamma$	$\alpha\delta$	$\alpha\beta\delta$		1	384
	5442	2111	I	α	β	$\alpha\beta$	$\gamma\delta$	$\alpha\gamma\delta$	$(\gamma\delta);[\alpha]$	4	96
	5442	1131	I	α	β	$\alpha\delta$	$\beta\gamma$	$\alpha\gamma\delta$	$[\alpha](\beta\delta)$	2	192
3221	4740	311	I	β	γ	$\alpha\beta$	$\alpha\gamma$	$\alpha\delta$	$(\beta\gamma)$	2	192
	4740	122	I	α	δ	$\alpha\beta$	$\alpha\gamma$	$\beta\gamma$	$(\beta\gamma)$	2	192
	5631	212	I	α	β	γ	$\alpha\beta$	$\alpha\gamma\delta$		1	384
	5631	121	I	α	β	δ	$\alpha\gamma$	$\alpha\beta\gamma$		1	384
3220	7620		I	α	β	γ	$\alpha\beta$	$\alpha\gamma$	$(\beta\gamma);[\alpha]$	4	96
3211	6630		I	α	β	γ	$\alpha\beta$	$\alpha\delta$	$[\alpha](\gamma\delta)$	2	192
2222	4641	2333	I	α	β	γ	δ	$\alpha\beta\gamma\delta$	$(\alpha\beta);(\alpha\gamma);(\alpha\delta)$	24	16
	4641	1111	α	β	γ	δ	$\alpha\beta$	$\gamma\delta$	$(\alpha\beta);(\alpha\gamma\beta\delta)$	8	48
	4641	1232	I	α	β	δ	$\alpha\gamma$	$\beta\gamma\delta$	$(\beta\delta)$	2	192
	4641	1212	I	α	γ	$\alpha\delta$	$\beta\gamma$	$\beta\delta$	$(\alpha\gamma)(\beta\delta)$	2	192
2221	4911		I	α	β	γ	δ	$\alpha\beta\gamma$	$(\alpha\beta);(\alpha\gamma)$	6	64
	5640		I	β	γ	δ	$\alpha\beta$	$\alpha\gamma$	$(\beta\gamma)$	2	192
	6531		I	α	β	γ	$\alpha\beta$	$\gamma\delta$	$(\alpha\beta)$	2	192
2211	6720		I	α	β	γ	δ	$\alpha\beta$	$(\alpha\beta);(\gamma\delta)$	4	96
											8008

Four Classes; Seven-fold Statements.

Letters	Distance	Basis	Type							Generators	Group Order	Conjugates
3333	0, 18, 03		I	$\alpha\beta$	$\alpha\gamma$	$\alpha\delta$	$\beta\gamma$	$\beta\delta$	$\gamma\delta$	$(\alpha\beta);(\alpha\gamma);(\alpha\delta)$	24	16
	2, 13, 42		α	β	γ	δ	$\alpha\beta$	$\alpha\gamma\delta$	$\beta\gamma\delta$	$(\alpha\beta);(\gamma\delta)$	4	96
	3990		α	β	γ	$\alpha\delta$	$\beta\delta$	$\gamma\delta$	$\alpha\beta\gamma$	$(\alpha\beta);(\alpha\gamma)$	6	64
	3, 10, 71		I	δ	$\alpha\beta$	$\alpha\gamma$	$\alpha\delta$	$\beta\gamma$	$\beta\gamma\delta$	$(\beta\gamma)$	2	192
	3, 12, 33		I	β	γ	δ	$\alpha\beta\gamma$	$\alpha\beta\delta$	$\alpha\gamma\delta$	$(\beta\gamma);(\beta\delta)$	6	64
	4881	2222	α	β	γ	δ	$\alpha\beta$	$\gamma\delta$	$\alpha\beta\gamma\delta$	$(\alpha\beta);(\alpha\gamma\beta\delta)$	8	48
	4881	2223	I	α	δ	$\alpha\beta$	$\beta\gamma$	$\alpha\gamma\delta$	$\beta\gamma\delta$		1	384
	4962	3232	I	β	δ	$\alpha\gamma$	$\alpha\delta$	$\beta\gamma$	$\alpha\beta\gamma\delta$	$(\alpha\gamma)(\beta\delta)$	2	192

Four Classes; Seven-fold Statements—(*Continued*).

Letters	Distance	Basis	Type							Generators	Group Order	Conjugates
	4962	1331	α	δ	$\alpha\beta$	$\alpha\gamma$	$\beta\gamma$	$\beta\delta$	$\gamma\delta$	$(\beta\gamma)$; $(\alpha\delta)$	4	96
	5772	2211	β	γ	δ	$\alpha\gamma$	$\alpha\delta$	$\beta\delta$	$\alpha\beta\gamma$		1	384
	5772	2222	I	β	γ	δ	$\alpha\delta$	$\alpha\beta\gamma$	$\alpha\beta\gamma\delta$	$(\beta\gamma)$	2	192
	5853		I	δ	$\alpha\gamma$	$\alpha\delta$	$\beta\gamma$	$\beta\delta$	$\alpha\beta\gamma$	$(\alpha\beta)$	2	192
	6663	2211	I	α	β	$\alpha\beta$	$\gamma\delta$	$\beta\gamma\delta$	$\alpha\gamma\delta$	$(\alpha\beta)$; $(\gamma\delta)$	4	96
	6663	1122	I	α	β	$\alpha\delta$	$\beta\gamma$	$\beta\gamma\delta$	$\alpha\gamma\delta$	$(\alpha\beta)(\gamma\delta)$	2	192
3332	3, 13, 32		I	α	$\alpha\beta$	$\alpha\gamma$	$\beta\gamma$	$\beta\delta$	$\gamma\delta$	$(\beta\gamma)$	2	192
	4, 10, 61	3323	I	α	β	γ	$\gamma\delta$	$\alpha\beta\gamma$	$\alpha\beta\delta$	$(\alpha\beta)$	2	192
	4, 10, 61	2212	α	β	γ	δ	$\alpha\gamma$	$\beta\gamma$	$\alpha\beta\delta$	$(\alpha\beta)$	2	192
	5871	1222	I	α	δ	$\alpha\beta$	$\alpha\gamma$	$\beta\gamma$	$\beta\gamma\delta$	$(\beta\gamma)$	2	192
	5871	1211	α	γ	δ	$\alpha\beta$	$\alpha\gamma$	$\beta\gamma$	$\beta\delta$	$(\alpha\gamma)$	2	192
	5871	2212	I	β	γ	$\alpha\gamma$	$\alpha\delta$	$\beta\delta$	$\alpha\beta\gamma$		1	384
	5871	2233	I	α	β	δ	$\alpha\gamma$	$\beta\gamma$	$\alpha\beta\gamma\delta$	$(\alpha\beta)$	2	192
	5952	2232	I	α	β	γ	$\alpha\beta$	$\alpha\gamma\delta$	$\beta\gamma\delta$	$(\alpha\beta)$	2	192
	5952	1232	I	α	γ	δ	$\alpha\beta$	$\alpha\beta\gamma$	$\beta\gamma\delta$		1	384
	5952	1121	α	β	γ	δ	$\alpha\beta$	$\gamma\delta$	$\alpha\beta\gamma$	$(\alpha\beta)$	2	192
	5952	2223	I	α	β	γ	δ	$\alpha\beta\gamma$	$\alpha\beta\gamma\delta$	$(\alpha\beta)$; $(\alpha\gamma)$	6	64
	6762	1121	I	α	δ	$\alpha\beta$	$\beta\gamma$	$\gamma\delta$	$\alpha\beta\gamma$		1	384
	6762	1123	I	α	β	$\alpha\gamma$	$\beta\gamma$	$\beta\delta$	$\alpha\gamma\delta$		1	384
	6762	2211	I	β	γ	$\alpha\gamma$	$\alpha\delta$	$\beta\gamma$	$\alpha\beta\delta$		1	384
	6762	1112	α	β	γ	$\alpha\beta$	$\alpha\gamma$	$\beta\delta$	$\gamma\delta$	$(\beta\gamma)$	2	192
	6762	2222	I	α	β	γ	$\alpha\delta$	$\beta\gamma$	$\alpha\beta\gamma\delta$	$(\beta\gamma)$	2	192
3331	4980		I	β	γ	$\alpha\beta$	$\alpha\gamma$	$\alpha\delta$	$\beta\gamma\delta$	$(\beta\gamma)$	2	192
	6960		α	β	γ	δ	$\alpha\beta$	$\alpha\gamma$	$\beta\gamma$	$(\alpha\beta)$; $(\alpha\gamma)$	6	64
	6, 10, 41		I	α	γ	$\alpha\beta$	$\alpha\gamma$	$\beta\gamma$	$\beta\delta$	$(\alpha\gamma)$	2	192
	7851	112	I	α	β	γ	$\alpha\beta$	$\gamma\delta$	$\alpha\beta\gamma$	$(\alpha\beta)$	2	192
	7851	111	I	α	β	δ	$\alpha\gamma$	$\beta\gamma$	$\alpha\beta\gamma$	$(\alpha\beta)$	2	192
	7851	122	I	α	β	γ	$\alpha\beta$	$\alpha\gamma$	$\beta\gamma\delta$	$(\beta\gamma)$	2	192
3330	9930		I	α	β	γ	$\alpha\beta$	$\alpha\gamma$	$\beta\gamma$	$(\alpha\beta)$; $(\alpha\gamma)$	6	64
3322	4, 13, 22		I	α	β	γ	δ	$\alpha\beta\gamma$	$\alpha\beta\delta$	$(\alpha\beta)$; $(\gamma\delta)$	4	96
	5970		I	α	β	γ	$\alpha\delta$	$\beta\delta$	$\alpha\beta\gamma$	$(\alpha\beta)$	2	192
	5, 10, 51		I	β	δ	$\alpha\beta$	$\alpha\gamma$	$\alpha\delta$	$\beta\gamma$		1	384
	6861	2233	I	α	β	γ	δ	$\alpha\beta$	$\alpha\beta\gamma\delta$	$(\alpha\beta)$; $(\gamma\delta)$	4	96
	6861	2112	I	β	γ	δ	$\alpha\beta$	$\alpha\gamma$	$\alpha\beta\delta$		1	384
	6861	1232	I	α	β	δ	$\alpha\beta$	$\alpha\gamma$	$\beta\gamma\delta$		1	384
	6861	1111	α	β	γ	δ	$\alpha\beta$	$\alpha\gamma$	$\beta\delta$	$(\alpha\beta)(\gamma\delta)$	2	192
	6942	1122	I	α	β	$\alpha\beta$	$\alpha\gamma$	$\beta\delta$	$\gamma\delta$	$(\alpha\beta)(\gamma\delta)$	2	192
	6942	1122	I	α	β	$\alpha\gamma$	$\alpha\delta$	$\beta\gamma$	$\beta\delta$	$(\alpha\beta)$; $(\gamma\delta)$	4	96
	7752	1211	I	α	β	γ	$\alpha\delta$	$\beta\gamma$	$\alpha\beta\delta$		1	384
	7752	1131	I	α	β	δ	$\alpha\beta$	$\gamma\delta$	$\alpha\beta\gamma$	$(\alpha\beta)$	2	192
3321	7950		I	α	β	δ	$\alpha\beta$	$\alpha\gamma$	$\beta\gamma$	$(\alpha\beta)$	2	192
	7, 10, 31		I	α	β	γ	δ	$\alpha\beta$	$\alpha\beta\gamma$	$(\alpha\beta)$	2	192
	8841		I	α	β	γ	$\alpha\beta$	$\alpha\delta$	$\beta\gamma$		1	384
3222	6960		I	β	γ	δ	$\alpha\beta$	$\alpha\gamma$	$\alpha\delta$	$(\beta\gamma)$; $(\beta\delta)$	6	64
	6, 10, 41		I	α	β	γ	δ	$\alpha\delta$	$\alpha\beta\gamma$	$(\beta\gamma)$	2	192
	7851		I	α	β	γ	$\alpha\beta$	$\alpha\delta$	$\gamma\delta$		1	384
3221	8940		I	α	β	γ	δ	$\alpha\beta$	$\alpha\gamma$	$(\beta\gamma)$	2	192
2222	8841		I	α	β	γ	δ	$\alpha\beta$	$\gamma\delta$	$(\alpha\beta)$; $(\alpha\gamma\beta\delta)$	8	48
												11,440

Four Classes; Eight-fold Statements.

Letters	Distance	Basis	Type							
4444	0, 24, 04		I	$\alpha\beta$	$\alpha\gamma$	$\alpha\delta$	$\beta\gamma$	$\beta\delta$	$\gamma\delta$	$\alpha\beta\gamma\delta$
	4, 12, 12, 0		I	γ	$\alpha\beta$	$\alpha\delta$	$\beta\delta$	$\alpha\beta\gamma$	$\alpha\gamma\delta$	$\beta\gamma\delta$
	4, 14, 82		I	$\alpha\gamma$	$\alpha\delta$	$\beta\gamma$	$\beta\delta$	$\gamma\delta$	$\alpha\beta\gamma$	$\alpha\beta\delta$
	6, 10, 10, 2		I	γ	$\alpha\gamma$	$\alpha\delta$	$\beta\delta$	$\alpha\beta\gamma$	$\alpha\beta\delta$	$\beta\gamma\delta$
	6, 12, 64		I	α	β	γ	$\alpha\beta\delta$	$\alpha\gamma\delta$	$\beta\gamma\delta$	$\alpha\beta\gamma\delta$
	8884	2662	I	β	γ	$\alpha\delta$	$\beta\gamma$	$\alpha\beta\delta$	$\alpha\gamma\delta$	$\alpha\beta\gamma\delta$
	8884	4444	I	α	β	$\alpha\delta$	$\beta\gamma$	$\alpha\gamma\delta$	$\beta\gamma\delta$	$\alpha\beta\gamma\delta$
4443	3, 18, 43		I	γ	$\alpha\beta$	$\alpha\gamma$	$\alpha\delta$	$\beta\gamma$	$\beta\delta$	$\alpha\beta\gamma\delta$
	5, 12, 10, 1		I	α	$\alpha\gamma$	$\beta\gamma$	$\beta\delta$	$\gamma\delta$	$\alpha\beta\gamma$	$\alpha\beta\delta$
	6, 11, 92	4443	β	γ	$\alpha\beta$	$\alpha\gamma$	$\alpha\delta$	$\beta\delta$	$\gamma\delta$	$\alpha\beta\gamma$
	6, 11, 92	3333	I	γ	δ	$\alpha\beta$	$\alpha\gamma$	$\beta\gamma$	$\alpha\beta\delta$	$\alpha\beta\gamma\delta$
	7, 10, 83	3331	I	α	β	$\alpha\beta$	$\gamma\delta$	$\alpha\beta\gamma$	$\alpha\gamma\delta$	$\beta\gamma\delta$
	7, 10, 83	3412	I	β	δ	$\alpha\gamma$	$\alpha\delta$	$\beta\gamma$	$\alpha\beta$	$\alpha\beta\gamma\delta$
4442	6, 12, 10, 0		I	β	γ	$\alpha\beta$	$\alpha\gamma$	$\alpha\delta$	$\alpha\beta\gamma$	$\beta\gamma\delta$
	6, 14, 62		I	γ	$\alpha\beta$	$\alpha\gamma$	$\alpha\delta$	$\beta\gamma$	$\beta\delta$	$\alpha\beta\gamma$
	8, 10, 82	3421	I	α	γ	$\alpha\gamma$	$\beta\gamma$	$\beta\delta$	$\alpha\beta\gamma$	$\alpha\beta\delta$
	8, 10, 82	4462	I	α	β	γ	$\alpha\beta$	$\gamma\delta$	$\alpha\beta\gamma$	$\alpha\beta\gamma\delta$
	8, 10, 82	4413	I	α	β	δ	$\alpha\gamma$	$\beta\gamma$	$\alpha\beta\gamma$	$\alpha\beta\gamma\delta$
4441	9, 12, 61		I	β	γ	$\alpha\beta$	$\alpha\gamma$	$\alpha\delta$	$\beta\gamma$	$\alpha\beta\gamma$
4440	12, 12, 40		I	α	β	γ	$\alpha\beta$	$\alpha\gamma$	$\beta\gamma$	$\alpha\beta\gamma$
4433	5, 14, 72		I	β	γ	$\alpha\beta$	$\alpha\gamma$	$\alpha\delta$	$\beta\delta$	$\alpha\beta\gamma\delta$
	6, 11, 10, 1	3322	I	β	γ	$\alpha\beta$	$\alpha\gamma$	$\alpha\delta$	$\alpha\beta\delta$	$\beta\gamma\delta$
	6, 11, 10, 1	6432	I	β	γ	$\alpha\beta$	$\alpha\delta$	$\beta\delta$	$\alpha\beta\gamma$	$\alpha\gamma\delta$
	6, 13, 63	3311	I	γ	$\alpha\beta$	$\alpha\gamma$	$\alpha\delta$	$\beta\gamma$	$\beta\delta$	$\alpha\beta\delta$
	6, 13, 63	3333	I	α	β	γ	$\alpha\beta$	$\alpha\beta\delta$	$\alpha\gamma\delta$	$\beta\gamma\delta$
	6, 13, 63	3133	I	β	γ	δ	$\alpha\beta$	$\alpha\beta\gamma$	$\alpha\beta\delta$	$\alpha\gamma\delta$
	6, 13, 63	1322	I	α	$\alpha\beta$	$\alpha\gamma$	$\alpha\delta$	$\beta\gamma$	$\beta\delta$	$\beta\gamma\delta$
	7, 10, 92	1422	I	α	δ	$\alpha\beta$	$\beta\gamma$	$\gamma\delta$	$\alpha\beta\gamma$	$\alpha\beta\delta$
	7, 10, 92	3411	I	α	γ	$\alpha\delta$	$\beta\gamma$	$\beta\delta$	$\alpha\beta\gamma$	$\alpha\beta\delta$
	7, 10, 92	4322	I	α	β	$\alpha\beta$	$\alpha\delta$	$\gamma\delta$	$\alpha\beta\gamma$	$\beta\gamma\delta$
	8983	4411	I	γ	δ	$\alpha\beta$	$\alpha\delta$	$\beta\gamma$	$\alpha\beta\gamma$	$\alpha\beta\delta$
	8983	4422	I	α	β	$\alpha\beta$	$\alpha\delta$	$\beta\gamma$	$\alpha\gamma\delta$	$\beta\gamma\delta$
	8983	6212	I	β	γ	$\alpha\beta$	$\alpha\gamma$	$\beta\delta$	$\alpha\beta\delta$	$\alpha\gamma\delta$
4432	7, 12, 81	3412	I	β	δ	$\alpha\beta$	$\alpha\gamma$	$\alpha\delta$	$\beta\gamma$	$\alpha\beta\delta$
	7, 12, 81	4433	I	α	β	γ	$\alpha\beta$	$\alpha\delta$	$\alpha\beta\gamma$	$\beta\gamma\delta$
	8, 11, 72	3211	I	β	δ	$\alpha\beta$	$\alpha\gamma$	$\alpha\delta$	$\beta\gamma$	$\alpha\beta\gamma$
	8, 11, 72	1322	I	α	β	δ	$\alpha\beta$	$\alpha\gamma$	$\alpha\beta\gamma$	$\beta\gamma\delta$
	8, 11, 72	4412	I	α	β	$\alpha\gamma$	$\alpha\delta$	$\beta\gamma$	$\beta\delta$	$\alpha\beta\gamma$
	8, 11, 72	1423	I	α	β	$\alpha\beta$	$\alpha\gamma$	$\alpha\delta$	$\beta\gamma$	$\beta\gamma\delta$
4431	10, 11, 61		I	α	β	δ	$\alpha\beta$	$\alpha\gamma$	$\beta\gamma$	$\alpha\beta\gamma$
4422	8, 12, 80		I	α	β	δ	$\alpha\beta$	$\alpha\gamma$	$\beta\gamma$	$\alpha\beta\delta$
	8, 14, 42		I	α	β	$\alpha\beta$	$\alpha\gamma$	$\alpha\delta$	$\beta\gamma$	$\beta\delta$
	10, 10, 62		I	α	β	δ	$\alpha\beta$	$\alpha\gamma$	$\beta\delta$	$\alpha\beta\gamma$
4333	4, 18, 33		I	α	$\alpha\beta$	$\alpha\gamma$	$\alpha\delta$	$\beta\gamma$	$\beta\delta$	$\gamma\delta$
	6, 12, 91	5222	I	β	γ	δ	$\alpha\beta$	$\alpha\gamma$	$\alpha\delta$	$\alpha\beta\gamma\delta$
	6, 12, 91	3223	I	α	β	δ	$\alpha\gamma$	$\beta\gamma$	$\alpha\beta\delta$	$\alpha\gamma\delta$
	7, 11, 82	3211	I	γ	δ	$\alpha\beta$	$\alpha\gamma$	$\alpha\delta$	$\beta\delta$	$\alpha\beta\gamma$
	7, 11, 82	6311	I	β	γ	δ	$\alpha\gamma$	$\alpha\delta$	$\alpha\beta\gamma$	$\alpha\beta\delta$
	7, 11, 82	4322	I	α	β	δ	$\alpha\gamma$	$\alpha\delta$	$\alpha\beta\gamma$	$\beta\gamma\delta$
	7, 11, 82	1222	I	α	δ	$\alpha\beta$	$\alpha\gamma$	$\alpha\delta$	$\beta\gamma$	$\beta\gamma\delta$

Four Classes; Eight-fold Statements—*Continued*).

Letters	Distance	Basis	Type							
	8, 10, 73	4211	I	α	δ	$\alpha\gamma$	$\alpha\delta$	$\beta\gamma$	$\beta\delta$	$\alpha\beta\gamma$
	8, 10, 73	2212	I	α	γ	$\alpha\beta$	$\alpha\delta$	$\beta\gamma$	$\gamma\delta$	$\alpha\beta\delta$
4332	7, 14, 52		I	α	γ	$\alpha\beta$	$\alpha\gamma$	$\alpha\delta$	$\beta\gamma$	$\beta\delta$
	8, 11, 81	3111	I	β	γ	δ	$\alpha\beta$	$\alpha\gamma$	$\alpha\delta$	$\alpha\beta\gamma$
	8, 11, 81	1223	I	α	β	γ	$\alpha\beta$	$\alpha\gamma$	$\alpha\delta$	$\beta\gamma\delta$
	8, 11, 81	4122	I	α	β	δ	$\alpha\beta$	$\alpha\gamma$	$\beta\gamma$	$\alpha\gamma\delta$
	9, 10, 72	4211	I	α	β	γ	$\alpha\gamma$	$\alpha\delta$	$\beta\gamma$	$\alpha\beta\delta$
	9, 10, 72	4111	I	α	β	δ	$\alpha\gamma$	$\alpha\delta$	$\beta\gamma$	$\alpha\beta\gamma$
	9, 10, 72	4113	I	α	β	γ	$\alpha\beta$	$\alpha\gamma$	$\beta\delta$	$\alpha\gamma\delta$
4331	10, 12, 51		I	α	β	γ	δ	$\alpha\gamma$	$\beta\gamma$	$\alpha\beta\gamma$
4322	9, 12, 61		I	α	β	δ	$\alpha\beta$	$\alpha\gamma$	$\alpha\delta$	$\beta\gamma$
4222	10, 12, 60		I	α	β	γ	δ	$\alpha\beta$	$\alpha\gamma$	$\alpha\delta$
3333	6, 12, 10, 0		I	β	γ	δ	$\alpha\beta$	$\alpha\gamma$	$\alpha\delta$	$\beta\gamma\delta$
	6, 14, 62	3311	I	γ	δ	$\alpha\beta$	$\alpha\gamma$	$\alpha\delta$	$\beta\gamma$	$\beta\delta$
	6, 14, 62	2233	I	α	β	γ	δ	$\alpha\gamma$	$\alpha\beta\delta$	$\beta\gamma\delta$
	8, 10, 82	2211	I	β	γ	δ	$\alpha\gamma$	$\alpha\delta$	$\beta\gamma$	$\alpha\beta\delta$
	8, 10, 82	2222	I	α	β	γ	δ	$\alpha\beta$	$\gamma\delta$	$\alpha\beta\gamma\delta$
	8, 10, 82	1111	α	β	γ	δ	$\alpha\beta$	$\alpha\gamma$	$\beta\delta$	$\gamma\delta$
3332	8, 12, 71	3111	I	β	γ	δ	$\alpha\beta$	$\alpha\gamma$	$\alpha\delta$	$\beta\gamma$
	8, 12, 71	2122	I	α	β	γ	δ	$\alpha\gamma$	$\beta\gamma$	$\alpha\beta\delta$
	9, 11, 62	2111	I	α	β	γ	δ	$\alpha\gamma$	$\beta\delta$	$\alpha\beta\delta$
	9, 11, 62	1112	I	α	β	δ	$\alpha\beta$	$\alpha\gamma$	$\beta\delta$	$\gamma\delta$
3331	10, 12, 60		I	α	β	γ	δ	$\alpha\beta$	$\alpha\gamma$	$\beta\gamma$
3322	10, 11, 61		I	α	β	γ	δ	$\alpha\beta$	$\alpha\gamma$	$\gamma\delta$

Four Classes; Eight-fold Statements.

Distance	Generators	Group Order	Conjugates
0, 24, 04	(all) ; $[\alpha\beta]$; $[\alpha\gamma]$; $[\alpha\delta]$	192	2
4, 12, 12, 0	$(\alpha\beta)$; $(\alpha\delta)$; [all]	48	8
4, 14, 82	$(\alpha\beta)$; $(\gamma\delta)$; $[\gamma\delta]$	8	48
6, 10, 10, 2	$[\alpha\delta](\beta\gamma)$; $[\beta\gamma\delta]$	4	96
6, 12, 64	$(\alpha\beta)$; $(\alpha\gamma)$; $[\alpha\beta\gamma\delta]$	12	32
8884	$(\beta\gamma)$; $(\alpha\delta)$; [all]	32	12
8884	$[\alpha](\alpha\beta\gamma\delta)$; $[\alpha](\beta\delta)$; $[\alpha\beta\gamma\delta]$	16	24
3, 18, 43	$[\alpha\gamma](\alpha\gamma)$; $[\beta\gamma](\beta\gamma)$;	6	64
5, 12, 10, 1	$[\beta\gamma](\beta\gamma)$	2	192
6, 11, 92	$(\beta\gamma)$	2	192
6, 11, 92	$(\alpha\beta)$	2	192
7, 10, 83	$(\alpha\beta)$	2	192
7, 10, 83		1	384
6, 12, 10, 0	$(\beta\gamma)$; $[\alpha\beta](\alpha\beta)$; $[\alpha\beta\gamma]$	12	32
6, 14, 62	$(\alpha\beta)$; $[\alpha\beta]$	4	96
8, 10, 82	$[\beta\gamma](\beta\gamma)$; $[\alpha]$	4	96
8, 10, 82	$(\alpha\beta)$; $[\alpha\beta]$	4	96
8, 10, 82	$(\alpha\beta)$; $[\alpha\beta\gamma]$	4	96
9, 12, 61	$(\beta\gamma)$; $[\alpha\beta](\alpha\beta)$	6	64
12, 12, 40	$(\alpha\beta)$; $(\alpha\gamma)$; [all]	48	8
5, 14, 72		1	384
6, 11, 10, 1	$[\alpha\beta](\alpha\beta)(\gamma\delta)$	2	192
6, 11, 10, 1	$[\alpha\beta](\alpha\beta)$	2	192
6, 13, 63	$(\alpha\beta)$; $[\alpha\beta](\gamma\delta)$	4	96

Four Classes; Eight-fold Statements—(*Continued*).

Distance	Generators	Group Order	Conjugates
6, 13, 63	$(\alpha\beta)$; $[\alpha\beta](\gamma\delta)$	4	96
6, 13, 63	$(\gamma\delta)$; $[\alpha\beta](\alpha\beta)$	4	96
6, 13, 63	$(\gamma\delta)$; $[\alpha\beta](\alpha\beta)$	4	96
7, 10, 92		1	384
7, 10, 92	$[\alpha](\gamma\delta)$	2	192
7, 10, 92	$[\beta](\gamma\delta)$	2	192
8983	$(\alpha\beta)(\gamma\delta)$; $[\alpha\beta](\alpha\beta)$	4	96
8983	$(\alpha\beta)(\gamma\delta)$; $[\alpha\beta](\gamma\delta)$	4	96
8983		1	384
7, 12, 81	$[\alpha\beta](\alpha\beta)$	2	192
7, 12, 81	$[\alpha\beta](\alpha\beta)$	2	192
8, 11, 72		1	384
8, 11, 72		1	384
8, 11, 72	$(\alpha\beta)$	2	192
8, 11, 72	$[\alpha\beta](\alpha\beta)$	2	192
10, 11, 61	$(\alpha\beta)$	2	192
8, 12, 80	$(\alpha\beta)$; $[\alpha](\gamma\delta)$; $[\alpha\beta]$	8	48
8, 14, 42	$(\alpha\beta)$; $(\gamma\delta)$; $[\alpha\beta]$	8	48
10, 10, 62	$(\alpha\beta)(\gamma\delta)$; $[\beta]$	4	96
4, 18, 33	$(\beta\gamma)$; $(\gamma\delta)$	6	64
6, 12, 91	$(\beta\gamma)$; $(\gamma\delta)$	6	64
6, 12, 91		1	384
7, 11, 82		1	384
7, 11, 82	$(\gamma\delta)$	2	192
7, 11, 82		1	384
7, 11, 82	$(\beta\gamma)$	2	192
8, 10, 73		1	384
8, 10, 73	$(\alpha\delta)$	2	192
7, 14, 52		1	384
8, 11, 81	$(\beta\gamma)$	2	192
8, 11, 81	$(\beta\gamma)$	2	192
8, 11, 81		1	384
9, 10, 72		1	384
9, 10, 72	$[\alpha](\beta\gamma)$	2	192
9, 10, 72	$[\alpha](\beta\gamma)$	2	192
10, 12, 51	$(\alpha\beta)$	2	192
9, 12, 61		1	384
10, 12, 60	$(\beta\delta)$; $(\gamma\delta)$; $[\alpha]$	12	32
6, 12, 10, 0	$(\beta\gamma)$; $(\gamma\delta)$	6	64
6, 14, 62	$(\alpha\beta)$; $(\gamma\delta)$	4	96
6, 14, 62	$(\alpha\gamma)$; $(\beta\delta)$	4	96
8, 10, 82		1	384
8, 10, 82	$(\alpha\beta)$; $(\alpha\gamma\beta\delta)$	8	48
8, 10, 82	$(\alpha\delta)$; $(\alpha\gamma\delta\beta)$	8	48
8, 12, 71	$(\beta\gamma)$	2	192
8, 12, 71	$(\alpha\beta)$	2	192
9, 11, 62	$(\beta\delta)$	2	192
9, 11, 62	$(\alpha\delta)$	2	192
10, 12, 60	$(\alpha\beta)$; $(\alpha\gamma)$	6	64
10, 11, 61	$(\alpha\gamma)(\beta\delta)$	2	192
			12,870

A Memoir on the Invariants of Biternary Quantics.[*]

BY O. E. GLENN.

An object of this paper is to apply transvection [†] as the fundamental process in methods which may be employed, in the case of any connex $f = a_x^m \alpha_u^n$, to find the complete system of invariant formations. The results possess, for this object, a generality not before attained. Use is made of analogies with methods current in the binary field, for instance a new result reached is a theory of constant symbolical moduli for biternary forms. The applications treated relate to the system of $a_x^2\alpha_u$. The system of $a_x\alpha_u$ is well known.[‡] By a process of investigation from one degree to the next higher we obtain the complete system of 32 concomitants of $a_x^2\alpha_u$ of the first three degrees. Maisano gave,[§] quite analogously, the complete system of the first five degrees of the ternary quartic. Another result reached is a system of 90 concomitants of $a_x^2\alpha_u$ which is relatively finite and complete with regard to a set of symbolical moduli. This result is comparable, in principle, with E. Noether's system [||] for the ternary quartic, which is also relatively finite and complete with respect to a set of moduli. We also obtain a formulary and theory which constitute a noteworthy generalization of the reducing series known in the binary theory as Gordan's series.[¶]

1. *Introduction.* A biternary transvectant T of two forms,

$$\phi = a_{1x}a_{2x}\cdots a_{rx}\alpha_{1u}\alpha_{2u}\cdots\alpha_{su},$$
$$\psi = b_{1x}b_{2x}\cdots b_{\rho x}\beta_{1u}\beta_{2u}\cdots\beta_{\sigma u},$$

is characterized by four indices i, j, k, l, and is the result of an analytical operation consisting of multiple polarization. The definitions formulated from this initial point of view are to be found in a previous paper by the present writer.[*] An equivalent definition is the following: Given two forms,

---

[*] Presented to the American Mathematical Society, May and September, 1926.

[†] Gordan, *Journal für Mathematik*, Vol. 69 (1868), p. 323; Cayley, *Mathematical Papers*, Vol. 7, p. 348; Isserlis, *American Journal of Mathematics*, Vol. 45 (1923), p. 54.

[‡] Clebsch and Gordan, *Mathematische Annalen*, Vol. 1 (1869), p. 359; Clebsch, *Vorlesungen über Geometrie*, 1 Aufl., p. 942.

[§] *Giornali di Battaglini*, Vol. 19 (1881), p. 198.

[||] *Journal für Mathematik*, Vol. 134 (1908), p. 23.

[¶] *Mathematische Annalen*, Vol. 5 (1872), p. 95; Grace and Young, *Algebra of Invariants*, 1st Ed. (1903), p. 56.

$$\phi = a_x^r \alpha_u^s, \quad \psi = b_x^\rho \beta_u^\sigma,$$

then, by definition,[†]

$$T = (\phi, \psi)[ijkl] = a_\beta^i b_\alpha^k (abu)^j (\alpha\beta x)^l \; a_x^{r-i-j} b_x^{\rho-j-k} \; \alpha_u^{s-k-l} \beta_u^{\sigma-i-l},$$

and we have, necessarily,

$$(1) \qquad i + j \leqq r, \quad k + l \leqq s, \quad i + l \leqq \sigma, \quad j + k \leqq \rho.$$

The integer $g = r + s + \rho + \sigma$, which is the sum of the order and the class of the product $\phi\psi$, is called the grade of T.

Accordingly a transvectant T is a polynomial in expressions formed by means of seven types of factors, viz., a_x, α_u, a_α, (abc), $(\alpha\beta\gamma)$, (abu), $(\alpha\beta x)$, types which can be produced within a single form (connex)

$$C = A a_{1x} \cdots a_{rx} \alpha_{1u} \cdots \alpha_{su},$$

by use of three operations of convolution, thus,

$$C_1 = A(a_1 a_2 u) a_{3x} \cdots a_{rx} \alpha_{1u} \cdots \alpha_{su},$$
$$C_2 = A(\alpha_1 \alpha_2 x) a_{1x} \cdots a_{rx} \alpha_{3u} \cdots \alpha_{su},$$
$$C_3 = A a_{1\alpha_1} a_{2x} \cdots a_{rx} \alpha_{2u} \cdots \alpha_{su}.$$

For example the factor $(a_1 a_2 a_3)$ is obtained from C_1 followed by the type C_3, after writing $(a_1 a_2 u) = (a_1 a_2)_u$. The result of applying, to C, ζ convolutions of the first type (C_1), η of the second and θ of the third will be designated in this paper by,

$$(2) \qquad \zeta\eta\theta/C.$$

2. *The definitive process of construction.* The theory of ternary transvection was treated, in its relation to simultaneous systems, in two previous papers by the present author. The system of the pair[‡] p_x^2, $a_x\alpha_u$ and that of the pair[§] $a_x\alpha_u$, $p_x\theta_u$ were found. That transvection is a definitive process for the generation of all concomitants of a single connex is a result of the following:

---

[*] *Transactions of the American Mathematical Society*, Vol. 17 (1916), p. 405.

[†] In previous papers this formula was written $(\phi, \psi)^{ij}_{kl}$. Cf. This JOURNAL, Vol. 48, p. 45.

[‡] *Transactions of the American Mathematical Society*, *loc. cit.*, p. 412.

[§] *American Journal of Mathematics*, Vol. 48 (1926), pp. 45 to 56.

THEOREM. *Every monomial mixed concomitant t of a connex* $\phi = a_x^m \alpha_u^n$, *(including invariants and covariants as special cases), is a term of a transvectant* $T = (P, Q)\,[ijkl]$, *where* $P = a_\alpha^p a_x^{m-p} \alpha_u^{n-p}$ *and Q is a monomial concomitant of a degree (in the coefficients of* ϕ*) one less than the degree of t.*

The most general form of t, omitting numerical factors, may be written in three parts,

$$t = a_\alpha^p a_x^\mu \alpha_u^\nu \times \Pi(abc)\Pi a_\beta \Pi b_\alpha \; \Pi(\alpha\beta\gamma)\Pi(abu)\Pi(\alpha\beta x) \\ \times F(b, c, \cdots, \beta, \gamma, \cdots),$$

in which a Π function represents the product of all factors in t of the indicated type and the symbols are equivalent according to the rule $a_x^m \alpha_u^n = b_x^m \beta_u^n = \cdots$. The first part contains those factors of t that involve a or α with no other symbols, the second part contains all symbolic factors that involve either a or α with any of $b, c, \cdots, \beta, \gamma, \cdots$, and F includes all factors that do not contain a or α.

In the second part of t we are to replace a by u and α by x but for practical reasons it is desirable first to replace a by v and α by y where v is cogredient to u and y is cogredient to x, obtaining a preliminary form,

$$t_{(1)} = a_\alpha^p a_x^\mu \alpha_u^\nu \times \Pi(bcv)\Pi\beta_v \Pi b_y \; \Pi(\beta\gamma y)\Pi(buv)\Pi(\beta xy) \\ \times F(b, \cdots ; \beta, \cdots).$$

As is well known,* $(buv) = b_x$, $(\beta xy) = \beta_u$. Suppose that the number of factors (bcv), β_v in $t_{(1)}$ is i, that there are j factors (buv), k of the types $(\beta\gamma y)$, b_y and l of the type (βxy). Then t is a term of

$$T_1 = (P, \Pi(bcu)\Pi_1 \beta_u \Pi_1 b_x \; \Pi(\beta\gamma x)\Pi_2 b_x \Pi_2 \beta_u F(b, c, \cdots ; \beta, \gamma, \cdots))\,[ijkl].$$

Note that the first product of factors β_u, i. e. $\Pi_1\beta_u$, was derived from $\Pi\beta_v$ while the second product $\Pi_2\beta_u$ was derived from $\Pi(\beta xy)$: Likewise in the case of $\Pi_1 b_x$, $\Pi_2 b_x$. The theorem is proved.

We refer to the process which produces $t_{(1)}$ from t as contraction with regard to $a_x^m \alpha_u^n$ or with reference to the pair of symbols (a, α).

In the analogous manner we may contract any monomial with regard to any set of symbolic pairs, $\{(a, \alpha), (b, \beta), \cdots\}$, which are involved in it.

The sum of the numerical coefficients in any T is unity and it follows from theorems in my former paper (*Transactions of the American Mathematical Society*), quoted above, that, if t is a term of T, the difference $t - T$ equals a linear combination of terms each of which is a term of a transvectant of grade $< g$ whose two forms are forms derived by convolution from ϕ and ψ respectively. If t' is a term of $t - T$ and if t' is a term of T', we have,

---

* **Grace and Young, *loc. cit.*, p. 258.**

$$(3) \qquad t = T + \sum ct' = T + \sum cT' + \sum dt'',$$

t'' being a term of a transvectant of grade less than that of T'. Since the grade can be reduced only a finite number of times the arbitrary monomial t is ultimately expressed as a linear combination of transvectants. Hence,

THEOREM. *By the operation of transvection we can generate all invariant formations in the system of a connex.*

3. *Inductive procedure relating to the degrees.* One method of procedure, if the problem is to find a fundamental system of $\phi = a_x^m \alpha_u^n$, is to construct an irreducible set, containing the maximum number, of degrees $\leqq d - 1$ and proceed by induction to the construction of a complete set of irreducible forms of degree d. This induction starts with the following complete set of degree one, $(n \leqq m)$:

$$(4) \qquad \phi_p = a_\alpha^p a_x^{m-p} \alpha_u^{n-p}, \quad (p = 0, \cdots, n).$$

Let the maximal set of degrees $\leqq d - 1$ be,

$$\gamma_1, \gamma_2, \cdots, \gamma_s.$$

Any form of degree d is a linear combination of monomials t and hence of transvectants,

$$T_1 = (\phi_p, G)[ijkl],$$

and G, being of degree $d - 1$, is a polynomial in $\gamma_1, \cdots, \gamma_s$.

It has been shown that if $[ijkl] = [m + q,\ n + r,\ o + s,\ p + t]$ and $\phi = AB$, $\psi = CD$, all factors being equal to non-symbolical quantics, while $(A, C)[mnop]$, $(B, D)[qrst]$ exist and are not zero, $T = (\phi, \psi)[ijkl]$ is reducible (*Transactions of the American Mathematical Society, loc. cit.*, p. 409).

Now the transvectant $(\phi_p, G)[ijkl]$ is a linear combination of the transvectants, of the same matrix, of ϕ_p and the various terms of G. Hence we have only to find the transvectants of the formula,

$$T_1 = (\phi_p, \gamma_1^{\sigma_1}\gamma_2^{\sigma_2} \cdots \gamma_s^{\sigma_s})[ijkl],$$

which are not reducible by the process of separation just described. It is known that the number of these in any case is finite. Hence when these are found the induction will be complete.

In § 7 of this paper this method is applied to the special orders (2, 1).

4. *Biternary reducing series.* Previous writers have considered forms of generalization[*] of the finite series of Gordan which expresses a doubly binary quantic as a power series in the determinant (xy). A modification of this series reduces it to a linear relation between transvectants[†] of the type $(f, (g, h)^r)^s$ and this form of the series may be effectively generalized for biternary transvectants T. The method leads to a very general reducing series by means of which reducible biternary transvectants of forms of numerical orders (m, n) may be reduced by substitutions in a formula. The method of proof of the series is also of an inductive nature such that it applies to the direct construction of the special forms of the reducing series which are required in order to reduce special transvectants. We therefore also arrive at an important practical process of performing reductions. We treat two cases of some complexity in full.

Let

$$f = a_x^2\alpha_u = b_x^2\beta_u, \quad g = a_\alpha a_x = b_\beta b_x.$$

Then,

$$(f, f)\,[0110] = (bcu)\,c_\beta b_x \gamma_u.$$

Consider the form,

$$T = (g, (f, f)\,[0110])\,[1000].$$

It has the term $t = a_\alpha c_\beta (abc)\gamma_u b_x$. But this is also a term of[‡]

$$S = -(f, (g, f)\,[0100])\,[1010].$$

Successive use of the theorem on the difference between a transvectant and one of its terms will give, therefore, an equality between two finite series whose terms are transvectants, and each series is equal to t. Thus, (cf. (3)),

$$pT + p_1T_1 + \cdots + p_sT_s = qS + q_1S_1 + \cdots + q_tS_t.$$

The order of arrangement of the quantics g, f, f in any T_i is the same as in T and the order in any S_i is the same as in S. To derive this series we have

$$\begin{aligned} (5) \qquad & T = (a_\alpha a_x, c_\beta b_x \gamma_u (bcu))\,[1000] = \tfrac{1}{2}\{t + a_\alpha c_\beta a_\gamma b_x (bcu)\}. \\ & T - t = \tfrac{1}{2} a_\alpha c_\beta b_x \{(bc)_u \gamma_a - (bc)_a \gamma_u\} = \tfrac{1}{2} a_\alpha c_\beta b_x (\{(bc)\gamma\}ua) = \tfrac{1}{2}s. \end{aligned}$$

---

[*] Study, *Methoden zur Theorie der ternären Formen* (1889), p. 54; Capelli, *Giornali di Battaglini*, Vol. 18 (1880), p. 17; E. Noether, *Journal für Mathematik*, Vol. 134 (1908), p. 29.

[†] Grace and Young, *loc. cit.*, p. 56.

[‡] Contract t with regard to $b_x^2\beta_u$.

Now s is a term of

$$T_1 = (a_\alpha a_x,\ c_\beta b_x(\{bc\}\gamma x))\,[0100] = \tfrac{1}{2}\{s + a_\alpha c_\beta (abu)(\{bc\}\gamma x)\},$$

and

$$\begin{aligned} T_1 - s &= \tfrac{1}{2}(a_\alpha a_x,\ (b\{(bc)\gamma\}u)c_\beta)\,[1000]u_x + \tfrac{1}{2}(a_\alpha a_x,\ (\{(bc)\gamma\}bu)c_\beta)\,[0000] \\ &= \tfrac{1}{2}T_2 u_x + \tfrac{1}{2}T_3. \end{aligned}$$

We obtain, likewise, from S,

$$\begin{aligned} S &= (b_x^2\beta_u,\ a_\alpha c_x \gamma_u (cau))\,[1010] = \tfrac{1}{2}\{t + a_\alpha c_\beta b_\gamma b_x (cau)\}. \\ S - t &= \tfrac{1}{2}a_\alpha c_\beta b_x\{(ca)_u\gamma_b - (ca)_b\gamma_u\} = \tfrac{1}{2}a_\alpha c_\beta b_x(\{(ca)\gamma\}ub) = \tfrac{1}{2}\tau. \end{aligned}$$

Also τ is a term of

$$S_1 = (b_x^2\beta_u, a_\alpha c_x(\{ca\}\gamma x))\,[0110] = \tfrac{1}{2}\{\tau + a_\alpha b_x(\{ca\}\gamma\beta)(bcu)\},$$

as is shown by contracting τ with regard to $b_x^2\beta_u$. Hence,

$$\begin{aligned} S_1 - \tau &= \tfrac{1}{2}a_\alpha b_x\{(\{ca\}\gamma)_\beta(bcu) - c_\beta(b\{(ca)\gamma\}u)\} \\ &= \tfrac{1}{2}a_\alpha b_x\{(\{(ca)\gamma\}bc)u_\beta + (\{(ca)\gamma\}cu)b_\beta\} \\ &= \tfrac{1}{2}(b_x^2\beta_u, a_\alpha(c\{(ca)\gamma\}u))\,[1000] + \tfrac{1}{2}(b_\beta b_x,\ a_\alpha(\{(ca)\gamma\}cu))\,[0000] \\ &= \tfrac{1}{2}S_2 + \tfrac{1}{2}S_3. \end{aligned}$$

Equating the value of t given by the formulas relating to T to the value of t obtained from S we get the reducing sequence * for T, S, viz.,

$$(6)\qquad T - \tfrac{1}{2}T_1 + \tfrac{1}{4}T_2 u_x + \tfrac{1}{4}T_3 = S - \tfrac{1}{2}S_1 + \tfrac{1}{4}S_2 + \tfrac{1}{4}S_3.$$

The analogy with Gordan's series may be emphasized by writing, according to the general theory, the second form in each transvectant T_i as a result of applying convolutions to $(f, f)\,[0110]$ and the second form in each S_i as a convolution from $(g, f)\,[0100]$, under the notation (2). It will be understood that the ζ, in $\zeta\eta\theta/C$, produces only factors of the type (abu) and the η, factors of the type $(\alpha\beta x)$ only. Our series may then be written,

$$\begin{aligned} &(g, (f, f)\,[0110])\,[1000] - \tfrac{1}{2}(g, 010/(f, f)\,[0110])\,[0100] \\ &\qquad + \tfrac{1}{4}(g, 110/(f, f)\,[0110])\,[1000]u_x \\ &\qquad + \tfrac{1}{4}(g, 110/(f, f)\,[0110])\,[0000] \\ &= -(f, (g, f)\,[0100])\,[1010] + \tfrac{1}{2}(f, 010/(g, f)\,[0100])\,[0110] \\ &\qquad - \tfrac{1}{4}(f, 110/(g, f)\,[0100])\,[1000] \\ &\qquad - \tfrac{1}{4}(001/f, 110/(g, f)\,[0100])\,[0000]. \end{aligned}$$

The notation used in this form of the series (only) is indefinite since if a

---

* Stroh, *Mathematische Annalen*, Vol. 31 (1888), p. 444.

symbolical product C involves a large number of factors of the various types there will be several ways of forming the convolutions represented by $\zeta 00/C$ for instance. For this reason we shall usually represent the forms in the transvectants by their actual expressions as symbolical products (cf. (5)).

The degree of complexity to be anticipated in a reducing series is illustrated better by the following instance:

$$f = a_x^2\alpha_u^2 = b_x^2\beta_u^2, \quad g = a_\alpha a_x \alpha_u = b_\beta b_x \beta_u,$$
$$(f, f)[1101] = a_\beta \alpha_u b_x (\alpha\beta x)(abu).$$

The transvectant,

$$T = (g, (f, f)[1101])[0110] = (c_\gamma c_x \gamma_u,\ a_\beta \alpha_u b_x (\alpha\beta x)(abu))[0110],$$

has the term,

$$t = a_\beta c_\gamma \alpha_u (\alpha\beta\gamma)(abu)(cbu).$$

By contraction with regard to $a_x^2\alpha_u^2$ it is shown that t is also a term of

$$S = -(f, (g, f)[0101])[1110] = (a_x^2\alpha_u^2,\ c_\gamma \beta_u b_x (\beta\gamma x)(cbu))[1110].$$

The series headed by T and S respectively are calculated by the methods shown in the previous problem. There are 17 terms in the resulting series, as follows:

$$\begin{aligned}
&(c_\gamma c_x \gamma_u,\ a_\beta \alpha_u (abu)(\alpha\beta x) b_x)[0110] - \tfrac{1}{2}(c_\gamma c_x \gamma_u,\ a_\beta \alpha_u (abu)(\{\alpha\beta\} bu))[1000] \\
&+ \tfrac{1}{2}(c_\gamma^2, a_\beta \alpha_u (abu)(\{\alpha\beta\} bu))[0000] - \tfrac{1}{6}(c_\gamma c_x \gamma_u, a_\beta b_a (abu)(\alpha\beta x))[0100] \\
&+ \tfrac{1}{6}(c_\gamma c_x \gamma_u, a_\beta \alpha_u b_x (\{\alpha\beta\} ab))[0100] \\
&= (a_x^2\alpha_u^2, c_\gamma \beta_u b_x (cbu)(\beta\gamma x))[1110] - \tfrac{1}{2}(a_x^2\alpha_u^2, c_\gamma b_x (\beta\gamma x)(\{bc\}\beta x))[0210] \\
&+ \tfrac{1}{6}(a_x^2\alpha_u^2, c_\gamma (\{bc\}\beta x)(\{\beta\gamma\} bu))[1100] + \tfrac{1}{6}(a_a a_x \alpha_u, c_\gamma b_x (\{(bc)\beta\}(\beta\gamma) u))[0100] \\
&+ \tfrac{1}{6}(a_x^2\alpha_u^2, c_\gamma b_x ((\beta\gamma)\{(bc)\beta\} u))[1100] - \tfrac{1}{2}(a_a a_x \alpha_u, c_\gamma \beta_u (bcu)(\{\beta\gamma\} bu))[1000 \\
&+ \tfrac{1}{6}(a_a a_x \alpha_u, c_\gamma (b\{\beta\gamma\} u)(\{bc\}\beta x))[0100] \\
&\qquad\qquad\qquad\qquad + \tfrac{1}{12}(a_a a_x \alpha_u, c_\gamma (bcu)(\beta\{(\beta\gamma) b\} x))[0100] \\
&- \tfrac{1}{2}(a_x^2\alpha_u^2, c_\gamma \beta_u (cbu)(\{\beta\gamma\} bu))[2000] \\
&\qquad\qquad\qquad\qquad + \tfrac{1}{12}(a_x^2\alpha_u^2, c_\gamma \beta_u (\{(\beta\gamma) b\}(cb) x))[1100] \\
&+ \tfrac{1}{12}(a_a a_x \alpha_u, c_\gamma \beta_u ((bc)\{(\beta\gamma) b\} x))[0100] \\
&\qquad\qquad\qquad\qquad + \tfrac{1}{12}(a_x^2\alpha_u^2, c_\gamma (cbu)(\{(\beta\gamma) b\}\beta x))[1100].
\end{aligned}$$

5. *The general series.* The series developed in the preceding section are special cases of something very general to which attention may now be given. Consider three connexes,

$$f = a_x^m \alpha_u^n, \quad g = b_x^\mu \beta_u^\nu, \quad h = c_x^\sigma \gamma_u^\tau.$$

The arbitrary transvectant of f and g is a monomial,

$$(f, g)[ijkl] = a_\beta^i \alpha_b^k (abu)^j (\alpha\beta x)^l a_x^\zeta b_x^\eta \alpha_u^\theta \beta_u^\iota,$$
$$\zeta = m - i - j, \quad \eta = \mu - j - k, \quad \theta = n - k - l, \quad \iota = \nu - i - l,$$

Suppose that p, q, r, s are fixed chosen integers which satisfy the conditions, (cf. (1)),

$$p+s \leqq j+\theta+\iota, \quad q+r \leqq l+\zeta+\eta, \quad p+q \leqq \sigma, \quad r+s \leqq \tau,$$

and consider any set of solutions of the linear diophantine equations,

$$(7) \quad \sigma_1+\sigma_2+\sigma_3=p, \quad s_1+s_2+s_3=q, \quad t_1+t_2+t_3=r, \quad \tau_1+\tau_2+\tau_3=s.$$

Under these conditions, Z being numerical,

$$\begin{aligned}(8) \qquad T &= (h, (f, g)[ijkl])[pqrs] \\ &= \sum Z a_\beta{}^i \alpha_b{}^k (abc)^{\sigma_1} c_a{}^{\sigma_2} c_\beta{}^{\sigma_3} \ (c\{\alpha\beta\}u)^{s_1} (cau)^{s_2} (cbu)^{s_3} \\ &\quad \times (\gamma\{ab\}x)^{\tau_1} (\gamma\alpha x)^{\tau_2} (\gamma\beta x)^{\tau_3} \ (\alpha\beta\gamma)^{t_1} \gamma_a{}^{t_2} \gamma_b{}^{t_3} (abu)^{j-\sigma_1-\tau_1} \\ &\quad \times (\alpha\beta x)^{l-s_1-t_1} \, a_x{}^{\zeta-s_2-t_2} \, b_x{}^{\eta-s_3-t_3} \alpha_u{}^{\theta-\sigma_2-\tau_2} \, \beta_u{}^{\iota-\sigma_3-\tau_3} \, c_x{}^{\sigma-p-q} \gamma_u{}^{\tau-r-s}.\end{aligned}$$

The terms of this summation are given by the totality of solutions in integers of the equations (7). For the term t from which we may develop a reducing series, we select the above summand, with $s_1=\tau_1=0$, and contract t with regard to $a_x{}^m \alpha_u{}^n$. Hence,

$$\begin{aligned} t_{(1)} &= \beta_v{}^i b_y{}^k (bcv)^{\sigma_1} c_y{}^{\sigma_2} (cuv)^{s_2} \ (\gamma xy)^{\tau_2} (\beta\gamma y)^{t_1} \gamma_v{}^{t_3} \\ &\quad \times (buv)^{j-\sigma_1} (\beta xy)^{l-t_1} (-1)^{s_2+\tau_2} F \cdot a_x{}^{\zeta-s_2-t_2} \alpha_u{}^{\theta-\sigma_2-\tau_2}, \\ F &= c_\beta{}^{\sigma_3} (cbu)^{s_3} \ (\gamma\beta x)^{\tau_3} \gamma_b{}^{t_3} \, b_x{}^{\eta-s_3-t_3} \beta_u{}^{\iota-\sigma_3-\tau_3} \, c_x{}^{\sigma-p-q} \gamma_u{}^{\tau-r-s}. \end{aligned}$$

Therefore the term t of T is also a term of the transvectant,

$$\begin{aligned}(9) \qquad S &= (f, (-1)^\xi (h, g)[\sigma_3, s_3+\sigma_1, t_3, \tau_3+t_1]) \\ &\quad \times [i+\sigma_1+t_2, j+s_2-\sigma_1, k+t_1+\sigma_2, l+\tau_2-t_1], \\ &\qquad\qquad (\xi=\sigma_1+t_1+s_2+\tau_2).\end{aligned}$$

Also since

$$(h, (f, g)[ijkl])[pqrs] = (-1)^{j+l} (h, (g, f)[kjil])[pqrs],$$

t is a term of

$$\begin{aligned}(10) \qquad R &= (g, (-1)^{\xi+j+l} (h, f)[\sigma_3, s_3+\sigma_1, t_3, \tau_3+t_1]) \\ &\quad \times [k+\sigma_1+t_2, j+s_2-\sigma_1, i+t_1+\sigma_2, l+\tau_2-t_1].\end{aligned}$$

Applying the methods of section 4, which were based on the theorem on the difference between a transvectant and one of its terms, we can immediately assert that the following reducing series exist, Γ, Δ, Ε being rational numbers:

$$\begin{aligned}(11) \qquad & T + \sum \mathrm{E} \, (00a/h, bcd/(f, g)[ijkl])[p_1 q_1 r_1 s_1] u_x{}^{m_1} \\ &= S + \sum \Delta \, (00\alpha/f, \beta\gamma\delta/(h, (-1)^\xi g)[\sigma_3, s_3+\sigma_1, t_3, \tau_3+t_1]) \ [p_2 q_2 r_2 s_2] u_x{}^{m_2} \\ &= R + \sum \Gamma \, (00\theta/g, \iota\kappa\lambda/(h, (-1)^{\xi+j+l} f)[\sigma_3, s_3+\sigma_1, t_3, \tau_3+t_1]) \ [p_3 q_3 r_3 s_3] u_x{}^{m}\end{aligned}$$

These are finite series of multiple entry (cf. (6)), arranged according to descending grades of the transvectants. Every transvectant in a series is of lower grade than the one which forms the leading term of that series.

Let the grades of T, S, R be τ, σ, ρ, respectively. We refer to the set (T, S, R), with its grade numeral (τ, σ, ρ), as a reducing triad.

By analogous reasoning if we contract a monomial t of degree ν in the coefficients of ν forms $f_1, \cdots, f_\nu$, with regard to the symbolical letters which appertain to those respective forms, we obtain a reducing ν-ad of transvectants $(T_1, \cdots, T_\nu)$, whose grade numeral is of the form $(\tau_1, \cdots, \tau_\nu)$.

When we come to the applications of these series to the problem of reducing transvectants, where $f = g = h = a_x^m \alpha_u^n$, not more than one form of each reducing triad need be retained as irreducible, since every transvectant of the first summation is of grade lower than T and every transvectant in the second summation is of lower grade than S and the series is finite.

If we define the *index*[*] of $(P, Q)\,[ijkl]\,u_x^m$ to be the number,

$$e = 2i + j + 2k + l - 2m,$$

the terms of the sequences can be arranged according to both decreasing grade and decreasing index. The difference between a transvectant and one of its terms is exclusively a sum of terms of transvectants of both lower grade and lower index.

6. *Reduction of ordered sets.* The transvectants of a given degree derived from a connex $a_x^m \alpha_u^n$ can be arranged as a linear aggregate,

$$T_1,\ T_2,\ \cdots,\ T_i,\ \cdots,\ T_j,\ \cdots \quad (j > i), \tag{12}$$

in which the grade of T_j is $\geqq$ the grade of T_i. In searching for the irreducible transvectants we examine each term of (12) beginning with T_1, testing the cases, in order, for reducibility by separation, by identities, etc. If T_i is a transvectant T and S precedes T in (12) or is reducible in terms of those which precede or possesses a term which is so reducible, then T_i is reduced. In any event not more than one transvectant of the triad (T, S, R) need be retained. The number of irreducible cases for a given degree will be finite.

7. *Fundamental system of concomitants of $a_x^2 \alpha_u$ of the first three degrees.* The list (4) for this quantic is $f = a_x^2 \alpha_u$, $g = a_\alpha a_x$.

---

[*] *Transactions of the American Mathematical Society, loc. cit.*, p. 409. In the formula for e $2i$ represents the number by which the grade of PQ is reduced by i convolutions. The other terms of e have like interpretations when we attach to u_α the property of an operator which raises the grade by 2.

The transvectants of degree 2 are all monomials. Those which are not reducible by separation are arranged as an ordered set in the following table (to be read by rows):

$(g,g)[0100]=a_\alpha b_\beta(abu)$, $(g,f)[1000]=a_\alpha a_\beta b_x^2$, $(g,f)[0100]=a_\alpha(abu)\beta_u b_x$,
$(f,f)[1000]=a_\beta\alpha_u a_x b_x^2$, $(f,f)[0100]=(abu)\alpha_u\beta_u a_x b_x$, $(f,f)[0200]=(abu)^2\alpha_u\beta_u$,
$(f,f)[1100]=a_\beta(abu)\alpha_u b_x$, $(f,f)[0010]=b_\alpha\beta_u a_x^2 b_x$, $(f,f)[0001]=(\alpha\beta x)a_x^2 b_x^2$,
$(f,f)[1010]=a_\beta b_\alpha a_x b_x$, $(f,f)[0101]=(abu)(\alpha\beta x)a_x b_x$, $(f,f)[0201]=(abu)^2(\alpha\beta x)$,
$(f,f)[0110]=b_\alpha(abu)\beta_u a_x$, $(f,f)[1110]=a_\beta b_\alpha(abu)$.

Referring to these quantics by their numbers in the set we note that a change of sign occurs in (1), (5), (9), (12), (14) when the pairs (a,α), (b,β) are interchanged. Hence these forms vanish. Likewise (8)=(4), (13)=(7). We have

$$(13)\qquad (f,f)[0101]=\begin{vmatrix} a_\alpha & a_\beta & a_x \\ b_\alpha & b_\beta & b_x \\ \alpha_u & \beta_u & u_x \end{vmatrix} a_x b_x = g^2u_x+2(4)-2fg-(10)u_x.$$

Hence (11) is reducible. There remain six concomitants, (2), (3), (4), (6), (7), (10), for which we use the notation, A, B, C, D, E, F.

The next table contains the ordered set of non-separable transvectants of degree 3. To obtain their explicit symbolical expressions we represent the first quantic in each as $f=c_x^2\gamma_u$, or $g=c_\gamma c_x$, and the second quantic in the form given in the above list for the degree 2. In accordance with the theory it is sufficient in most developments to write one term only of each transvectant. Thus, $(f,F)[0110]=a_\beta a_\gamma b_\alpha(cbu)c_x$.

$(g,A)[0100]$, $(g,F)[0100]$, $(g,B)[0100]$, $(g,B)[1000]$, $(g,E)[0100]$, $(g,E)[1000]$,
$(g,C)[0100]$, $(g,C)[1000]$, $(g,D)[1000]$, $(f,g^2)[0200]$, $(f,g^2)[0110]$, $(f,A)[0100]$,
$(f,A)[0200]$, $(f,A)[0010]$, $(f,A)[0110]$, $(f,F)[0100]$, $(f,F)[0200]$, $(f,F)[0010]$,
$(f,F)[0110]$, $(f,B)[0010]$, $(f,B)[1010]$, $(f,B)[2010]$, $(f,B)[0001]$, $(f,B)[1001]$,
$(f,B)[0101]$, $(f,B)[1101]$, $(f,B)[1000]$, $(f,B)[2000]$, $(f,B)[1100]$, $(f,B)[0100]$,
$(f,E)[0001]$, $(f,E)[1001]$, $(f,E)[0101]$, $(f,E)[1101]$, $(f,E)[0010]$, $(f,E)[1010]$,
$(f,E)[2010]$, $(f,E)[1000]$, $(f,E)[2000]$, $(f,E)[0100]$, $(f,E)[1100]$, $(f,gf)[0210]$,
$(f,C)[0001]$, $(f,C)[0101]$, $(f,C)[0201]$, $(f,C)[0010]$, $(f,C)[1010]$, $(f,C)[0110]$,
$(f,C)[1110]$, $(f,C)[0210]$, $(f,C)[1000]$, $(f,C)[1100]$, $(f,C)[0100]$, $(f,C)[0200]$,
$(f,D)[1000]$, $(f,D)[2000]$, $(f,D)[0001]$, $(f,D)[1001]$, $(f,D)[2001]$, $(f,f^2)[1001]$,
$(f,f^2)[1101]$, $(f,f^2)[2000]$, $(f,f^2)[2010]$, $(f,f^2)[0210]$.

The actual operation of reducing series in the problem of the discovery of all reducible transvectants in this set will now be shown in detail in a score of cases. The series (6), of which (11) is the general expression, give all reductions in this problem, including those types for which Clebsch and Gordan employed a special identity. The number of irreducible transvectants is 24. These are collected, finally, in explicit symbolical form, in a separate list, one term, only, of each transvectant being written.

(a) Consider the case,

$$S = (f, -B)[1010] = -(f, (g, f)[0100])[1010] = a_\alpha c_\beta (abc) \gamma_u b_x.$$

Contracting t with regard to $a_x{}^2 \alpha_u$ we find it to be a term, also, of

$$T = (g, -E)[1000] = -(g, (f, f)[1100])[1000].$$

Likewise t is a term of $(f, B)[2000] = R$ on the basis of contraction with regard to $c_x{}^2 \gamma_u$. The first series of (11) is that given in (6). The reducing triad is

$$\{T, S, R\}, \tag{14}$$

with the grade numeral (4, 6, 6). Hence S, R are reducible in terms of T and other transvectants, T_1, T_2, T_3, S_1, S_2, $\cdots$ of grade < 6: The calculation does not reduce T, in fact it is ultimately retained in the irreducible system.

The transvectants of the aggregate being ordered it is unnecessary to compute each reducing sequence in the explicit form (6). It is sufficient to determine the reducing triad by contraction except in some cases where one or more members of the triad vanish identically and in certain other cases where the methods are augmented by special processes.

(b) $(g, E)[0100] = a_\beta c_\gamma (abu)(cbu) \alpha_u$. Contraction with regard to $a_x{}^2 \alpha_u$ and $b_x{}^2 \beta_u$, respectively, gives the reducing triad,

$$\{(g, E)[0100], \ (f, B)[1100], \ (f, fg)[0210]\},$$

for which the grade numeral is (4, 6, 7). The first form is not reduced but when the grades 6, 7 are reached in the problem of the discovery of the irreducible transvectants of the aggregate the other two forms of the triad are reduced in terms of $(g, E)[0100]$ and other transvectants of grade less than that of the transvectant considered.

(c) $(f, A)[0200] = a_\alpha a_\beta (cbu)^2 \gamma_u$. The reducing triad is

$$\{(f, A)[0200], \ (g, D)[1000], \ (f, fg)[0210]\},$$

with the grade numeral (5, 5, 7). We retain the second transvectant. Instead, we could retain the first.

(d) $(f, A)[0100] = a_\alpha a_\beta (cbu)\gamma_u b_x c_x$. Contracting with reference to $a_x{}^2\alpha_u$, we find

$$S = (a_\alpha a_x,\ (cbu)\beta_u\gamma_u b_x c_x)[1000] = 0.$$

The reducing triad is

$$\{(f, A)[0100],\ 0,\ -(f, fg)[0110]\},$$

with the grade numeral (5, 6, 7). We resort to a special process. Since,

$$S = \tfrac{1}{2}\{a_\alpha (acb)\beta_u\gamma_u b_x c_x + t + a_\alpha a_\gamma (cbu)\beta_u b_x c_x\},$$

in which the first term is zero, there results

$$S - t = a_\alpha (cbu)(\{\gamma\beta\}au) b_x c_x.$$

Contracting the latter with regard to $a_x{}^2\alpha_u$ it is found to be a term of

$$(a_\alpha a_x,\ (\beta\gamma x)(bcu) b_x c_x)[0100] = (g,\ g^2 u_x + 2C - 2gf - F u_x)[0100].$$

Now $(g, C)[0100]$, of grade 5, is one of the forms retained as irreducible while $(g, g^2)[0100]$, $(g, gf)[0100]$, $(g, F)[0100]$ are lower forms in the problem of the grade 5. Hence t is reduced. No form is retained from the reducing triad.

(e) $(f, F)[0110] = a_\beta a_\gamma b_\alpha (cbu) c_x$. The reducing triad is

$$\{(f, F)[0110],\ (f, (cbu)\beta_u\gamma_u b_x c_x)[2010],\ -(f, C)[1110]\},$$

for which the grade number is (5, 8, 7). The second form is zero but the interval from 5 to 8 for the grades is too large for successful use of the method employed in (d). The first form is retained and the last one is reduced.

(f) $(f, E)[2010] = a_\beta b_\gamma c_\alpha (abc)$. This transvectant is reproduced by contraction in any one of the three possible ways. Hence the three reducing series (11) are identical and there is no reduction. In fact this is the only pure invariant* of degree 3 and the invariant of lowest degree of f.

(g) $(f, E)[1101] = a_\beta (abc)(\gamma\alpha x)(cbu)$. We find the reducing triad,

$$\{(f, E)[1101],\ (f, -D)[2001],\ (f, C)[1110]\},$$

with the grade numeral (6, 7, 7). The first form is not reduced by this.

Expansion gives

---

* There is an invariant of degree 6, $(dea)(def)c_\alpha c_\gamma a_\delta b_\epsilon b_\zeta f_\beta$.

$$t = \begin{vmatrix} a_\gamma & a_\alpha & a_x \\ b_\gamma & b_\alpha & b_x \\ c_\gamma & c_\alpha & c_x \end{vmatrix} (cbu) a_\beta = a_\beta (cbu) a_\gamma b_\alpha c_x + a_\beta (cbu) b_\gamma c_\alpha a_x$$

$$+ a_\beta (cbu) c_\gamma a_\alpha b_x - a_\beta (cbu) c_\gamma b_\alpha a_x - a_\beta (cbu) a_\gamma c_\alpha b_x - a_\beta (cbu) b_\gamma a_\alpha c_x .$$

By contraction we find these six monomials to be terms of the respective transvectants,

$$(f, F)[0110],\ (f, E)[1010],\ (g, B)[1000],$$
$$(g, F)[0100],\ (f, F)[0110],\ (g, E)[1000].$$

These are all of grade < 6 excepting the second and, since it is of grade 6 and is one of the forms retained as irreducible, $(f, E)[1101]$ and, therefore, each form of the reducing triad, is reduced.

The calculations which were necessary for the preparation of the following table,[*] involved the reduction of over fifty transvectants.

TABLE I.

The Complete System of f of the First Three Degrees.

Degree 1.

$$f = a_x^2 \alpha_u , \quad g = a_\alpha a_x$$

Degree 2.

$$(g, f)[1000] = a_\alpha a_\beta b_x^2 , \quad (g, f)[0100] = a_\alpha (abu) \beta_u b_x$$
$$(f, f)[1000] = a_\beta \alpha_u a_x b_x^2 , \quad (f, f)[0200] = (abu)^2 \alpha_u \beta_u$$
$$(f, f)[1100] = a_\beta (abu) \alpha_u b_x , \quad (f, f)[1010] = a_\beta b_\alpha a_x b_x$$

Degree 3.

$(g, A)[0100] = a_\alpha a_\beta c_\gamma (cbu) b_x ,$
$(g, B)[0100] = a_\alpha c_\gamma (abu)(cbu) \beta_u ,$
$(g, E)[1000] = a_\beta c_\alpha c_\gamma (abu) b_x ,$
$(g, D)[1000] = c_\alpha c_\gamma (abu)^2 \beta_u ,$
$(f, F)[0100] = a_\beta b_\alpha (cbu) \gamma_u a_x c_x ,$
$(f, F)[0010] = a_\beta a_\gamma b_\alpha b_x c_x^2 ,$
$(f, E)[0001] = a_\beta (abu)(\gamma \alpha x) b_x c_x^2 ,$
$(f, E)[1010] = a_\beta b_\gamma c_\alpha (abu) c_x ,$
$(f, E)[1000] = a_\beta (abc) \alpha_u \gamma_u b_x c_x ,$
$(f, E)[1100] = a_\beta c_\alpha (abu)(cbu) \gamma_u ,$
$(f, C)[0010] = a_\beta a_\gamma \alpha_u b_x^2 c_x^2 ,$
$(f, C)[0200] = a_\beta (cbu)^2 \alpha_u \gamma_u a_x ,$

$(g, F)[0100] = a_\beta b_\alpha c_\gamma (cau) b_x$
$(g, E)[0100] = a_\beta c_\gamma (abu)(cbu) \alpha_u$
$(g, C)[0100] = a_\beta c_\gamma (cau) \alpha_u b_x^2$
$(f, A)[0010] = a_\alpha a_\beta b_\gamma b_x c_x^2$
$(f, F)[0200] = a_\beta b_\alpha (cau)(cbu) \gamma_u$
$(f, F)[0110] = a_\beta a_\gamma b_\alpha (cbu) c_x$
$(f, E)[0010] = a_\beta b_\gamma (abu) \alpha_u c_x^2$
$(f, E)[2010] = a_\beta b_\gamma c_\alpha (abc)$
$(f, E)[0100] = a_\beta (abu)(cbu) \alpha_u \gamma_u c_x$
$(f, C)[0001] = a_\beta (\gamma \alpha x) a_x b_x^2 c_x^2$
$(f, C)[0100] = a_\beta (cau) \alpha_u \gamma_u b_x^2 c_x$
$(f, D)[2000] = (abc)^2 \alpha_u \beta_u \gamma_u$

---

[*] The form u_x of zero degree belongs to the list.

8. *Theory of symbolic moduli.* Simplification and further condensation of notations in connection with the processes which relate to the discovery of complete systems is attained by means of a theory of moduli.

Definitions. If a system (A) of connexes is finite and complete and (B) is finite and complete then the system (C) which consists of all terms of all transvectants $(A, B)\,[ijkl]$ is also finite and complete (*Transactions of the American Mathematical Society, loc. cit.*, 411).

A system such as (C) is said to be relatively finite with respect to a set of constant symbolic moduli a_α, $a_\alpha a_\beta$, (abc), $\cdots$, when every form of (C) is expressible as a polynomial in a finite set of forms (C) together with forms which contain at least one of the moduli as a factor.

A multiple convolution in a product N of powers of forms (A) may be a polynomial. We express this by $/N = \sum //N$ and each form $//N$ may be obtained singly by the properly chosen convolution $\zeta\eta\theta/N$.

A system, as (C), is said to be relatively complete with regard to a set of moduli when every form $//N$ obtained by convolution from a product of forms (C) is a polynomial in forms (C) together with terms containing a modulus as a factor.

Special Lemma 1. *The two forms* $f = a_x^2\alpha_u$, $D = (abu)^2\alpha_u\beta_u$, *of Table I constitute a system which is relatively finite and complete modulis* a_α, b_γ, $(abc)^2$.

Since the forms are finite in number we need only to investigate the forms obtainable by the processes of convolution between products.

No process of convolution is applicable to g. From $D^q f^r$ we obtain only expressions $D^s f^t$ modulis a_α, a_β, (abc), $(\alpha\beta\gamma)$, as is evident from Table I together with the following considerations relating to convolutions from D^q. From D alone we obtain,

$$
\begin{aligned}
010/D &= (abu)^2(\alpha\beta x) = 0,\\
010/D &= (abu)(\{ab\}\alpha x)\beta_u = (abu)(\{ab\}\beta x)\alpha_u = B + E,\\
020/D &= (\{ab\}\alpha x)(\{ab\}\beta x) = 2A - g^2 - F.
\end{aligned}
$$

$$(D, D)\,[000l] = \sum(abu)(\alpha\gamma x)P + \sum(\{ab\}\tau x)Q, \quad (\tau = (cd),\ \gamma \text{ or } \delta).$$

All terms of both summations contain factors a_α, b_γ or a_τ.

We next prove that (abc) is a *reducible modulus*, that is, any monomial which contains it can be transformed into a linear expression in monomials each containing a factor a_α or b_γ or $(abc)^2$.

If any monomial $t = (abc)\,P$ be contracted with regard to the three pairs (a, α), (b, β), (c, γ) it is seen to be a term of,

$$T = (/Q, R)\,[ijkl],$$

where R does not contain the symbols of these pairs and Q is the leader of the form-sequence* which appertains to (abc), viz., $Q = (abc)\,a_x b_x c_x \alpha_u \beta_u \gamma_u = 0$. Hence

$$t = \sum I(/Q, /R)\,[ijkl]\,u_x^m, \tag{15}$$

I being numerical. All transvectants of the summation in which no actual convolution occurs in Q vanish. All convolutions in Q result in factors a_α, b_γ excepting $G = (abc)^2 \alpha_u \beta_u \gamma_u$ and three instances equivalent, by interchanges of symbols, to

$$H = /Q = (abc)\,(abu)\,\alpha_u \beta_u \gamma_u c_x.$$

But,

$$H - c_\alpha (abu)^2\, \beta_u \gamma_u c_x = a_\alpha (abu)\,(bcu)\,\beta_u \gamma_u c_x - b_\alpha (abu)\,(acu)\,\beta_u \gamma_u c_x.$$

A like argument deletes the factor $(\alpha\beta\gamma)$ from the modular system. The lemma is proved.

COROLLARY. A monomial which contains a factor $a_\alpha (abc)$ can be transformed into a sum each term of which contains a constant composed of two factors chosen from the types a_α, b_γ.

Special Lemma 2. The quantics

$$g = a_\alpha a_x,\ C = a_\beta a_x b_x^2 \alpha_u,\ E = a_\beta (abu)\,\alpha_u b_x,\ G = (abc)^2 \alpha_u \beta_u \gamma_u,$$

each containing one and only one of the moduli a_α, b_γ, $(abc)^2$ *as a factor, constitute a system which is relatively finite and complete modulis:*

$$(S):\ a_\alpha a_\beta,\ a_\beta b_\alpha,\ a_\alpha b_\gamma,\ a_\beta c_\delta,\ a_\alpha b_\beta,\ a_\beta (cde)^2,\ (abc)^2 (def)^2.$$

To prove this it is only necessary to identify the results of convolution in the list which follows, from which convolutions which vanish identically have been omitted. Any convolution in gG contains $a_\alpha a_\beta$. A convolution in any product of powers of g, C, E, G, other than gG, contains one of the last five moduli as a factor.

$$(16)\quad \begin{aligned} &100/C = a_\beta (abu)\,\alpha_u b_x = E, && 001/C = a_\beta b_\alpha a_x b_x = F, \\ &001/C = a_\alpha a_\beta b_x^2 = A, && 110/C = 010/E = a_\beta (\{ab\}\alpha x)\,b_x = A - F. \end{aligned}$$

---

[*] Cf. § 9.

9. *The modular form-sequence* (*Formenreihe, Noether*). A quantic which has a given constant symbolic factor, for example $P = a_\alpha b_\gamma (abu) \beta_u c_x^2$, is a member of a sequence of forms which are obtainable from a given one of the sequence, called the leading form, and which all have this same constant as their only constant factor. We use only *outer* indices j, l of the matrix $[ijkl]$ in constructing a form-sequence and omit convolutions which would bring an additional constant factor into a form. The form-sequence of which P is a term is,

$$(17) \qquad L = a_\alpha b_\gamma a_x b_x c_x^2 \beta_u,\ P,\ a_\alpha b_\gamma (acu) \beta_u b_x c_x,\ a_\alpha b_\gamma (bcu) \beta_u a_x c_x,$$
$$a_\alpha b_\gamma (acu)(bcu) \beta_u.$$

The form $a_\alpha b_\gamma (\{ab\} \beta x) c_x^2$, for instance, is not included since it equals $a_\alpha b_\gamma a_\beta b_x c_x^2 - a_\alpha b_\gamma b_\beta a_x c_x^2$.

The rank of a modulus. The factors of a modulus are of three types, a_r, (abc), $(\alpha\beta\gamma)$. However it is usual to write $(abc) = a_{(bc)} = a_s$, $(\alpha\beta\gamma) = t_\alpha$, thus reducing all types to one. The number of factors of the type a_s in a modulus,

$$M = a_s^\rho b_t^\sigma \cdots n_w^\tau,$$

is the rank of M. Thus the modular systems used in the special lemmas 1, 2 are not homogeneous in rank.

General Lemma. If a finite system (A) *of connexes,* $A_1, A_2, \cdots$, *all of which are concomitants of* $f = a_x^m \alpha_u^n$, *includes f and is relatively complete for a set* (G) *of constant symbolic moduli,* $G_1, G_2, \cdots$, *and if, also a finite system* (B) *of connexes* $B_1, B_2, \cdots$; *concomitants of f, is relatively complete for a set* (Γ) *of moduli* $\Gamma_1, \Gamma_2, \cdots$, *and contains the leading form of the form-sequence whose quantics have as their only constant symbolic factor the modulus* G_i, *for each case* $i = 1, 2, \cdots$, *then the system* (C) *derived by transvection from* (A) *and* (B) *is relatively finite and complete, modulo* (Γ).

The proof is in two parts.

Proposition I. If $//P$ is an expression derived by convolution from a power P of f, any term t of $\sigma = (//P, B)[ijkl]$ can be expressed (mod (Γ)) as an aggregate of transvectants of the type $\tau = (A, B)[ijkl]$ where A, B are forms from the respective systems (A), (B) and A is of a degree $\leqq$ the degree of P.

The proof is by induction with reference to advancing degrees of P, initiated by the case ($k \neq 0$, a numerical constant), $(ku_x^m, B)[0000]$ where $//P$ is of zero degree and the proposition is evidently true.

$$X = C_1^{\gamma_1} C_2^{\gamma_2} \cdots C_q^{\gamma_q},$$

is a term of a transvectant $(//A, //B)[ijkl]$ where $//A$, $//B$ are derived by convolution from forms (A), (B) respectively. Hence,

$$//X = (//A, //B)[ijkl] + \sum I(///A, ///B)[i_1 j_1 k_1 l_1] u_x^m$$
$$\equiv \sum (A', B')[pqrs] u_x^n \quad (\text{mod } (\Gamma)),$$

where A' is a form (A) of degree $\leqq$ the degree of $//A$. All transvectants of the latter summation are polynomials in a finite set,

$$C_1, C_2, \cdots, C_q, u_x.$$

Hence the system (C) is relatively complete, as well as finite, modulo (Γ).

10. *The special investigations resumed.* The general Lemma, used in connection with the methods relating to the special lemmas 1, 2, enables us to construct systems of concomitants which are relatively finite and complete with respect to a set of moduli of rank $\geqq \nu$ from a system which is relatively finite and complete with respect to a set of rank $\geqq \nu - 1$. Since the number of such forward steps is finite[*] an absolutely finite and complete system will be reached. For example the system derived by transvection from $\{f, D\}$ and $\{g, C, E, G\}$ is relatively finite and complete modulo (S). We can also increase the second set g, C, E, G by adjunction of other forms, as A and F, making corresponding alterations in the set of moduli.

An outstanding problem is how to reduce a modular system to the minimum number. In the binary theory, due to Gordan, there is one modulus in each system of moduli, that for the rank $2k$ being $(ab)^{2k}$ and that for the next higher rank, $(2k+2)$, being $(ab)^{2k+2}$. The theory underlying the binary transformations is a lemma due to Jordan: If $v_1 + v_2 + v_3 = 0$, then any product of powers of v_1, v_2, v_3, of order n can be expressed linearly in terms of such products as contain an exponent $\geqq \frac{2}{3}n$. For use in the binary theory the equality is,

$$(ab)c_x + (bc)a_x + (ca)b_x = 0.$$

The generalization of Jordan's lemma to four variables has been made by Wood and by A. Young.[†] If

$$v_1 + v_2 + v_3 + v_4 = 0,$$

for example,

---

[*] Grace and Young, *loc. cit.*, p. 274.

[†] *Proceedings of the London Mathematical Society*, Series 2, Vol. 1 (1901), p. 345; Vol. 22 (1923), p. 185.

$$(19) \qquad (abc)d_x - (bcd)a_x + (cda)b_x - (dab)c_x = 0,$$

any monomial of degree n in v_1, v_2, v_3, v_4 can be expressed linearly in terms of such as contain an exponent $\geqq \frac{1}{2}n$. Young's paper deals with perpetuants which are free from u_1, u_2, u_3. No extensions of Jordan's lemma which would be applicable to mixed concomitants have been made.

Modular theory of the system of $f = a_x \alpha_u$. The form f constitutes a system which is relatively finite and complete modulis a_α, a_β. The form $f_1 = b_\alpha a_x b_u$ constitutes a system which is relatively finite and complete modulis $a_\beta b_\alpha$ $(= i_1)$, $b_\alpha d_\gamma$. Since $i = a_\alpha$ is an invariant and f_1 contains as a factor the modulus a_β $(= b_\alpha)$, the General Lemma states that the transvectant system,

$$(20) \qquad (f^r, f_1^s)\,[ijkl],$$

with i, is relatively finite and complete modulis i_1, $b_\alpha d_\gamma$. At this point it was necessary to test all transvectants (20), not reducible by separation, for reducibility as members of reducing ν-ads. The irreducible cases are i_2, g, h,

$$i_2 = a_\beta b_\gamma c_\alpha, \quad g = b_\alpha(\beta\gamma x)a_x c_x, \quad h = b_\alpha(acu)\beta_u\gamma_u.$$

The modulus $b_\alpha d_\gamma$ contains four symbols. Any form which contains it as a factor is of a degree, in the coefficients of f, $\gtreqless 4$ and hence is reducible in terms of the irreducible forms already found. Thus $b_\alpha d_\gamma$ is a reducible modulus. Since i_1 is an invariant the following system* is absolutely complete:

$$u_x, \; i, \; i_1, \; i_2, \; f, \; f_1, \; g, \; h.$$

11. *The relatively finite and complete system of* $f = a_x^2 \alpha_u$, (mod (S)). The set of symbolic moduli (Γ) of the second rank for $a_x^2\alpha_u$ is the set (S) of Special Lemma 1. The transvectants,

$$(21) \qquad (f^n D^p, g^r C^s E^t G^u)\,[ijkl],$$

which are not reducible, form a relatively finite † and complete system (mod (S)). Hence we first determine the non-separable transvectants of this formula, (cf. Table II). The biternary reducing series (11), while still useful in performing reductions, is not infallible as in the problem of investigating from degree to degree for the irreducible concomitants. For, a reducing ν-ad will here usually contain transvectants which do not occur in the system (21). An example is the triad,

---

[*] Due to Clebsch and Gordan, *Mathematische Annalen*, Vol. 1 (1869), p. 359.

[†] E. Noether, *Journal für Mathematik*, Vol. 134 (1908), p. 90.

$$\{(f,E)[0010],\ (f,C)[1100],\ (-f,f^2)[1110]\}.$$

This is analogous to the corresponding situation in the binary theory. The number of non-separable transvectants of the formula (21) exceeds 250. The following list includes all which have no modulus (S) as a factor, omitting a small number which were reduced (mod (S)) by the biternary series. In constructing the table we write all pairs of products of quantics for which (21) is of a given grade > 3. The totality of matrices $[ijkl]$ which may be used with a given pair of products is constructed from the integers which satisfy (1). It is then an elementary exercise in tactic to identify the cases which are not separable. An example of reduction by series is afforded by the pair,

$$T = (f,C)[1100], \quad S = (f,E)[0010],$$

which were shown above to be terms of the same triad. From (11),

$$T + \sum \epsilon(/f,/C)[ijkl]u_x^m = S + \sum \delta(/f,/E)[pqrs]u_x^n,$$

and from this formula there follows a relation (cf. (16)),

$$T \equiv S + \sum I(f,E)[abcd]u_x^e \quad (\text{mod } (S)).$$

If one term, only, of each transvectant (cf. Table I) is written in the table the equalities are congruencial modulis transvectants of this same list, of lower grade, of forms derived from the given forms by convolution.

THEOREM. *The irreducible transvectants of Table II constitute a system for $a_x^2\alpha_u$, which is relatively finite and complete modulo (S), when the forms f, D, g, C, E, G, u_x are added thereto.*

TABLE II.

Grade 4. $(f,g)[0010]$, $(f,g)[0100]$. *Grade* 5. $(D,g)[0010]$

Grade 6.

$(f,E)[0001]$, $(f,E)[1001]$, $(f,E)[0101]$, $(f,E)[1101]$, $(f,E)[0010]$
$(f,E)[1010]$, $(f,E)[2010]$, $(f,E)[1000]$, $(f,E)[0100]$, $(f,E)[1100]$
$(f,G)[1000]$, $(f,G)[2000]$, $(f,G)[1001]$, $(f,G)[0001]$, $(f,G)[2001]$

Grade 7.

$(f,C)[0001]$, $(f,C)[0101]$, $(f,C)[0201]$, $(f,C)[0010]$, $(f,C)[1010]$
$(f,C)[0100]$, $(f,C)[0200]$, $(D,E)[0010]$, $(D,E)[0001]$, $(D,E)[0002]$
$(D,E)[0011]$, $(D,E)[0012]$, $(D,G)[0001]$, $(D,G)[0002]$, $(D,G)[0003]$
$(f,gG)[1100]$, $(f,gG)[1010]$, $(f,gG)[0101]$, $(f,gG)[1101]$, $(f,gG)[2010]$

Grade 8.

$(D, C)[0010]$, $(D, C)[0001]$, $(D, C)[0020]$, $(D, C)[0021]$, $(D, C)[0011]$
$(D, C)[0030]$, $(D, C)[0031)$, $(D, gG)[0011]$, $(D, gG)[0012]$, $(D, gG)[0013]$

Grade 9.

$(f^2, E)[0011]$, $(f^2, E)[1011]$, $(f^2, E)[0002]$, $(f^2, E)[0102]$, $(f^2, G)[3000]$
$(f^2, G)[0002]$, $(f^2, G)[1002]$

Grade 10.

$(f^2, C)[0020]$, $(f^2, C)[1020]$, $(f^2, C)[0120]$, $(f^2, C)[0300]$, $(f^2, C)[1120]$
$(f^2, C)[0011]$, $(f^2, C)[0111]$, $(f^2, C)[0211]$, $(f^2, C)[0301]$, $(f^2, C)[1200]$
$(f^2, C)[1210]$, $(f^2, C)[1300]$, $(fD, E)[1011]$, $(fD, E)[0102]$, $(f^2, gG)[3100]$
$(f^2, gG)[3010]$, $(f^2, gG)[0102]$, $(f^2, gG)[1102]$, $(fD, G)[1002]$

Grade 11.

$(fD, C)[1020]$, $(fD, C)[0120]$, $(fD, C)[1030]$, $(fD, C)[0111]$, $(fD, C)[1120]$
$(fD, C)[0121]$, $(fD, C)[0211]$, $(fD, gG)[1102]$

Grade 12. $(f^3, E)[0012]$, $(f^3, G)[0003]$. *Grade* 16. $(f^4, C)[0031]$

Grade 13.

$(f^3, C)[0030]$, $(f^3, C)[1030]$, $(f^3, C)[0021]$, $(f^3, C)[0121]$, $(f^3, gG)[0103]$.

12. *Conclusion.* We add some data concerning the modular systems of rank $\geqq 3$ and the next step of the general method of finding an absolutely complete system for $a_x{}^2\alpha_u$. Some progress can be made toward reducing a modulus such as $a_\beta c_\delta$ in terms of other products by use of the jacobian identity (19). But, in view of the extent of Table II it is obvious that the next transvectant system will yield several thousand non-separable cases. We can encompass this system by theoretical considerations. We readily find forms which have the respective moduli of (S) as their only constant factors, as follows: $a_\alpha b_\beta : g^2$; $a_\alpha a_\beta : A$; $a_\beta b_\alpha : F$; $a_\alpha b_\gamma$: The form-sequence (17) led by $L = gC$; $a_\beta c_\delta$: The form-sequence $(B_1, \cdots, B_{10})$ led by $B_1 = C^2$; $a_\beta (cde)^2$: The form-sequence of CG; $(abc)^2(def)^2$: The form-sequence led by G^2. If these quantics be written $B_1, \cdots, B_\tau$, and those of Table II as $A_1, \cdots, A_\sigma$, the irreducible transvectants of the formula,

$$(22) \qquad (A_1{}^{a_1} \cdots A_\sigma{}^{a_\sigma},\ B_1{}^{b_1} \cdots B_\tau{}^{b_\tau})\,[ijkl],$$

compose a system which is relatively finite and complete (mod (Γ)), (Γ) being the set of moduli with respect to which the system $B_1, \cdots, B_\tau$ is relatively finite and complete. The set (Γ) is of rank $\geqq 3$. If it should consist of pure invariants only, the formula (22) would give the fundamental system.

THE UNIVERSITY OF PENNSYLVANIA.

Transitive Substitution Groups Containing Regular Subgroups of Lower Degree.

By F. E. Johnston.

---

§ I. *Introduction.*

In 1831 E. Galois noted [*] that in any substitution group the totality of substitutions each of which leaves invariant a particular letter constitute a subgroup. This subgroup, which in a modern notation is denoted by the symbol G_1 when the original group G is transitive (and in what follows it is assumed that such is the case), was first studied in detail by A. L. Cauchy about 1845. Many of his results now seem trivial, but some of them are significant. In particular, he first demonstrated [†] the fundamental theorem that the order of G_1 equals the order of G divided by its degree. He showed [‡] further that when the G_1 of a transitive group is the direct product of transitive constituents the given group is imprimitive, a fact which is worth noting, since it is along a line which remained undeveloped until G. A. Miller recently treated [§] the subject of transitive groups whose subgroups consisting of the totality of operators which omit a particular letter are the direct product of transitive constituents. Cauchy also proved [‖] that if a transitive group G of degree n contains a cyclic substitution of order n, the number of such substitutions is a multiple of the order of G_1, a theorem of interest, since it is a special case of the more general one which states [¶] that in every group the number of letters in a cycle and its conjugates equals the order of the group. He announced erroneously that a simply transitive group is imprimitive unless all the constituents of G_1 are of the same degree.[**]

From 1845 until the recent work of G. A. Miller there was no direct attempt to study the properties of substitution groups by means of their G_1's, although some theorems relating thereto appeared as by-products of other investigations. E. Mathieu showed [*] that for every value of p^m where p is a prime there is a doubly transitive group of degree p^m and order $p^m(p^m-1)$ contained in a triply transitive group of degree p^m+1 and order $(p^m+1)p^m(p^m-1)$. He demonstrated [†] likewise the existence of the two well-known five-fold transitive groups on 12 and 24 letters respectively. C. Jordon proved [‡] that in a simply transtive primitive group each prime that divides the order of one constituent of G_1 will divide the order of every such constituent. F. Rudio demonstrated [§] the fact that if G_1 omits more than one letter G is imprimitive, a theorem which he said had been communicated to him by Frobenius. H. W. Kuhn showed [‖] that the number of substitutions on the letters of any transitive group which are commutative with each of its substitutions equals the number of letters that are left unchanged by G_1. Shortly afterward it was proved [¶] that if the number of letters left unchanged by G_1 is α, then G_1 is invariant under an intransitive subgroup of G of order αg_1 (g_1 being the order of G_1), which has a regular transitive constituent on the letters omitted by G_1.

---

[*] *Manuscrits d'Evariste Galois* (1908), page 39.

[†] *Oeuvres Complètes D'Augustin Cauchy*, First Series, Vol. 9, page 295.

[‡] *Ibid.*, pages 317-320.

[§] *Proceedings of the National Academy of Sciences*, Vol. 2, page 150.

[‖] *Oeuvres Complètes d'Augustin Cauchy*, First Series, Vol. 9, page 469.

[¶] *American Journal of Mathematics*, Vol. 44, page 122.

[**] G. A. Miller, *Bibliotheca Mathematica*, Series 3, Vol. 10, page 321.

G. A. Miller has considered [**] those groups whose G_1's are the direct products of transitive constituents showing, among other things, that in such a case if G_1 has a non-regular transitive constituent of lowest degree, there can be only one constituent of lowest degree; that if there is more than one constituent of lowest degree, these constituents are simply isomorphic regular groups, the degree of each of them being equal to the number of letters omitted by G_1; that if G_1 contains constituents of more than one degree, every constituent, except possibly those of lowest degree, is non-regular and the degree of every such constituent equals the sum of the degrees of all the constituents of lower degree increased by the number of letters omitted by G_1.

The same author treated [***] also groups whose G_1's are of prime order or transitive, showing that the regular group or odd prime order p is always the G_1 of one, and only one, transitive group of degree $2p$; that in the special case when p is of the form $2^\alpha-1$ it is also the G_1 of one, and only one, group of degree $p+1$, and that it can not be the G_1 of any other transitive group;

---

[*] *Liouville's Journal*, Second Series, Vol. 5, pages 9-42.

[†] *Ibid.*, Vol. 6, pages 241-323.

[‡] *Traité des Substitutions*, page 284.

[§] *Crelle's Journal*, Vol. 102, page 3.

[‖] *American Journal of Mathematics*, Vol. 23, page 173.

[¶] G. A. Miller, *Bulletin of the American Mathematical Society*, Vol. 14, page 19.

[**] *Proceedings of the National Academy of Sciences*, Vol. 2, pages 150-152.

[***] *Transactions of the American Mathematical Society*, Vol. 27, pages 137-145.

that the regular group of order 2 is the G_1 of the octic group and of the symmetric group of degree 3 and of no other group; that the number of transitive groups of degree $2k+1$ which have for their G_1's the group of order 2 and degree $2k$ is the same as the number of abstract abelian groups of order $2k+1$, k being an arbitrary positive integer; that every two different subgroups which are separately composed of all the substitutions which omit one letter of a transitive group of degree n generate a group of degree n; that a regular group of order 2^α, $\alpha > 2$, can not appear as the G_1 of a transitive group of degree $2^\alpha + 2$; that if a regular abelian group of order M is the G_1 of a transitive group of degree $m+h$, then m/h equals 1 or 2.

It is the object of this paper to consider further those transitive groups whose G_1's are regular, or more particularly the properties of those regular groups which may present themselves as the G_1's of transitive groups. Since the number of letters omitted by G_1 is a divisor of the degree of G as well as of the degree of G_1, it follows that if a regular group of degree m appears as the G_1 of a group G, then G is of degree $m + m/k$ where k is a divisor of m. We shall confine ourselves to those groups in which $m/k > 1$, that is to imprimitive groups. It is known[*] that every regular group of degree m may present itself as the G_1 of at least one transitive group of degree $2m$ and order $2m^2$; that is, if k is 1 there is no restriction on G_1. In § 2 of this article we shall determine those groups regular on m letters which may appear as the G_1's of groups of degree $m + m/2$; and in § 3 we shall consider some of the general properties of such regular groups where $k > 2$, proving in particular that there is no upper limit to the values which k may assume.

§ II. REGULAR GROUPS OF DEGREE m WHICH MAY APPEAR AS THE G_1'S OF GROUPS OF DEGREE $m + m/2$, $m/2 > 1$.

We know that G_1 is invariant under a subgroup of G which we shall call H_1. In the present case H_1 is of order $m^2/2$ and has two transitive constituents, one on the letters of G_1 which we shall denote by K_1, and a second which is regular on the $m/2$ letters of G which do not appear in G_1 and which we shall denote by C_1. In H_1, G_1 corresponds to the identity of C_1 but evidently to no other operator; and so K_1 is a subgroup of the holomorph of G_1, each defective[†] substitution omitting exactly $m/2$ letters. In other words the subgroup of K_1, which consists of all the substitutions of K_1 omitting a particular letter, and which we shall call G_{11} is a substitution group on $m/2$ letters; and since it is simply isomorphic with C_1 it is regular. Moreover, each coset of K_1 written with regard to G_1 contains exactly one substitution from G_{11} and one form is conjugate G_{11}' the substitutions having no letter in common. C_1 is therefore abelian, since from a coset of K_1 corresponding to any particular operator of C_1 we may choose a substitution on $m/2$ letters, while from the coset of K_1 corresponding to any other arbitrary operator of C_1 we may choose a substitution on the other $m/2$ letters of K_1, and hence their product in one order is identical with their product in the reverse order. Again, since G_{11} must appear as one of the constituents of a subgroup of index 2 in G_1 — this subgroup consisting of a simple isomorphism between G_{11} and G_{11}' — it follows that no regular substitution group of degree m can be the G_1 of a group of degree $m + m/2$ unless it involve an abelian subgroup of index 2, which is simply isomorphic with a subgroup of the group of isomorphisms I of this regular group of degree m. In particular, *no cyclic regular group of degree* m *except that of degree* 4 *can be the* G_1 *of a group of degree* $m + m/2$, $m/2 > 1$, since the order of the group of isomorphisms of a cyclic group of even order is always less than half the order of the cyclic group, unless the cyclic group have for its order 2^a, in which case the group of isomorphisms although of half the order of the cyclic group, is always non-cyclic whenever $\alpha > 2$. It is known that the cyclic regular group of degree 4 appears as the G_1 of exactly one group of degree 6.

---

[*] G. A. Miller, *Transactions of the American Mathematical Society*, Vol. 27, page 137.

[†] In speaking of substitution groups of degree m we shall use the term defective substitution to denote any substitution of degree α where $1 < \alpha < m$.

Let us consider what abelian groups, outside the cyclic group mentioned, can appear as the G_1 of a transitive group of degree $m + m/2$. We have seen that G_1 contains a subgroup of index 2 consisting of a simple isomorphism between G_{11} and G_{11}'. In each coset of K_1 there appears some subsitution of G_{11}, say s and also a substitution of G_{11}', the latter substitution being that constituent of the substitution of G_1 whose other constituent is the inverse of s. Furthermore, these two defective substitutions in each coset of K_1 are conjugate under G_1; and since G_1 is abelian, it follows that every cycle in a substitution of G_1 is conjugate with every other cycle in that substitution. Hence s must be of order 2; otherwise we could pick out a cycle of G_1 such that the number of letters in it and its conjugates would exceed the order of G_1, which is impossible. Hence G_1 is of order 2^a and either of type $(1, 1, 1, \cdots)$ or of type $(2, 1, 1, \cdots)$. This condition is sufficient as well as necessary. For suppose we have a regular group of order 2^a and type $(1, 1, 1, \cdots)$ which we shall denote by G_1, although as yet we have not shown that it may actually possess the properties of the groups heretofore denoted by that symbol.

The group of isomorphisms of G_1 involves a subgroup of order $2^{\alpha-1}$ which leaves fixed each operator of a subgroup L of index 2 in G_1, while it multiplies successively an operator t of G_1 which is not in L by all the operators of L. This subgroup of I, therefore, when considered as a subgroup of the holomorph of G_1, omits the $m/2$ letters of one constituent of L and so is exactly the G_{11} referred to in the previous discussion. Hence G_1 may have associated with it a group similar to H_1 above. If we now extend this H_1 by an operator which together with the group of order $2^{\alpha-1}$ consisting of a simple isomorphism between G_{11} and C_1 will generate a group similar to G_1, we get a transitive group of degree $m + m/2$, under which the three subgroups C_1, G_{11} and G_{11}' are permuted according to the symmetric group of degree 3, H_1 corresponding to a subgroup of order 2. Hence the transitive group thus generated is of order $3m^2/2$. So we have a transitive group G of order $3m^2/2$ and degree $3m/2$. Its subgroup consisting of all the substitutions that omit a particular letter will be of order m and is therefore G_1. Moreover, from the method of construction we see that all such groups formed with a given G_1 are conjugate. The same reasoning would apply throughout if G_1 had been assumed to be of type $(2, 1, 1, \cdots)$. Hence we have the

THEOREM: *The necessary and sufficient condition that a regular abelian group of degree m appear as the G_1 of a group of degree $m + m/2$, $m/2 > 1$, is that it be of order 2^α and either of type $(1, 1, 1, \cdots)$ or of type $(2, 1, 1, \cdots)$; and every such group appears as the G_1 of exactly one group.*

It was noted above that, as a consequence of the fact that 2^α has primitive root only if $\alpha < 3$, we could say at once that the only regular cyclic group of degree m that could appear as the G_1 of a group of degree $m + m/2$, $m/2 > 1$ is the cyclic group of order 4. We can now show, conversely, that a number of the form 2^α can have primitive roots only if $\alpha = 1$ or 2. Otherwise, the I of a regular cyclic group G_1 of order 2^α, $(\alpha > 2)$ would be cyclic. Since the operator of order 2 in G_1 is characteristic, $2^{\alpha-2}$ of its transpositions would have to be permuted cyclically among themselves by a substitution of order $2^{\alpha-1}$ in the I of G_1. Since the subgroup L of index 2 in G_1 is characteristic, these $2^{\alpha-2}$ transpositions in G_1 are on the letters of one of the two regular constituents of L. Moreover, this substitution of order $2^{\alpha-1}$ in I can not involve any letters from the other constituent of L; for if it did the average number of letters in the holomorph of G_1 would be greater than $n - 1$. So I would be simply one of the constituents of L and by an argument similar to that employed above such a G_1 could appear as the subgroup consisting of the totality of operators that omit a particular letter of a group of degree $m + m/2$, which is impossible. Hence the theorem.*

Let us now consider the non-abelian groups which will satisfy the condition under discussion. As before the two defective operators of each coset of K_1 are conjugate under G_1; and if s represents an operator of G_{11} which appears in a particular coset of K_1, then the other defective operator of this coset appears in G_1 in a substitution whose other constituent is s^{-1}. Hence every operator of G_1 outside its abelian subgroup of index 2 transforms each operator of this subgroup into its inverse. So G_1 is either a generalized dihedral group or a generalized dicyclic group. Moreover, we can show, just as before, that this condition is sufficient as well as necessary. Indeed this same argument has already been used to determine the groups of isomorphisms of the generalized dihedral and the generalized dicyclic groups.† Hence we have the

THEOREM: *The necessary and sufficient condition that a non-abelian regular group of degree m appear as the G_1 of group of degree $m + m/2$, $m/2 > 1$ is that it be either a generalized dihedral group or a generalized dicyclic group; and every such group appears as the G_1 of exactly one group.*

§ III. REGULAR GROUPS OF DEGREE m WHICH MAY APPEAR AS THE G_1'S OF GROUPS OF DEGREE $m + m/k$, $k > 2$, $m/k > 1$.

It is known that no abelian group of degree m can be the G_1 of a group of degree $m + m/k$, $k > 2$, $m/k > 1$. That such a non-abelian group may exist is evident from the group of isomorphisms of the non-cyclic group of order 9 when the former group is written on eight letters corresponding to the operators of order 3 in the latter group. There results a group of order 48 whose G_1 is the regular, non-cyclic group of order 6.‡

Considering further the case where k is 3 and retaining the notation of the last section, we notice that such a G_1 is invariant under a subgroup in G of order $m^2/3$, this subgroup H_1 arising from an m to 1 isomorphism between two transitive constituents: C_1 which is regular on the $m/3$ letters omitted by G_1, and K_1 which is a subgroup of the holomorph of G_1. Each defective substitution of K_1 omits exactly $m/3$ letters. If we denote by G_{11}

---

* C. F. Gauss, *Disquisitiones Arithmeticae*, § 89.

† G. A. Miller, *Philosophical Magazine*, Vol. 231, page 223.

‡ *Transactions of the American Mathematical Society*, Vol. 27, p. 145.

the subgroup of K_1 consisting of all the operators of K_1 which leave invariant a particular letter we see that G_{11} consists of a simple isomorphism between two regular constituents, each of order $m/3$; for it is simply isomorphic with the regular group C_1 of order $m/3$ and each of its operators, other than the identity, involves $2m/3$ letters; we see further that it is invariant under a subgroup of K_1 of order $m^2/9$, this latter group having a regular constituent C_{11} of order $m/3$ on the letters of K_1 omitted by G_{11}. There is, therefore, in G_1 a subgroup of index 3 which we denote by L. It is simply isomorphic with a subgroup of the group of isomorphism of G_1 and has 3 simply isomorphic regular constituents, one of them being C_{11}. Furthermore, as the argument is perfectly general, we have the

THEOREM: *A regular group of degree m can not appear as the G_1 of a group of degree $m + m/k$, $m/k > 1$, $k > 1$ unless it contains a subgroup of index k which is simply isomorphic with a subgroup of the group of isomorphisms of this regular group.*

Returning to the special case where k is 3, let us denote by L' that part of L which remains after dropping the constituent C_{11}. Now L' is a simple isomorphism between two regular constituents which we shall denote by L_1, and L_2. It has been shown that G_{11} is of order $m/3$ and each of its substitutions is commutative with every substitution of L'. Consequently G_{11} is either a one-to-one isomorphism between the conjoint of L_1 and the conjoint of L_2 or it is such that half of its substitutions interchange the letters of L_1 and L_2. We proceed to consider the second of these alternatives.

Since C_1 and C_{11} are simply isomorphic groups, so likewise are L' and G_{11}, these latter having no common operator except the identity and every operator of one being commutative with every operator of the other. Therefore $\{G_{11}, L'\}$ is of order $m^2/9$. Those operators of G_{11} which do not interchange L_1 and L_2 form an invariant subgroup of G_{11} of index 2. This subgroup, which we shall denote by M, consists of a simple isomorphism between a subgroup of index 2 of the conjoint of L_1 and a subgroup of index 2 of the conjoint of L_2. If we write $\{G_{11}, L'\}$ with regard to the invariant subgroup L', the multiplying operator in each coset being an operator of G_{11}, we see that each coset corresponding to an operator of M (except the coset L' itself) contains exactly two operators of degree $m/3$, one operator in each coset being on the letters of one of the conjugates of C_{11} under K_1 and the other operator in each coset being on the letters of the other conjugate of C_{11} under K_1. If one of the operators from M should be such that those of its constituents which are on the letters of L_1 actually constitute a substitution of L_1, then every operator of M is such that all its cycles on the letters of L_1 actually constitute a substitution of L_1. We proceed to prove that this is impossible. If it were true, half the substitutions of L_1 would be in its central and consequently L_1, and therefore C_1, would be abelian. So every coset of K_1 written with regard to G_1 would go into itself when transformed by the operators of K_1. Suppose we consider a coset corresponding to one of the operators of M and consider in it a substitution of degree $2m/3$ which would omit the letters of L_1. Let us transform this substitution by one of the substitutions of G_{11} which interchanges the letters of L_1 and L_2. As this latter substitution omits the letters of C_{11}, it transforms the former substitution into itself; for the substitution in this coset on the letters of C_{11} and L_1 could not have its constituents on the letters of C_{11} the same as those constituents on the letters of C_{11} in the other substitution. This, however, is impossible, since the former substitution omits the letters of L_1 and is transformed into one omitting the letters of L_2. So we see that no substitution of M can be such that the totality of its cycles on the letters of L_1 will constitute a substitution of L_1. The same remarks apply to the letters of L_2. In other words, L_1 must have a subgroup of index 2 corresponding to a similar subgroup in its conjoint, the two subgroups having nothing in common except the identity. Furthermore, this shows that no substitution of the subgroup of index 2 of L_1 is invariant under L_1. For the subgroup of its conjoint corresponding to its subgroup of index 2 contains no substitution of L_1; but these are all the substitutions of the conjoint which do not interchange the two systems of letters of the subgroup of index 2 in L_1.

Returning to the subgroup L of G_1, we see that it contains a subgroup of index 2 which has 6 simply isomorphic regular constituents which we shall denote by A, B, C, D, E, and F, where A and B are subgroups of L_1, C and D subgroups of L_2, and E and F subgroups of C_{11}. The group $\{G_{11}, L'\}$ is of degree $2m/3$ and order $m^2/9$; moreover its subgroup G_{111} consisting of all the substitutions which omit a particular letter is, as we saw, of degree $m/3$ and order $m/6$; it contains 2 simply isomorphic regular constituents. The substitutions of G_{111} can not interchange the letters of L_1 and L_2 since they do not involve all those letters. Hence one of the constituents of G_{111} is on letters of L_1 and the other on letters of L_2. Moreover, that one which is on the letters of L_1 involves half the letters of L_1; and since the letters of A and B are systems of imprimitivity of L_1, this constituent must be on the letters of one of them—say A, and similarly the other constituent must be on the letters of C or D—say C. Again, they are actually the conjoints of A and C

respectively, for they must be commutative with each substitution of $\{A \sim B \sim C \sim D\}$.* Moreover, the inverse of each substitution must be among the constituents of $\{A \sim B \sim C \sim D\}$ and consequently they are the groups A and C. That is, A, B, C, D, E, F are abelian groups. Likewise the 2 transitive constituents of the conjugate of G_{111} under $\{G_{11}, L'\}$ are B and D. We see, therefore, that M is a simple isomorphism between the four groups A, B, C, D,—different, of course, from that which appears in L. The subgroup $\{A \sim B \sim C \sim D \sim E \sim F\}$ is invariant under G_1 and these 6 subgroups give us systems of imprimitivity which are permuted under G_1 according to the regular non-cyclic group of order 6. The subgroup L permutes them thus: $AB \cdot CD \cdot EF$. The subgroup of G_1 containing a conjugate of C_{11} under K_1 will permute them according to a similar substitution of order 2; and likewise the subgroup containing the other conjugate of C_{11} under K_1. Moreover every coset of K_1 for which the multiplying operator is from M will permute them in a similar manner, while each of the remaining cosets of K_1 will permute them according to the other 6 operators in the subgroup of order 12 in the holomorph of the non-cyclic group of order 6. We know that each of the subgroups A, B, C, D, E, F, is abelian and that L_1 contains invariantly the abelian subgroup $\{A \sim B\}$, no operator of which except the identity is invariant under L_1. The group L_1 is, therefore, a generalized dihedral group and A is of odd order.†

Hence, considering G_1 as isomorphic with the non-cyclic regular group of order 6 according to which A, B, C, D, E, and F are permuted, we see that every operator in the cosets corresponding to these operators of order 2 is of order 2. Furthermore, these operators of order 2 interchange the 2 cosets corresponding to the 2 operators of order 3 and so none of them is commutative with any operator in the subgroup of G_1 of index 2 corresponding to the subgroup of order 3 in the group of order 6. Hence we see that G_1 is a generalized dihedral group. Moreover, the subgroup of index 2 of G_1 is of order 3^{α} and G_{11} is a subgroup of its group of isomorphisms of order $2 \cdot 3^{\alpha-1}$. Hence G_1 must be a generalized dihedral group whose subgroup of index 2 is of order 3^{α} and type $(1, 1, 1 \cdots)$.

That such a group can exist for all values of α is evident from the following considerations: represent the group of isomorphisms of an abelian group of order 3^a and type $(1, 1, 1, \cdots)$ transitively on $3^a - 1$ letters. Its subgroup consisting of the totality of operators which omit a particular letter will be transitive, of degree $3^a - 3$ and order $(3^a - 3) \cdots (3^a - 3^{a-1})$; and its subgroup consisting of all the operators which omit a particular letter will in turn be of degree $(3^a - 9)$ and order $(3^a - 9) \cdots (3^a - 3^{a-1})$. Proceeding in this way we arrive at a regular group of degree $3^a - 3^{a-1}$ which is the G_1 of a transitive group of degree $3^a - 3^{a-2}$.

---

* In general, if we are given r simply isomorphic groups, $\Gamma_1, \Gamma_2, \Gamma_3, \ldots, \Gamma_r$, we shall denote the groups consisting of a simple isomorphism between them by the symbol $\{\Gamma_1 \sim \Gamma_2 \sim \Gamma_3 \sim \ldots \sim \Gamma_r\}$.

† Burnside, *Theory of Groups* (1911), page 90.

We have so far been considering the case where half the substitutions of G_{11} interchange the letters of L_1 and L_2. We shall show in what follows that in the other case, that is, where G_{11} consists of a simple isomorphism between the conjoint of L_1 and the conjoint of L_2, G_1 can have no subgroup of index 2. Hence we have the

THEOREM: *The necessary and sufficient condition that a regular group of degree m which contains a subgroup of index 2 shall appear as the G_1 of a group of degree $m + m/3$, $m/3 > 1$ is that it shall be a generalized dihegral group of order $2 \cdot 3^a$ whose abelian subgroup of index 2 is of type* $(1, 1, 1, \cdots\cdots)$.

As the existence proof employed above is perfectly general we have at once the additional

THEOREM: *For every prime value of k there exists an infinitude of regular groups of degree m which may appear as the G_1's of groups of degree $m + m/k$, $m/k > 1$.*

For composite values of k the matter is not so simple. Indeed, it is not certain that there is any composite value of k and a corresponding value of m for which a regular group of degree m may appear as the G_1 of a group of degree $m + m/k$, $m/k > 1$.

We have now to consider for $k = 3$ the possibilities when G_{11} consists of a simple isomorphism between the conjoints of L_1 and L_2. Since G_{11} omits the letters of C_{11}, each substitution of G_{11} is commutative with each substitution of L'; and since in the present case the substitutions of G_{11} do not interchange the two systems of L', it follows that G_{11} is an isomorphism between L_1 and L_2,—different, of course, from that which they enjoy in L'; consequently L_1, L_2, L, G_{11}, C_{11}, C_1 are all simply isomorphic abelian groups and G_1 contains an abelian subgroup of index 3 which is simply isomorphic with a subgroup of the group of isomorphisms of G_1. Since the letters of

L_1, L_2, C_{11} form systems of imprimitivity of G_1, L is an invariant subgroup of G_1 and its three systems are permuted by G_1 according to the group of order 3. Furthermore, since G_{11} is a subgroup of the group of isomorphisms of G_1 which leaves invariant each operator of L and transforms into itself each of the cosets of G_1 with regard to L—for it leaves the three systems of imprimitivity invariant—all the operators of G_1 outside of L are of the same order. So the number of Sylow subgroups of G_1 each having for its order a power of 3 is k, the order of G_1 being $3^a k$, k prime to 3. If G_1 contained a subgroup of index 2, the number of its Sylow subgroups each having for its order a power of 3 would be $k/2$. As this is impossible, such a G_1 could not have a subgroup of index 2.

Since every operator of G_1 outside of L has for its order a power of 3, L can have no operator invariant under G_1 unless that operator be of order 3^a ($\alpha \geqq 0$). Hence we have the

THEOREM: *A regular group of degree m containing no subgroup of index 2 can appear as the G_1 of a group of degree $m + m/3$ only if this regular group contains an invariant abelian subgroup of index 3 which is simply isomorphic with a subgroup in the group of isomorphisms of the given group of degree m and which contains no operator invariant under the given regular group except the identity and, possibly, operators whose orders are powers of* 3.

As examples of such regular groups we have the non-abelian group of order 21, the non-abelian group of order 39, the tetrahedral group, the group of order 48 which contains an abelian subgroup of order 16 and type (1,1,1,1) and permutes these 15 substitutions of order 2 according to 5 cycles of 3 each, the group of order 36 which contains an invariant abelian subgroup of order 12 whose invariants are 6 and 2 and whose three operators of order 2 are permuted cyclically by the entire group, and the non-abelian group of order 27 containing no operator of order 9. The first two groups mentioned are examples of the

THEOREM: *Every non-abelian regular group of order $3p$ where p is any odd prime appears as the G_1 of a transitive group of degree $4p$.*

To prove this we have only to note that the group of isomorphisms of every such group G_1 contains an operator of order p which leaves invariant every substitution of the given group of order $3p$ except those of order 3 and permutes these latter in 2 cycles each of order p. Hence the holomorph of

G_1 contains a subgroup of order $3p^2$ whose subgroup consisting of the totality of substitutions which leave invariant a particular letter is of order p, consisting of a simple isomorphisms between 2 of the constituents of the subgroup of order p of G_1. This subgroup may serve as the K_1 mentioned above. It has three systems of imprimitivity which we may denote by A, B, and C, each involving p letters and it permutes them according to the cyclic group on three letters. We may construct a $3p$ to 1 isomorphism between K_1 and a regular group on p additional letters thereby forming H_1 which will have a subgroup of order p involving 3 constituents, A, B, and one on the p added letters which we may denote by D. We may extend this group by means of a substitution of order 3 which gives a non-abelian group G_1' conjugate with G_1. The three systems of imprimitivity A, B, and D of G_1' are permuted by it according to the cyclic group of degree 3. Hence G_1 and G'_1 generate a transitive group of degree $4p$ which permutes the four systems A, B, C, and D according to the alternating group and its subgroup consisting of all its operators that leave invariant a particular letter is the given G_1.

UNIVERSITY OF ILLINOIS.
URBANA, ILLINOIS.

Generalized Green's Functions for Compatible Differential Systems.

By W. W. Elliott.

1. *Introduction.* If the fundamental homogeneous differential system [*]

$$
\begin{aligned}
&L(u) = l_n(x)u^{(n)}(x) + \cdots + l_1(x)u'(x) + l_0(x)u(x) = 0;\\
\text{(I)}\quad &U_1(u) = \alpha_{11}u(a) + \alpha_{12}u'(a) + \cdots + \alpha_{1n}u^{(n-1)}(a) + \beta_{11}u(b) + \\
&\qquad \cdots + \beta_{1n}u^{(n-1)}(b) = 0,\\
&\quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad .\\
&U_n(u) = \alpha_{n1}u(a) + \alpha_{n2}u'(a) + \cdots + \alpha_{nn}u^{(n-1)}(a) + \beta_{n1}u(b) + \\
&\qquad \cdots + \beta_{nn}u^{(n-1)}(b) = 0,
\end{aligned}
$$

is incompatible, its adjoint [†] system

$$
\begin{aligned}
&M(v) = (-1)^n \frac{d^n}{dx^n}[l_n(x)v(x)] + \cdots - \frac{d}{dx}[l_1(x)v(x)] + l_0(x)v(x)\\
&\quad = 0; \quad V_1(v) = \gamma_{11}v(a) + \cdots + \gamma_{1n}v^{(n-1)}(a) + \delta_{11}v(b) + \cdots\\
\text{(II)}\quad &\qquad + \delta_{1n}v^{(n-1)}(b),\\
&\quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad .\\
&V_n(v) = \gamma_{n1}v(a) + \cdots + \gamma_{nn}v^{(n-1)}(a) + \delta_{n1}v(b) + \cdots\\
&\qquad + \delta_{nn}v^{(n-1)}(b),
\end{aligned}
$$

is also incompatible,[‡] and they both possess Green's functions in the ordinary sense.[§]

The Green's function, $G(xy)$, for system (I) is defined by the following conditions,

(III) (a) $\quad G(xy),\ \partial G(xy)/\partial x, \cdots, \partial^{n-1}G(xy)/\partial x^{n-1},$

exist and are continuous for $a \leq x \leq b$, $a \leq y \leq b$,

(b) $\quad \partial^{n-1}G(xy)/\partial x^{n-1},\ \partial^n G(xy)/\partial x^n,$

---

[*] M. Bôcher, *Annals of Mathematics*, ser. 2, Vol. 13 (1911), p. 71.

[†] M. Bôcher, *Les Méthodes de Sturm*, p. 34.

[‡] M. Bôcher, *Les Méthodes de Sturm*, p. 34.

[§] M. Bôcher, *Annals of Mathematics*, ser. 2, Vol. 13 (1911), p. 71.

exist and are continuous for $a \leqq x < y,\ y < x \leqq b$,

$$\text{(c)}\qquad \partial^{n-1}G(y+,y)/\partial x^{n-1} - \partial^{n-1}G(y-,y)/\partial x^{n-1} = 1/l_n(y),$$

$$\text{(d)}\qquad L_x[G(xy)] = 0, \quad a \leqq x < y,\ y < x \leqq b,$$

$$\text{(e)}\qquad U_p[G(xy)] = 0, \quad [p = 1, 2, \cdots, n].$$

These conditions determine a unique function, which is given by the explicit formula,[*]

$$\text{(IV)}\quad G(xy) = \frac{\begin{vmatrix} H(xy) & u_1(x) & \cdots & u_n(x) \\ U_1(H) & U_1(u_1) & \cdots & U_1(u_n) \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ U_n(H) & U_n(u_1) & \cdots & U_n(u_n) \end{vmatrix}}{\begin{vmatrix} U_1(u_1) & \cdots & U_1(u_n) \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ U_n(u_1) & \cdots & U_n(u_n) \end{vmatrix}},$$

in which

$$\text{(V)}\quad H(xy) = \frac{\operatorname{sgn}\,(x-y)}{2l_n(y)} \frac{\begin{vmatrix} u_1(y) & \cdots & u_n(y) \\ u_1'(y) & \cdots & u_n'(y) \\ \cdot & \cdot & \cdot \\ u_1^{(n-2)}(y) & \cdots & u_n^{(n-2)}(y) \\ u_1(x) & \cdots & u_n(x) \end{vmatrix}}{\begin{vmatrix} u_1(y) & \cdots & u_n(y) \\ u_1'(y) & \cdots & u_n'(y) \\ \cdot & \cdot & \cdot \\ u_1^{(n-1)}(y) & \cdots & u_n^{(n-1)}(y) \end{vmatrix}},$$

where

$$\operatorname{sgn} x = (1,\ x < 0;\ 0,\ x = 0;\ -1,\ x > 0).$$

Green's functions of this general type have been studied by Burkhardt, Bôcher, Birkhoff, Bounitzky, and many others.[†]

Among the properties of the function $G(xy)$ the following are perhaps the most fundamental,

---

[*] For formulas like these, except for change in notation, see Westfall's dissertation, Göttingen (1905).

[†] G. D. Birkhoff, *Transactions of the American Mathematical Society*, Vol. 9 (1908), p. 373; M. E. Bounitzky, *Journal de Mathématiques* (1909), p. 65; M. Bôcher, *Bulletin of the American Mathematical Society*, Vol. 7 (1901), p. 297; *Annals of Mathematics*, ser. 2, Vol. 13 (1911), p. 71; M. H. Burkhardt, *Bulletin de la Société Mathématique de France*, Vol. 22 (1894), p. 71.

(A) The solution of the semi-homogeneous system

$$\text{(VI)} \qquad L(u) = f(x), \quad U_p(u) = 0 \qquad (p = 1, \cdots, n),$$

$f(x)$ being any continuous function, is given by[*]

$$u(x) = \int_a^b G(xs)\, f(s)\, ds.$$

Also, conversely, if for every continuous $f(x)$ the solution of (VI) can be written in the form

$$u(x) = \int_a^b G_1(xs) f(s)\, ds,$$

then

$$G_1(xs) = G(xs).$$

(B) If $H(xy)$ denotes the Green's function for System (II), then

$$H(xy) = G(yx).\dagger$$

The existence of generalized Green's functions for certain special types of compatible differential systems has been previously shown.[‡]

It is the purpose of this paper to demonstrate the existence and characteristic properties of generalized Green's functions for general compatible differential systems of type I.

2. *Existence of Generalized Green's Functions.* The fundamental homogeneous differential system considered will be of the form,[§]

$$\begin{aligned} L(u) &= l_n(x)u^{(n)}(x) + \cdots + l_1(x)u'(x) + l_0(x)u(x) = 0; \\ U_1(u) &= \alpha_{11}u(a) + \alpha_{12}u'(a) + \cdots + \alpha_{1n}u^{(n-1)}(a) + \beta_{11}u(b) \\ &\qquad + \beta_{12}u'(b) + \cdots + \beta_{1n}u^{(n-1)}(b) = 0, \\ &\cdots\cdots\cdots\cdots \\ U_n(u) &= \alpha_{n1}u(a) + \alpha_{n2}u'(a) + \cdots + \alpha_{nn}u^{(n-1)}(a) + \beta_{n1}u(b) \\ &\qquad + \beta_{n2}u'(b) + \cdots + \beta_{nn}u^{(n-1)}(b) = 0, \end{aligned}$$

where it is assumed $l_n(x), \cdots, l_0(x)$ are continuous functions, real or complex, of the real variable x in the interval $a \leq x \leq b$, $l_n(x)$ does not vanish in this interval, each $l_k(x)$ has a continuous derivative of order k, and the n sets of $2n$ constants, denoted by the α's and β's, are linearly independent.

---

[*] M. Bôcher, *Les Méthodes de Sturm*, p. 103.

[†] M. Bôcher, *Les Méthodes de Sturm*, p. 104.

[‡] D. Hilbert, *Göttinger Nachrichten* (1904), p. 213; F. Klein, *Mathematische Annalen*, Vol. 21 (1882), p. 141; W. Westfall, *Annals of Mathematics*, ser. 2, Vol. 10 (1909), p. 177; M. E. Bounitzky, *Journal de Mathématiques* (1909), p. 65; A. Myller, Inaugural-Dissertation, Göttingen (1906).

[§] G. D. Birkhoff, *Transactions of the American Mathematical Society*, Vol. 9 (1908), p. 373.

The above system will be denoted briefly by

$$L(u)=0,\quad U_p(u)=0 \qquad (p=1,\cdots,n). \tag{1}$$

The system adjoint to (1) will be denoted by

$$M(v)=0,\quad V_p(v)=0 \qquad (p=1,\cdots,n). \tag{2}$$

System (1) is called self-adjoint[*] if $L(u)=M(u)$, and if the sets of expressions

$$[V_1(u),\cdots,V_n(u)] \quad\text{and}\quad [U_1(u),\cdots,U_n(u)]$$

are linearly dependent.

System (1) is called Hermitian if

$$\bar{L}(u)=M(u),$$

and if the sets of expressions

$$[V_1(u),\cdots,V_n(u)] \quad\text{and}\quad [\bar{U}_1(u),\cdots,\bar{U}_n(u)]$$

are linearly dependent, the bars denoting conjugates of the coefficients only.

A Hermitian differential system obviously reduces to a self-adjoint system if all the coefficients are real. All results for Hermitian differential systems will then include those for self-adjoint systems if the coefficients are real.

We shall assume that system (1) is compatible of order k. It follows[†] that system (2) is compatible of order k. A complete set of k linearly independent solutions of (1) and (2) will be denoted respectively by

$$u_1(x),\cdots,u_k(x) \quad\text{and}\quad v_1(x),\cdots,v_k(x).$$

The general solution of (1) may be written in the form

$$u(x)=\sum_{i=1}^{k} A_i u_i(x),$$

where the A_i's are arbitrary constants.

On the assumption that (1) is compatible, the semi-homogeneous system

$$L(u)=f(x),\quad U_p(u)=0 \qquad (p=1,\cdots,n), \tag{3}$$

---

[*] M. Bôcher, *Les Méthodes de Sturm*, p. 39.

[†] M. Bôcher, *Les Méthodes de Sturm*, p. 34.

$f(x)$ being any continuous function, will have a solution with a continuous nth derivative if and only if [*]

$$\text{(4)} \qquad \int_a^b v_p(x)f(x)\,dx = 0 \qquad (p = 1, \cdots, k).$$

If (4) is satisfied, from the analogy of the ordinary [†] Green's function we are led to seek the existence of a continuous function $G(xs)$ such that every solution of (3) may be written in the form

$$\text{(5)} \qquad u(x) = \sum_{i=1}^{k} A_i u_i(x) + \int_a^b G(xs)f(s)\,ds.$$

If there exists such a continuous $G(xs)$, then we shall call $G(xs)$ a *Generalized Green's Function* for the compatible differential system (1).

Let $K(xy)$ be a function satisfying the following conditions,[‡]

(6) (a) $$K(xy),\ \partial K(xy)/\partial x, \cdots, \partial^{n-2}K(xy)/\partial x^{n-2},$$

exist and are continuous for $a \leqq x \leqq b$, $a \leqq y \leqq b$,

(b) $$\partial^{n-1}K(xy)/\partial x^{n-1}, \quad \partial^n K(xy)/\partial x^n,$$

exist and are continuous for $a \leqq x < y$, $y < x \leqq b$,

(c) $$\partial^{n-1}K(y+, y)/\partial x^{n-1} - \partial^{n-1}K(y-, y)/\partial x^{n-1} = 1/l_n(y),$$

(d) $$L_x[K(xy)] = 0, \quad a \leqq x < y, \quad y < x \leqq b.$$

Such functions exist and are given by the formula §

$$\text{(7)} \qquad K(xy) = \frac{\operatorname{sgn}(x-y)}{2l_n(y)} \cdot \frac{\begin{vmatrix} u_1(y) & \cdots & u_n(y) \\ u_1'(y) & \cdots & u_n'(y) \\ \cdots & \cdots & \cdots \\ u_1^{n-2}(y) & \cdots & u_n^{n-2}(y) \\ u_1(x) & \cdots & u_n(x) \end{vmatrix}}{\begin{vmatrix} u_1(y) & \cdots & u_n(y) \\ u_1'(y) & \cdots & u_n'(y) \\ \cdots & \cdots & \cdots \\ u_1^{(n-1)}(y) & \cdots & u_n^{(n-1)}(y) \end{vmatrix}}$$

---

[*] M. Bôcher, *Les Méthodes de Sturm*, p. 36.

[†] See Introduction.

[‡] These conditions are the same as III (a, b, c, d) in the Introduction.

§ G. D. Birkhoff, *Transactions of the American Mathematical Society*, Vol. 9 (1908), p. 373.

THEOREM I. *A generalized Green's function for the compatible differential system* (1) *exists and may be expressed by the formula*[*]

$$G(xs) = \frac{\begin{vmatrix} K(xs) & u_{k+1}(x) & \cdots & u_n(x) \\ U_1(K) & U_1(u_{k+1}) & \cdots & U_1(u_n) \\ \cdot & \cdot & \cdot & \cdot \\ U_{n-k}(K) & U_{n-k}(u_{k+1}) & \cdots & U_{n-k}(u_n) \end{vmatrix}}{\begin{vmatrix} U_1(u_{k+1}) & \cdots & U_1(u_n) \\ \cdot & \cdot & \cdot \\ U_{n-k}(u_{k+1}) & \cdots & U_{n-k}(u_n) \end{vmatrix}}.$$

By way of proof, let

$$u_0(x) = \int_a^b K(xs)f(s)\,ds;$$

then

$$L[u_0(x)] = f(x).$$

Every solution of $L(u) = f(x)$ is given by

$$u(x) = u_0(x) + \sum_{i=1}^{n} c_i u_i(x), \tag{8}$$

where the constants c_i are arbitrary. We seek to determine these constants so that $u(x)$ will satisfy the boundary conditions.

Since it is assumed that (1) is compatible of order k and (3) has a solution the rank [†] of each of the matrices

$$\left\| \begin{matrix} U_1(u_1) & \cdots & U_1(u_n) \\ \cdot & \cdot & \cdot \\ U_n(u_1) & \cdots & U_n(u_n) \end{matrix} \right\|, \tag{9}$$

and

$$\left\| \begin{matrix} U_1(u_1) & \cdots & U_1(u_n) & -U_1(u_0) \\ \cdot & \cdot & \cdot & \cdot \\ U_n(u_1) & \cdots & U_n(u_n) & -U_n(u_0) \end{matrix} \right\|, \tag{10}$$

is $(n-k)$.

Assume the solutions so arranged that $u_p(x)$ $[p = 1, \cdots, k]$, satisfy the boundary conditions and that the $(n-k)$-rowed determinant in the upper right hand corner of (9) fails to vanish. Then the system of equations

---

[*] This does not imply that all generalized Green's functions for System (1) are given by this formula. In order to obtain this formula, certain arrangements of solutions and the boundary conditions are assumed. These are made clear in the proof.

[†] M. Bôcher, *Les Méthodes de Sturm*, p. 21.

$$\begin{array}{c} c_{k+1}U_1(u_{k+1}) + \cdots + c_nU_1(u_n) = -U_1(u_0), \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ c_{k+1}U_n(u_{k+1}) + \cdots + c_nU_n(u_n) = -U_n(u_0), \end{array}$$

is obtained by applying the boundary conditions to (8) and assuming they are satisfied by $u(x)$. These equations may be solved for c_i's $(i = k+1, \cdots, n)$, and upon substituting the values thus found in (8) and simplifying, we get

$$u(x) = \sum_{i=1}^{k} c_iu_i(x) + \frac{\begin{vmatrix} \int_a^b K(xs)f(s)\,ds & u_{k+1}(x) & \cdots & u_n(x) \\ U_1\left[\int_a^b K(xs)f(s)\,ds\right] & U_1(u_{k+1}) & \cdots & U_1(u_n) \\ \cdot & \cdot & \cdot & \cdot \\ U_{n-k}\left[\int_a^b K(xs)f(s)\,ds\right] & U_{n-k}(u_{k+1}) & \cdots & U_{n-k}(u_n) \end{vmatrix}}{\begin{vmatrix} U_1(u_{k+1}) & \cdots & U_1(u_n) \\ \cdot & \cdot & \cdot \\ U_{n-k}(u_{k+1}) & \cdots & U_{n-k}(u_n) \end{vmatrix}}$$

Since the derivatives of the function $\int_a^b K(xs)f(s)\,ds$ up through the $(n-1)$th may be obtained correctly by formally differentiating under the integral sign, the equation above may be written

$$u(x) = \sum_{i=1}^{k} c_iu_i(x) + \int_a^b \frac{\begin{vmatrix} K(xs) & u_{k+1}(x) & \cdots & u_n(x) \\ U_1(K) & U_1(u_{k+1}) & \cdots & U_1(u_n) \\ \cdot & \cdot & \cdot & \cdot \\ U_{n-k}(K) & U_{n-k}(u_{k+1}) & \cdots & U_{n-k}(u_n) \end{vmatrix}}{\begin{vmatrix} U_1(u_{k+1}) & \cdots & U_1(u_n) \\ \cdot & \cdot & \cdot \\ U_{n-k}(u_{k+1}) & \cdots & U_{n-k}(u_n) \end{vmatrix}} f(s)\,ds,$$

which demonstrates that the function

$$(11) \qquad G(xs) = \frac{\begin{vmatrix} K(xs) & u_{k+1}(x) & \cdots & u_n(x) \\ U_1(K) & U_1(u_{k+1}) & \cdots & U_1(u_n) \\ \cdot & \cdot & \cdot & \cdot \\ U_{n-k}(K) & U_{n-k}(u_{k+1}) & \cdots & U_{n-k}(u_n) \end{vmatrix}}{\begin{vmatrix} U_1(u_{k+1}) & \cdots & U_1(u_n) \\ \cdot & \cdot & \cdot \\ U_{n-k}(u_{k+1}) & \cdots & U_{n-k}(u_n) \end{vmatrix}}$$

may be taken as a generalized Green's function for the compatible differential system (1).

Any function given by formula (11) will satisfy the properties of the ordinary Green's function given by III (a, b, c, d) in the introduction and

$$U_p[G(xs)] = 0 \qquad [p = 1, \cdots, (n-k)].$$

Also it is seen that the ordinary Green's function may be considered as a special case of $G(xs)$ when the order of compatibility of (1) is zero.

THEOREM II. *The generalized Green's function for the compatible system* (1) *is not unique. If $G_1(xs)$ is any one generalized Green's function, then every generalized Green's function $G(xs)$ is given by the formula,*

$$G(xs) = G_1(xs) + \sum_{i=1}^{k} \phi_i(x) v_i(s) + \sum_{i=1}^{k} u_i(x) \psi_i(s),$$

where the functions ϕ_i and ψ_i are arbitrary continuous functions.

Let

$$f(x) = \begin{vmatrix} g(x) & \bar{v}_1(x) & \cdot & \cdot & \bar{v}_k(x) \\ \int_a^b v_1(s) g(s)\, ds & v_{1,1} & \cdot & \cdot & v_{1,k} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \int_a^b v_k(s) g(s)\, ds & v_{k,1} & \cdot & \cdot & v_{k,k} \end{vmatrix}, \tag{12}$$

where $g(x)$ is any continuous function and

$$v_{p,q} = \int_a^b v_p(s)\, \bar{v}_q(s)\, ds.$$

On multiplying (12) by $v_p(x)$ $(p = 1, \cdots, k)$, and integrating with respect to x from a to b, it is seen that $f(x)$ satisfies condition (4).

Let $G(xs)$ and $G_1(xs)$ be any generalized Green's functions * for system (1), and form the difference

$$D(xs) = G(xs) - G_1(xs). \tag{13}$$

If equation (13) is multiplied by $f(s)$, integrated and simplified by making use of equation (5) it is seen that

$$\int_a^b D(xs) f(s)\, ds = \sum_{i=1}^{k} B_i u_i(x), \tag{14}$$

the B_i's being arbitrary constants.

---

* These functions may or may not be distinct.

From (12) and (14) it follows that

$$\int_a^b \Phi(xs)g(s)\,ds = \sum_{i=1}^{k} B_i u_i(x), \tag{15}$$

where

$$\Phi(xs) = \begin{vmatrix} D(xs) & \int_a^b D(xs)\bar{v}_1(s)\,ds & \cdot & \cdot & \int_a^b D(xs)\bar{v}_k(s)\,ds \\ v_1(s) & v_{1,1} & \cdot & \cdot \cdot & v_{1,k} \\ \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot & \cdot & \cdot & \cdot\ \cdot\ \cdot \\ v_k(s) & v_{k,1} & \cdot & \cdot \cdot & v_{k,k} \end{vmatrix} \tag{16}$$

On multiplying (15) by $\bar{u}_p(x)$ $(p = 1, \cdots, k)$, and integrating with respect to x from a to b we obtain the following equations

$$\begin{aligned} \int_a^b\!\!\int_a^b \bar{u}_1(x)\,\Phi(xs)g(s)\,ds\,dx &= \sum_{i=1}^{k} B_i \int_a^b u_i(x)\bar{u}_1(x)\,dx, \\ &\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ \int_a^b\!\!\int_a^b \bar{u}_k(x)\,\Phi(xs)g(s)\,ds\,dx &= \sum_{i=1}^{k} B_i \int_a^b u_i(x)\bar{u}_k(x)\,dx. \end{aligned} \tag{17}$$

The determinant of the coefficients of the B_i's in system (17) does not vanish since it is the gramian of k linearly independent functions. Solving these equations for the B_i's and substituting the values thus found in (15) we obtain, after simplification,

$$\int_a^b \begin{vmatrix} \Phi(xs) & u_1(x) & \cdot\ \cdot\ \cdot & u_k(x) \\ \int_a^b \bar{u}_1(t)\Phi(ts)\,dt & u_{1,1} & \cdot\ \cdot\ \cdot & u_{k,1} \\ \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot \\ \int_a^b \bar{u}_k(t)\Phi(ts)\,dt & u_{1,k} & \cdot\ \cdot\ \cdot & u_{k,k} \end{vmatrix} g(s)\,ds = 0,$$

where $g(s)$ is any arbitrary continuous function.

Then,

$$\begin{vmatrix} \Phi(xs) & u_1(x) & \cdot\ \cdot\ \cdot & u_k(x) \\ \int_a^b \bar{u}_1(t)\,\Phi(ts)\,dt & u_{1,1} & \cdot\ \cdot\ \cdot & u_{k,1} \\ \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot \\ \int_a^b \bar{u}_k(t)\,\Phi(ts)\,dt & u_{1,k} & \cdot\ \cdot\ \cdot & u_{k,k} \end{vmatrix} = 0. \tag{18}$$

The coefficient of $\Phi(xs)$ in (18) does not vanish. Hence (18) may be solved for Φ, giving[*]

$$\Phi(xs) = \sum_{i=1}^{k} u_i(x)\alpha_i(s). \tag{19}$$

---

[*] The expressions for the functions α_i $(i = 1, \ldots, k)$ may be obtained from the above solution.

Upon substituting the value of $\Phi(xs)$ given by (16) in (19) and solving for $D(xs)$, we get

$$D(xs) = \sum_{i=1}^{k} \phi_i(x)v_i(s) + \sum_{i=1}^{k} u_i(x)\psi_i(s), \tag{20}$$

which gives the general form of the function D.

The function $G(xs)$ must be then of the form

$$G(xs) = G_1(xs) + \sum_{i=1}^{k} \phi_i(x)v_i(s) + \sum_{i=1}^{k} u_i(x)\psi_i(s). \tag{21}$$

If (21) is multiplied by $f(s)$ and integrated with respect to s from a to b, it is seen that if $G_1(xs)$ is any one generalized Green's function, $G(xs)$ will also serve as a generalized Green's function when $\phi_i(x)$ and $\psi_i(s)$ $(i = 1, \cdots, k)$, are any arbitrary continuous functions.

3. *Generalized Green's Functions with Property of Symmetry.* It is desirable to define a generalized Green's function which shall possess for adjoint systems the same property of symmetry as the ordinary Green's function.[*] With this end in view, we prove several lemmas.

LEMMA I. *If $K(xy)$ is any function satisfying conditions III (a, b, c, d) of the introduction, and $v(x)$ is any solution of the equation $M(v) = 0$, then*

$$\sum_{p=1}^{2n} U_p\,[K(xy)]\; V_{2n+1-p}[v(x)] = v(y).$$

To prove this, apply Green's theorem,[†] letting $v = v(x)$ and $u = K(xy)$. We get

$$\int_a^b \{v(x)L_x[K(xy)] - K(xy)M[v(x)]\}\,dx = [P(K,v)]_a^{y-} + [P(K,v)]_{y+}^{b}$$

Since the integral vanishes,

$$[P(K,v)]_a^{y-} + [P(K,v)]_{y+}^{b} = 0.$$

Whence

$$P[(K,v)]_a = [P(K,v)]_{y-}^{y+} = l_n(y)v(y)\left[\frac{\partial^{n-1}}{\partial x^{n-1}}K(x,y)\right]_{y-}^{y+} = v(y).$$

---

[*] See Introduction.

[†] For further details of such a proof, see M. Bôcher, *Les Méthodes de Sturm*, p. 104.

But

$$[P(K,v)]_a^b = \sum_{p=1}^{2n} U_p\,[K(xy)]\,V_{2n+1-p}(v).$$

Therefore

$$\sum_{p=1}^{2n} U_p\,[K(xy)]\,V_{2n+1-p}(v) = v(y),$$

as we wished to show.

Define the function $N(xy)$ by the formula

$$N\left\{\begin{matrix} v_1(y) & \cdots & v_k(y) \\ \psi_1(x) & \cdots & \psi_k(x) \end{matrix}\right\} = \frac{\begin{vmatrix} 0 & \psi_1(x) & \cdots & \psi_k(x) \\ v_1(y) & \int_a^b v_1(s)\psi_1(s)\,ds & \cdots & \int_a^b v_1(s)\psi_k(s)\,ds \\ \cdot & \cdot & \cdot & \cdot \\ v_k(y) & \int_a^b v_k(s)\psi_1(s)\,ds & \cdots & \int_a^b v_k(s)\psi_k(s)\,ds \end{vmatrix}}{\begin{vmatrix} \int_a^b v_1(s)\psi_1(s)\,ds & \cdots & \int_a^b v_1(s)\psi_k(s)\,ds \\ \cdot & \cdot & \cdot \\ \int_a^b v_k(s)\psi_1(s)\,ds & \cdots & \int_a^b v_k(s)\psi_k(s)\,ds \end{vmatrix}}$$

where the ψ_i's are any functions whatever, which give the determinant in the denominator of the above fraction a value different from zero.*

It is readily seen that *the function N is independent of linear combinations of the linearly independent solutions* $(v_1, \cdots, v_k)$, *and also of any particular linear combination of the* ψ_i*'s after this set of functions has been chosen.*

LEMMA II. *The system*

$$(22)\qquad L[K_1(xy)] = N(xy), \quad U_p[K_1(xy)] = -U_p(K), \quad (p = 1, \cdots, n),$$

has a solution in x which possesses a continuous nth derivative taken with respect to x.

Solutions of the above system with a continuous nth derivative exist if †

$$(23)\qquad \int_a^b v_i(x)\,N(xy)\,dx = -\sum_{p=1}^{n} U_p(K)\,V_{2n+1-p}(v_i) \qquad (i = 1, \cdots, k).$$

---

* If this condition is satisfied, the v_i's and ψ_i's may be transformed linearly into biorthogonal sets.

† M. Bôcher, *Les Méthodes de Sturm*, p. 36.

But

$$(24)\qquad \int_a^b v_i(x)N(xy)\,dx = \frac{\begin{vmatrix} 0 & \int_a^b v_i(s)\psi_1(s)\,ds & \cdots & \int_a^b v_i(s)\psi_k(s)\,ds \\ v_1(y) & \int_a^b v_1(s)\psi_1(s)\,ds & \cdots & \int_a^b v_1(s)\psi_k(s)\,ds \\ \cdot & \cdot & \cdot & \cdot \\ v_k(y) & \int_a^b v_k(s)\psi_1(s)\,ds & \cdots & \int_a^b v_k(s)\psi_k(s)\,ds \end{vmatrix}}{\begin{vmatrix} \int_a^b v_1(s)\psi_1(s)\,ds & \cdots & \int_a^b v_1(s)\psi_k(s)\,ds \\ \cdot & \cdot & \cdot \\ \int_a^b v_k(s)\psi_1(s)\,ds & \cdots & \int_a^b v_k(s)\psi_k(s)\,ds \end{vmatrix}}$$

$$= \frac{\begin{vmatrix} -v_i(y) & 0 & \cdots & 0 \\ v_1(y) & \int_a^b v_1(s)\psi_1(s)\,ds & \cdots & \int_a^b v_1(s)\psi_k(s)\,ds \\ \cdot & \cdot & \cdot & \cdot \\ v_k(y) & \int_a^b v_k(s)\psi_1(s)\,ds & \cdots & \int_a^b v_k(s)\psi_k(s)\,ds \end{vmatrix}}{\begin{vmatrix} \int_a^b v_1(s)\psi_1(s)\,ds & \cdots & \int_a^b v_1(s)\psi_k(s)\,ds \\ \cdot & \cdot & \cdot \\ \int_a^b v_k(s)\psi_1(s)\,ds & \cdots & \int_a^b v_k(s)\psi_k(s)\,ds \end{vmatrix}}$$

$$= -v_i(y) \qquad (i = 1, \cdots, k).$$

Since

$$\sum_{p=1}^{2n} U_p(K)\, V_{2n+1-p}(v_i) = \sum_{p=1}^{n} U_p(K)\, V_{2n+1-p}(v_i) \qquad (i = 1, \cdots, k),$$

from (24) and lemma (I) it is seen that condition (23) is satisfied and therefore system (22) has a solution.

The solution K_1 of (22) is not unique but is determined except for added solutions of the homogeneous system corresponding to (22).

Lemmas (I) and (II) hold for the adjoint systems when v_i is replaced by u_i and ψ_i by ϕ_i $(i = 1, \cdots, k)$, if the ϕ_i's are any functions* which give the determinant,

$$\begin{vmatrix} \int_a^b u_1(s)\phi_1(s)\,ds & \cdots & \int_a^b u_1(s)\phi_k(s)\,ds \\ \cdot & \cdot & \cdot \\ \int_a^b u_k(s)\phi_1(s)\,ds & \cdots & \int_a^b u_k(s)\phi_k(s)\,ds \end{vmatrix}$$

a value different from zero.

---

* The ϕ_i's and ψ_i's used in this section should not be confused with those used in Section (2).

Let

(25) $$G(xy) = K(xy) + K_1(xy).$$

Among the solutions chosen for K_1 there is always one for which

(26) $$\int_a^b \phi_i(x) G(xy)\,dx = 0 \qquad (i = 1, \cdots, k).$$

Hence the following theorem has been demonstrated:

THEOREM III. *If the system* (1) *is compatible, there exists a function satisfying the following conditions:*

(27) (a) $$G(xy),\ \partial G(xy)/\partial x,\ \cdots,\ \partial^{n-2} G(xy)/\partial x^{n-2}$$

exist and are continuous for $a \leq x \leq b,\ a \leq y \leq b,$

(b) $$\partial^{n-1} G(xy)/\partial x^{n-1}, \quad \partial^n G(xy)/\partial x^n$$

exist and are continuous for $a \leq x < y,\ y < x \leq b,$

(c) $$\partial^{n-1} G(y+, y)/\partial x^{n-1} - \partial^{n-1} G(y-, y)/\partial x^{n-1} = 1/l_n(y),$$

(d) $$L_x[G(xy)] = N(xy), \quad a \leq x < y, \quad y < x \leq b,$$

(e) $$U_p[G(xy)] = 0 \quad (p = 1, \cdots, n),$$

(f) $$\int_a^b \phi_i(x) G(xy)\,dx = 0 \quad (i = 1, \cdots, k).$$

The function $G(xy)$ satisfying these conditions depends both on the given differential system and on the choice of the ϕ_i's and ψ_i's.

THEOREM IV. *The function $G(xy)$ satisfying conditions* (27) *is a generalized Green's function for the compatible differential system* (1).

By way of proof, let $u_0(x)$ be defined by

$$u_0(x) = \int_a^b G(xs) f(s)\,ds,$$

where $f(s)$ satisfies condition (4). Upon applying the operators[*] $L,\ U_i$ $(i = 1, \cdots, n)$, and taking into account condition (4), we get

$$L(u_0) = f(x), \quad U_p(u_0) = 0 \qquad (p = 1, \cdots, n),$$

which shows that u_0 satisfies the semi-homogeneous system (3). Then every solution of the semi-homogeneous system (3) may be written in the form

$$u(x) = \sum_{i=1}^{k} c_i u_i(x) + \int_a^b G(xs) f(s)\,ds,$$

---

[*] All details of differentiation are here omitted.

which demonstrates the theorem.

Also, it is readily seen that $\int_a^b \phi_i(x)u_0(x)\,dx = 0 \ (i = 1, \cdots, k)$.

COROLLARY I. *If* $u_0(x) = \int_a^b G(xs)f(s)\,ds$, *where* $f(s)$ *is any continuous function, then* $u_0(x)$ *satisfies the system*

$$\left\{\begin{array}{l} L(u_0) = f(x) + \int_a^b N(xs)f(s)\,ds, \\ U_p(u_0) = 0 \quad (p = 1, \cdots, n), \\ \int_a^b \phi_i u_0\,dx = 0 \quad (i = 1, \cdots, k). \end{array}\right. \tag{28}$$

COROLLARY II. *If, for each continuous* $f(x)$, *a solution of system* (28) *is given in the form*

$$u_0(x) = \int_a^b G_0(xs)f(s)\,ds$$

then

$$G_0(xs) = G(xs).$$

Unless explicitly stated otherwise all generalized Green's functions considered in the remainder of this paper will satisfy conditions of type (27). We shall denote by $H(xy)$ the generalized Green's function for system (2).

THEOREM V. *If* $n > 1$, *for any explicit choice of* ϕ_i *and* ψ_i $(i = 1, \cdots, k)$,

$$H(xy) \equiv G(yx).$$

By way of proof,[*] let ξ and η be any two points in the interval (ab). For convenience suppose $\xi < \eta$. Apply Green's theorem letting $u = G(x\xi)$, and $v = H(x\eta)$, choosing as limits of integration a, $\xi-$, then $\xi+$, $\eta-$, and finally $\eta+$, b. Since the integrand is a continuous function of x, we get

$$\begin{aligned}\int_a^b \{H(x\eta)L[G(x\xi)] - G(x\xi)M[H(x\eta)]\}\,dx \\ = [P(G,H)]_a^{\xi-} + [P(G,H)]_{\xi+}^{\eta-} + [P(G,H)]_{\eta+}^{b} \\ = [P(G,H)]_a^{b} - [P(G,H)]_{\xi-}^{\xi+} - [P(G,H)]_{\eta-}^{\eta+} .\end{aligned}$$

But

---

[*] The method of proof used here is the same as that used by M. Bôcher in dealing with the ordinary Green's function; *Les Méthodes de Sturm*, p. 104.

$$\int_a^b H(x\eta)L[G(x\xi)]dx$$

$$= \int_a^b H(x\eta) \frac{\begin{vmatrix} 0 & \psi_1(x) & \cdots & \psi_k(x) \\ v_1(\xi) & \int_a^b v_1(s)\psi_1(s)ds & \cdots & \int_a^b v_1(s)\psi_k(s)ds \\ \cdot & \cdot & \cdot & \cdot \\ v_k(\xi) & \int_a^b v_k(s)\psi_1(s)ds & \cdots & \int_a^b v_k(s)\psi_k(s)ds \end{vmatrix}}{\begin{vmatrix} \int_a^b v_1(s)\psi_1(s)ds & \cdots & \int_a^b v_1(s)\psi_k(s)ds \\ \cdot & \cdot & \cdot \\ \int_a^b v_k(s)\psi_1(s)ds & \cdots & \int_a^b v_k(s)\psi_k(s)ds \end{vmatrix}} dx = 0.$$

Likewise

$$\int_a^b G(x\xi)M[H(x\eta)]dx = 0.$$

Also

$$[P(G,H)]_a^b = \sum_{p=1}^{2n} U_p(G)\ V_{2n+1-p}(H) = 0.$$

And since all derivatives of G and H of lower order than $(n-1)$ are continuous functions for all values of x

$$[P(G,H)]_{\xi-}^{\xi+}$$

$$= \left[l_n(x) \left\{ \frac{\partial^{n-1}}{\partial x^{n-1}} G(x\xi)H(x\eta) + (-1)^{n-1}G(x\xi) \frac{\partial^{n-1}}{\partial x^{n-1}} H(x\eta) \right\} \right]_{\xi-}^{\xi+} = H(\xi\eta),$$

$$[P(G,H)]_{\eta-}^{\eta+}$$

$$= \left[l_n(x) \left\{ \frac{\partial^{n-1}}{\partial x^{n-1}} G(x\xi)H(x\eta) + (-1)^{n-1}G(x\xi) \frac{\partial^{n-1}}{\partial x^{n-1}} H(x\eta) \right\} \right]_{\eta-}^{\eta}$$

$$= l_n(\eta) \{(-1)^{n-1}G(\eta\xi)(-1)^n/l_n(\eta)\} = -G(\eta\xi).$$

It follows that

$$H(\xi\eta) = G(\eta\xi) \qquad \xi < \eta.$$

By similar proof

$$H(\xi\eta) = G(\eta\xi) \qquad \xi > \eta.$$

Then from continuity

$$H(\xi\eta) = G(\eta\xi) \qquad \xi = \eta.$$

Hence, $$H(xy) \equiv G(yx).$$

COROLLARY I. *If* $n = 1$, $H(xy) = G(yx) \quad x \neq y$.

COROLLARY II. *For any chosen set of* ϕ_i*'s and* ψ_i*'s the function satisfying conditions* (27) *is unique.*

Since it has been assumed that the determinants

$$\begin{vmatrix} \int_a^b v_1(s)\psi_1(s)\,ds \cdots \int_a^b v_1(s)\psi_k(s)\,ds \\ \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \\ \int_a^b v_k(s)\psi_1(s)\,ds \cdots \int_a^b v_k(s)\psi_k(s)\,ds \end{vmatrix}, \quad \left| \begin{matrix} \int_a^b u_1(s)\phi_1(s)\,ds \cdots \int_a^b u_1(s)\phi_k(s)\,ds \\ \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \\ \int_a^b u_k(s)\phi_1(s)\,ds \cdots \int_a^b u_k(s)\phi_k(s)\,ds \end{matrix} \right.$$

do not vanish the ψ_i's and the ϕ_i's may be respectively transformed linearly into sets biorthogonal with the v_i's and u_i's $(i=1, \cdots, k)$. If this is done some of the preceding formulas will simplify considerably. The function N is then given by the formula

$$N\begin{pmatrix} v_1, & \cdots, & v_k \\ \psi_1, & \cdots, & \psi_k \end{pmatrix} = -\sum_{i=1}^{k} \psi_i v_i,$$

and $G(xy)$ will satisfy the equation

$$L_x[G(xy)] = -\sum_{i=1}^{k} \psi_i(x) v_i(y).$$

As a special example, consider the system

$$\begin{aligned} u'' &= 0, \\ u(0) + u(1) &= 0, \\ u'(0) - u'(1) &= 0. \end{aligned}$$

This system is compatible of order one, and a solution is $u_1(x) = 1 - 2x$. Its adjoint system is

$$\begin{aligned} v'' &= 0, \\ v(0) - v(1) &= 0, \\ v'(0) + v'(1) &= 0, \end{aligned}$$

which has a solution $v_1(x) = 1$.

If $\phi_1(x) = \bar{u}_1(x) = 1 - 2x$, and $\psi_1(x) = \bar{v}_1(x) = 1$, the generalized Green's functions for the two systems are respectively

$$\begin{aligned} G_1(xy) &= -x^2/2 - x/2 - 2xy^3 + 3xy^2 + y^3 - 3y^2/2 + y/2, & x &\leq y, \\ &= -x^2/2 + x/2 - 2xy^3 + 3xy^2 + y^3 - 3y^2/2 - y/2, & x &\geq y. \\ H_1(xy) &= -3x^2/2 + 3x^2y + x^3 - 2x^3y - y^2/2 + y/2 - x/2, & x &\leq y, \\ &= -3x^2/2 + 3x^2y + x^3 - 2x^3y - y^2/2 - y/2 + x/2, & x &\geq y. \end{aligned}$$

If $\phi_1(x) = \psi_1(x) = x$, the generalized Green's functions become

$$\begin{aligned} G_2(xy) &= -x^3/3 + 3xy - 2xy^3 + y^3 - 6x/5 - y + 4/15, & x &\leq y, \\ &= -x^3/3 + 3xy - 2xy^3 + y^3 - x/5 - 2y + 4/15, & x &\geq y. \\ H_2(xy) &= x^3 - 2x^3y + 3xy - y^3/3 - y/5 - 2x + 4/15, & x &\leq y, \\ &= x^3 - 2x^3y + 3xy - y^3/3 - 6y/5 - x + 4/15, & x &\geq y. \end{aligned}$$

If we choose $\phi_1(x) = x^2$, and $\psi_1(x) = x$, the corresponding generalized Green's functions are

$$\begin{aligned} G_3(xy) &= -x^3/3 - xy^4 + 2xy + y^4/2 - x - y/2 + 1/6, & x &\leq y, \\ &= -x^3/3 - xy^4 + 2xy + y^4/2 - 3y/2 + 1/6, & x &\geq y. \\ H_3(xy) &= x^4/2 - x^4y + 2xy - y^3/3 - 3x/2 + 1/6, & x &\leq y, \\ &= x^4/2 - x^4y + 2xy - y^3/3 - x/2 - y + 1/6, & x &\geq y. \end{aligned}$$

An Expansion Involving p Inseparable Parameters Associated with a Partial Differential Equation.*

BY CHESTER C. CAMP.

1. *Introduction and Formal Setting.* The partial differential equation

$$(1) \qquad \sum \partial u/\partial x_l + (\lambda h + k)u = 0,\dagger$$

may be solved by Lagrange's method provided h and k each consist of integrable functions of the several single variables and polynomials in more than one variable.‡ It is clear that one may then solve (1) in the form

$$(2) \qquad u = e^{-\lambda H - K} u_1$$

where $h = DH + \sum a_l(x_l)$, $k = DK + \sum g_l(x_l)$, $D \equiv \sum \partial/\partial x_l$

and u_1 is a solution of the equation

$$(3) \qquad \sum [\partial u_1/\partial x_l + (\lambda a_l + g_l)u_1] = 0.$$

It may be shown that the general solution of (3) is

$$(4) \qquad u_1 = exp\ [-\lambda \sum A_l(x_l) - \sum G_l(x_l)] \cdot \phi(x_1 - x_2, x_1 - x_3, \cdots, x_1 - x_p),$$

where $A_l(x_l) = \int a_l(x_l)dx_l$ and $G_l(x_l) = \int g_l(x_l)dx_l$,

and l runs from 1 to p.

If ϕ is so chosen that $u_1 = \prod X_l(x_l)$, equation (3) leads to the differential system

$$(5) \qquad \begin{aligned} &X_1' + (\lambda a_1 + g_1 - \textstyle\sum' \mu_l)X_1 = 0. \\ &X_l' + (\lambda a_l + g_l + \mu_l)X_l = 0, \qquad (l = 2, 3, \cdots, p). \end{aligned}$$

---

[*] Presented to the Society, Sept. 9, 1926.

[†] When l runs from 1 to p the sum is indicated by Σ and the product by Π, when l runs from 2 to p the sum is indicated by Σ', when l runs from 3 to p the sum is indicated by Σ''.

[‡] They might contain such terms as $x_1^2 x_2^3 \sin x_3$, which may be integrated as to x_1 under the condition $dx_1 = dx_2 = \cdots = dx_p$ by first writing

$$\int x_1^2 x_2^3 \sin x_3 dx_3 = -x_1^2 x_2^3 \cos x_3 + \int (3x_1^2 x_2^2 + 2x_1 x_2^3) \cos x_3 dx_3.$$

Further integrations by parts diminish the degrees of the multipliers of $\cos x_3$ or $\sin x_3$ to zero.

In case H and K each satisfy the boundary conditions

$$(6)\qquad [u(x_1, x_2, \cdots, x_p)]_{x_l=\pi} = [u(x_1, x_2, \cdots, x_p)]_{x_l=-\pi} \quad (l = 1, 2, \cdots, p)$$

they will not affect the principal parameter values of λ for the system (1), (6); namely

$$(7)\qquad \lambda_{m_1, m_2, \ldots, m_p} = \sum (m_l i - G_l)/\sum A_l$$

where m_l is an integer, $2\pi G_l = G_l(\pi) - G_l(-\pi)$, and $2\pi A_l = A_l(\pi) - A(-\pi)$, $(l = 1, 2, \cdots, p)$.

Let us assume that each $a_l(x_l)$ is either identically zero or of constant sign, also that $\sum A_l \neq 0$. The case in which every $g_l(x_l) \equiv 0$ will be considered first since the proof of convergence is thereby simplified; secondly, the case in which some $g_j(x_j) \not\equiv 0$; and finally that in which h and k may each contain functions of more than one variable.

It is to be noticed that the system (5) involves parameters which are tangled up so to speak; i. e., it is not possible to separate them so as to have p ordinary differential equations each involving a single parameter.

One may readily show that the adjoint system with similar boundary conditions has the same principal parameter values; e. g., for the simplest case the adjoint system

$$(8)\qquad \begin{aligned} &-Y_1' + (\lambda a_1 - \textstyle\sum' \mu_l) Y_1 = 0, \\ &-Y_l' + (\lambda a_l + \mu_l) Y_l = 0, \qquad (l = 2, 3, \cdots, p) \end{aligned}$$

$$(9)\qquad Y_l(\pi) = Y_l(-\pi), \qquad (l = 1, 2, \cdots, p)$$

has the characteristic values

$$(10)\qquad \begin{aligned} &\lambda^* = \textstyle\sum m_l i/\sum A_l \\ &\mu_l = \lambda^*(\textstyle\sum A_l - A_l) - (\sum m_l - m_l) i = -\lambda^* A_l + m_l i, \\ &\qquad\qquad (l = 2, 3, \cdots, p). \end{aligned}$$

By using two sets of characteristic values which differ in at least one of the parameters we may adapt (5) to one and (8) to the other using a double asterisk to indicate the solutions associated with the second set, thus

$$(11)\qquad \begin{aligned} &X_1'^* + (\lambda^* a_1 - \textstyle\sum' \mu_l^*) X_1^* = 0, \\ &X_l'^* + (\lambda^* a_l + \mu_l^*) X_l^* = 0, \qquad (l = 2, 3, \cdots, p); \end{aligned}$$

$$(12)\qquad \begin{aligned} &-Y_1'^{**} + (\lambda^{**} a_1 - \textstyle\sum' \mu_l^{**}) Y_1^{**} = 0, \\ &-Y_l'^{**} + (\lambda^{**} a_l + \mu_l^{**}) Y_l^{**} = 0, \qquad (l = 2, 3, \cdots, p). \end{aligned}$$

Multiplying the jth of equations (11), (12) by Y_j^{**}, $-X_j^*$, respectively, adding, integrating from $-\pi$ to π and using (9) and $X_j^*(\pi) = X_j^*(-\pi)$, one obtains

$$(13)\qquad (\lambda^* - \lambda^{**}) \int_{-\pi}^{\pi} a_1 X_1^* Y_1^{**} dx_1 - \Sigma' (\mu_l^* - \mu_l^{**}) \int_{-\pi}^{\pi} X_1^* Y_1^{**} dx_1 = 0,$$

$$(\lambda^* - \lambda^{**}) \int_{-\pi}^{\pi} a_j X_j^* Y_j^{**} dx_j + (\mu_j^* - \mu_j^{**}) \cdot \int_{-\pi}^{\pi} X_j^* Y_j^{**} dx_j = 0, \quad (j = 2, 3, \cdots, p).$$

Since the sets of characteristic values are not identical, it is necessary that the determinant of (13) vanish. This gives the conjugacy condition

$$(14)\qquad \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} (\Sigma a_l) \Pi X_l^* Y_l^{**} dx_l = 0.$$

If one assumes the expansion

$$(15)\qquad f(x_1, x_2, \cdots, x_p) = \sum_{\substack{m_i = -\infty \\ i = 1, 2, \ldots, p}}^{\infty} C_{m_1, m_2, \ldots, m_p}\, u_1^*(x; m_1, m_2, \cdots, m_p)$$

where f is made up of a finite number of pieces, each real, continuous, and possessing a continuous partial derivative in each variable, and the argument x in the principal solution u_1^* is written in place of $x_1, x_2, \cdots, x_p$, then formally † if $v_1^* \equiv \Pi Y_l^*(x_l)$,

$$C_{m_1, m_2, \ldots, m_p} = \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} f(s_1, \cdots, s_p) \sum a_l(s_l) v_1^* (s; m_1, \cdots, m_p) \cdot \Pi\, ds_l / (2\pi)^p \sum A_l.$$

Let us show that the formal series actually converges to the so-called mean value of f.

2. *Convergence Proof.* Case I, $g_l(x_l) \equiv 0$, $(l = 1, 2, \cdots, p)$. It is important in setting up the analog of the Green's function to make a transformation of parameters as follows:

$$(16)\qquad \begin{cases} \nu_1 = \lambda A_1 - \Sigma' \mu_l \\ \nu_l = \lambda A_l + \mu_l, \qquad (l = 2, 3, \cdots, p). \end{cases}$$

One can then readily prove that one term of the series (15) is the residue as to $\nu_1, \nu_2, \cdots, \nu_p$ at a set of values corresponding to a set of characteristic values for $\lambda, \mu_2, \mu_3, \cdots, \mu_p$, of the function

---

† Cf. an article by Carmichael, "Boundary Value and Expansion Problems," *American Journal of Mathematics*, Vol. 43, No. 4 (Oct. 1921), p. 260.

$$(17)\quad \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} f(s_1, \cdots, s_p) \sum a_l(s_l) G(x; s; \lambda, \mu_2, \cdots, \mu_p) \cdot \prod ds$$

where

$$(18)\quad G \equiv \exp\, \Big[\int_{x_1}^{s_1} (\lambda a_1 - \textstyle\sum' \mu_l)\, dx_1 + \sum \int_{x_l}^{s_l} (\lambda a_l + \mu_l)\, dx_l \Big] / (\textstyle\sum A_l) \prod (1 - e^{-2\pi\nu_l}), \quad s_l < x_l);$$

and when $s_j > x_j$ for one or more values of j the factor $(1 - e^{-2\pi\nu_j})$ in the denominator for each such j is to be replaced by $(e^{2\pi\nu_j} - 1)$.

The convergence is studied by the powerful and elegant method of Birkhoff adapted to multiple series by which one takes the limit of a p-fold contour integral I whose curves, whether circles or rectangles in the several ν-planes, enclose an indefinitely increasing number of characteristic values. One is led therefore to investigate one of the 2^p pieces in the form

$$(19)\quad I^{x_1, x_2, \ldots, x_p}_{-\pi, -\pi, \ldots, -\pi} = \int_{\Gamma_p} \int_{\Gamma_{p-1}} \cdots \int_{\Gamma_1} \int_{-\pi}^{x_1} \cdots \int_{-\pi}^{x_p} f\Big(\sum_{l=1}^{p} a_l \Big) G \prod ds_l d\nu_l / (2\pi i)^p.$$

Replace $\sum a_l$ by

$$(\textstyle\sum A_l) \{ [\lambda a_1 - \sum' \mu_l] + \sum' [\lambda a_l + \mu_l] \} / \sum \nu_l$$

thus separating the $2p$-fold integral into p pieces. If one integrates each by parts using the brackets in order to integrate exponentials, one obtains as the integrands of the contour integrals, exclusive of the factor

$$1/(2\pi i)^p\, (\textstyle\sum \nu_l) \prod (1 - e^{-2\pi\nu_l}),$$

$$\begin{aligned}(20)\quad & \int_{-\pi}^{x_p} \int_{-\pi}^{x_{p-1}} \cdots \int_{-\pi}^{x_2} \exp\, \Big[\textstyle\sum' \displaystyle\int_{x_l}^{s_l} (\lambda a_l + \mu_l)\, dx_l \Big] \cdot \\ & \qquad \{ f(x_1 - 0, s_2, \cdots, s_p) - f(-\pi, s_2, \cdots, s_p) \cdot \\ & \qquad\qquad \exp\, \Big[- \int_{-\pi}^{x_1} (\lambda a_1 - \textstyle\sum' \mu_l)\, dx_1 \Big] \} dx_2 \cdots dx_p \\ & + \sum_{j=2}^{p} \int_{-\pi}^{x_p} \int_{-\pi}^{x_{p-1}} \cdots \int_{-\pi}^{x_{j+1}} \int_{-\pi}^{x_{j-1}} \cdots \int_{-\pi}^{x_1} \\ & \exp\, \Big[\sum_{l \neq j}' \int_{x_l}^{s_l} (\lambda a_l + \mu_l)\, dx_l + \int_{x_1}^{s_1} (\lambda a_1 - \textstyle\sum' \mu_l)\, dx_1 \Big] \cdot \\ & \qquad \Big[f(s_1, \cdots, s_p) \exp \int_{x_j}^{s_j} (\lambda a_j + \mu_j)\, dx_j \Big]_{-\pi}^{x_j} \prod_{l=1}^{p} ds_l / ds_j \\ & - \int_{-\pi}^{x_p} \cdots \int_{-\pi}^{x_1} u_1(x_1, \cdots, x_p)\, v_1(s_1, \cdots, s_p)\, \Big(\sum \partial f / \partial s_l \Big) \prod ds_l + \epsilon.\end{aligned}$$

Here $\epsilon = 0$ if f is continuous. In case of a discontinuity, in x_1 for example, there will be an extra term in the integration by parts, which would involve a factor S $\exp\,[-\int_{x}^{x_1} (\lambda a_1 - \sum \mu_l)\,dx_1]$ in lieu of

$$f(-\pi, s_2, \cdots, s_p) \exp\,[-\int_{-\pi}^{x_1} (\lambda a_1 - \sum \mu_l)\,dx_1]$$

where S is the saltus or fluctuation of f in x_1 at the point $x_1 = x$. By a subsequent argument such a piece contributes zero to the limit. The analogous terms are not mentioned explicitly in a previous paper by the author [*] but are included by the notation $O(1/\lambda^2)$ or $O(1/\lambda)$, whereas such terms are not considered in a paper by Carman [†] although they can as readily be disposed of there.

The integrands of the exponentials are now written in the form $\lambda(a_l - A_l) + \nu_l$, $(l = 1, 2, \cdots, p)$, and one integrates $p - 1$ times by parts (each time with respect to a different variable), using the exponentials $\exp\,[\nu_l(s_l - x_l)]$, each integral of (20) previously obtained as an upper limit in the first integration by parts. One item of the result is

$$(21) \qquad \frac{1}{(2\pi i)^p} \int_{\Gamma_p} \cdots \int_{\Gamma_1} \frac{f(x_1 - 0, x_2 - 0, \cdots, x_p - 0)}{\prod \nu_l (1 - e^{-2\pi\nu_l})} \prod d\nu_l$$

Which is equivalent in the limit as $|\nu_l|$ becomes infinite to

$$f(x_1 - 0, x_2 - 0, \cdots, x_p - 0)/2^p,$$

as is seen by using circular contours with centers at the origins of the complex planes and radii $\frac{1}{2} + m_i$, where m_i is a positive integer. This is also a consequence of the following lemma.

$$\text{Lemma: ‡ If } 0 < h < 1, \text{ then } \underset{|z|\to\infty}{L} \frac{1}{2\pi i} \int_{C_{|z|}} \left| \frac{e^{-2\pi hz}\,dz}{z(1 - e^{-2\pi z})} \right| = 0 = \underset{|z|\to\infty}{L} \frac{1}{2\pi i} \int_{C_{|z|}} \left| \frac{e^{2\pi hz}\,dz}{z(e^{2\pi hz} - 1)} \right|$$

where $C_{|z|}$ represents a circle in the z-plane with center at the origin and radius $|z|$. A vertical rectangle with center at the origin and increasing altitude may be substituted for the circle. For $h = 0$, or $h = 1$ the limits of the first integral are $\frac{1}{2}$, or $-\frac{1}{2}$, respectively.

---

[*] "Multiple Fourier Series Expansions," *Transactions of the American Mathematical Society*, Vol. 25 (1923), pp. 123-134.

[†] "Convergence of Fourier Series," *Bulletin of the American Mathematical Society* (Oct. 1924).

[‡] For details of proof one may refer to the papers by Carman and the author cited above, or to an article by Birkhoff and Langer, "Boundary Problems and Developments Associated with a System of Ordinary Linear Differential Equations of the First Order," *Proceedings of the American Academy*, Vol. 58, No. 2 (Apr. 1923), pp. 112-114.

Use is made of this to show also that the contribution of each of the other integrals to the limit is zero.

Since the characteristic values are all pure imaginaries it will be sufficient to employ rectangular contours of distinct widths δ_i held fixed. The limit as each altitude becomes infinite represents the sum of an infinite series. This consists of the sum of a series of residues and hence the same sum will be obtained by using circular contours in which the several radii $|\nu_l|$ become infinite. Let the radii be so chosen that each $|\nu_l|$ and $|\sum \nu_l|$ become infinite together. The lengths of the rectangles may be chosen so as to include the same principal parameter values as the circular contours, the distance of the contours in either case from the nearest characteristic values being kept bounded below.

The third contour integral of (20) may be shown to converge in the limit to zero. To prove this one compares the infinite series represented by it with the simpler series obtained by removing according to the second mean value theorem the exponentials of the form $|\exp \lambda \int_{x_l}^{s_l} (a_l - A_l)\, dx_l|$. If $|\sum \partial f/\partial s_l|$ is now replaced by an upper bound M, the remaining integral can readily be shown by direct integration and the above lemma to contribute zero.

The product of the exponentials is bounded uniformly over the rectangles suggested above and hence the original series converges to zero for any interior point. For points on the boundary the integral still converges to zero on account of the factor $\sum \nu_l$ in the denominator, which becomes infinite for circular contours.

To show that the parts of (20) involving the lower limits converge to zero for an interior point one compares the series they represent with simpler series. As an example, in the first one, we remove the exponential factor whose index is $\sum' \int_{x_l}^{s_l} \lambda (a_l - A_l)\, dx_l - \lambda \int_{-\pi}^{x_1} (a_1 - A_1)\, dx_1$, take absolute values, and replace $f(-\pi, s_2, \cdots, s_p)$ by an upper bound. The resulting contour integral converges by the lemma and the fact that $|\sum \nu_l| \to \infty$ when $|\nu_1| \to \infty$. The original series will converge to zero and hence its contour integral will also over the rectangular or circular contours selected above.

The same argument holds for the remaining integrals of (20) and the analogous ones obtained in the $p-1$ successive integrations by parts. In the first of these one obtains the form

$$(22)\quad \frac{-1}{(2\pi i)^p}\int_{\Gamma_p}\cdots\int_{\Gamma_1}\quad\int_{-\pi}^{x_p}\cdots\int_{-\pi}^{x_2}\exp\,[\nu_2(s_2-x_2)+\\ \Sigma''\int_{x_l}^{s_l}(\lambda a_l+\mu_l)\,dx_l]\cdot\partial/\partial s_2\,\{f(x_1-0,\,s_2,\cdots,\,s_p)\cdot\\ \exp\int_{x_2}^{s_2}\lambda(a_2-A_2)\,dx_2\}\prod ds_l d\nu_l/N_2(\nu)\,ds_1,$$

where $N_2(\nu)\equiv\nu_2(\sum\nu_l)\prod\,[1-\exp\,(-2\pi\nu_l)]$.

For this consider a comparison series whose contour integral is

$$(23)\qquad \frac{-1}{(2\pi i)^p}\int_{\Gamma_p}\cdots\int_{\Gamma_1}\int\,|\exp\nu_2(s'_2-x_2)\,|\cdot\\ \int_{-\pi}^{x_p}\cdots\int_{-\pi}^{x_2}|\exp\,[-\nu_1A_2(s_2-x_2)/\sum A_l+\Sigma'''\,\nu_l(s_l-x_l)]\\ [\partial f/\partial s_2+\lambda(a_2-A_2)f]\,|\prod ds_l\,|\,d\nu_l\,|/|\,N_2(\nu)\,ds_1$$

where $-\pi<s_2'\leqq x_2$ by the mean value theorem, and where certain exponential factors which are bounded on the rectangular contours have been removed. But $s_2'\neq x_2$ unless ν_2 is on the axis of imaginaries. In such a case one omits from the contour Γ_2 small arcs over which the contour integral will contribute ϵ.

Since the last bracket of (23) divided by $\sum\nu_l$ is bounded, it is clear that the limit of (23) over a circle in the ν_2-plane and rectangular contours in the others will be zero.

The integral (22) is seen by the previous reasoning to contribute zero also. A like method applies to the other integrals of this form and to analogous forms in the other integrations by parts. The simpler comparison series converge to zero. One may change the one circular contour to a rectangle. The exponential multipliers in each case are bounded uniformly on the rectangular contours. Hence the original series converge to zero in the limit.

In case $A_2=a_2=0$ in (23) one of the exponents vanishes and the factor $\sum\nu_l$ is used instead of ν_1 in the denominator of the contour integral.

For an interior point each of the other parts of I may be shown to contribute analogous results, so that the whole integral has the limit $\sum f(x_1\pm0,\,x_2\pm0,\cdots,\,x_p\pm0)/2^p$, in other words the so-called mean value of f at the point $(x_1,\,x_2,\cdots,\,x_p)$. For points on the boundary the series will still converge to the mean value if we regard f as periodic in each argument with period 2π.

Case II, some $g_j(x_j) \not\equiv 0$. The presence of $\sum g_l(x_l)$ as a multiplier of u_1 gives for characteristic values

$$(24) \qquad \begin{cases} \lambda^* = \sum (m_l i - G_l)/\sum A_l, \\ \mu_l{}^* = m_l i - \lambda^* A_l - G_l, \qquad (l = 2, 3, \cdots, p). \end{cases}$$

The transformation of parmeters analogous to (16) is

$$(25) \qquad \begin{cases} \nu_1 = \lambda A_1 - \sum' \mu_l + G_1, \\ \nu_l = \lambda A_l + \mu_l + G_l, \qquad (l = 2, 3, \cdots, p). \end{cases}$$

$$(26) \qquad u_1 = \exp \{-\sum [\lambda A_l(x_l) + G_l(x_l)] + \sum' \mu_l(x_1 - x_l)\} = 1/\nu_1,$$

hence the analog of (19) will contain the extra factor $\exp \sum [G_l(s_l) - G_l(x_l)]$ in the integrand. Since each g_l is integrable $G_l(x_l)$ is continuous. It is clear that the f with the multiplier $\exp \sum G_l(s_l)$ may be regarded as a new f and that for any interior point the sum of the series analogous to that in (15) will be the same mean value as before.

Case III, DH or $DK \not\equiv 0$. For the more general form

$$(27) \qquad \sum \partial u/\partial x_l + \{\lambda[\sum a_l(x_l) + DH(x_1, \cdots, x_p)] \\ + \sum g_l(x_l) + DK(x_1, \cdots, x_p)\}u = 0$$

and the same boundary conditions (6) the general solution is given by (2) and (4). Let us first choose u_1 in the form $\prod X_l(x_l)$ as in (26). It is easy to prove that there will still be suitable values for λ^* in case H and K each satisfy (6). Then the presence of DH and DK will not alter the characteristic values. The conjugacy condition is now that in which $\sum a_l(s_l)$ is augmented by $DH(s_1, s_2, \cdots, s_p)$.

The denominator of the Green's function is not altered since

$$(28) \qquad \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} DH \prod ds_l = 0.$$

In the convergence proof one has the extra factor

$$(29) \qquad \exp \{\lambda[H(s_1, s_2, \cdots, s_p) - H(x_1, \cdots, x_p)] \\ + K(s_1, \cdots, s_p) - K(x_1, \cdots, x_p)\}$$

in the integrand, besides the new factor of the conjugacy condition. By separating the parts of $DH(s_1, \cdots, s_p)$ and integrating by parts one clearly obtains for one part of $I^{x_1, \cdots, x_p}_{-\pi, \ldots, -\pi}$ the same element of the mean value as before, since for $s_l = x_l$, $(l = 1, 2, \cdots, p)$ the value of (29) is unity. To show that the other pieces contribute zero for any interior point it is sufficient to use as a comparison series that formed by multiplying the several terms of the series for Case II by (29) and then to take absolute values. This factor is bounded uniformly on the rectangular contours for increasing altitudes. Hence the limit is zero. On the boundaries, if f is not regarded as periodic, certain parts of the mean value will require multipliers which are functions of the arguments that vary on the particular boundary.

Finally, it is possible to multiply the solution u_1 as given in (26) by a function ϕ with arguments as in (4) provided ϕ is periodic of period 2π in each. We take $v = 1/u$, then the contour integral will have the additional factor $\phi(x_1 - x_2, \cdots, x_1 - x_p)/\phi(s_1 - s_2, \cdots, s_1 - s_p)$ where f/ϕ may again be looked upon as a new f. If ϕ does not vanish obviously the series will again converge to the mean value at any interior point. We may therefore state the most general result in the following

THEOREM: *Let $f(x_1, x_2, \cdots, x_p)$ be made up of a finite number of pieces, each real, continuous, and possessing a continuous partial derivative in each argument, and let H, K and ϕ satisfy* (6), *while DH, DK consist of polynomials in the p variables or polynomials each multiplied by an integrable function of a single variable, $g_l(x_l)$ is integrable, each $a_l(x_l)$ is integrable and either identically zero or of constant sign, $-\pi \leqq x_l \leqq \pi$, but $\sum \int_{-\pi}^{\pi} a_l(x_l)\, dx_l \neq 0$, also $\phi \neq 0$, and $\partial\phi/\partial x_l$ is continuous; then the expansion*

$$\sum_{\substack{m_i=-\infty \\ i=1, 2, \ldots, p}}^{\infty} C_{m_1, \ldots, m_p} \quad u^*(x; m_1, \cdots, m_p),$$

where u^* is the most general characteristic solution of (27), (6), *$v^* = 1/u^*$, and*

$$(30) \qquad C_{m_1, m_2, \ldots, m_p} = \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} f v^* (\sum a_l + DH) \prod ds_l / \sum A_l,$$

converges to the so-called mean value of f at any interior point. In any interior closed region in which f and its first partial derivatives are continuous the series converges uniformly to f.

It is evident that the generalization involving ϕ can be applied to the multiple-Birkhoff series † and to other expansions associated with partial differential equations and boundary conditions.

---

† Cf. *Transactions of the American Mathematical Society*, Vol. 25, No. 3, pp. 338-342.

Moreover, it is believed that the convergence proof by the contour integral method for all such series can be carried through only in case the number of independent parameters equals the number of independent variables employed. For example, if in (4) ϕ had been chosen as unity the characteristic values (7) of λ would be simply $(mi - \sum G_l)/\sum A_l$ and the corresponding expansion for f would involve but one parameter. Another such expansion is given by Carmichael[*] for $f(x, y)$ with just one parameter λ. It is desirable to investigate further whether either of these formal expansions is valid.

UNIVERSITY OF NEBRASKA,
LINCOLN, NEBR.

---

[*] *American Journal of Mathematics, loc. cit.*, p. 259.

On the Asymptotic Evaluation of Functions Defined by Contour Integrals.

BY D. M. WRINCH.

INTRODUCTION.

1. In the present paper, the author attempts to give unity to the investigations of the asymptotic expansions for large values of n, of functions defined by the integral

$$I = \int e^{nf(z)} F(z)\, dz$$

taken along paths in the z-plane. The problem of grouping paths in such a way that the asymptotic expansions of the functions, defined by the integral I along paths belonging to the same group are identical, is first considered. The asymptotic evaluation of the integral I, along any one of a large class of paths is then effected by means of the discussion of the value of the integral I along certain standard paths.

This procedure is precisely similar to that adopted in the theory of complex integration. Cauchy's theorem gives specifications * of a group of paths associated with two points, such that the value of the integral

$$\int_C F(z)\, dz$$

is the same, whichever path of the group C may be. Next, the value of the integral taken along any one of a wide class of paths may be obtained by the discussion of the value of the integral taken along certain standard paths. The calculus of residues belongs to this part of the subject.

---

* Thus, e. g. Riemann considered the class of paths running from a point z_1 to a second point z_2 which is such that at all points on each path, P and Q, the real and imaginary parts of $F(z)$ have continuous derivates with respect to x and y such that

$$\partial P/\partial x = \partial Q/\partial y, \quad \partial Q/\partial x = -\partial P/\partial y.$$

Again, in Goursat's form of Cauchy's theorem, each member of the class of paths under consideration runs from a point z_1 to a second point z_2, and at all points on each path and at all points between any two of the paths, $F'(z)$ exists. Either of these specifications yields a group of paths associated with two points which is such that the value of the integral

$$\int_C F(z)\, dz$$

is the same whichever path of the group C may be. Cp. Watson, 'Complex Integration and Cauchy's Theorem,' *Cambridge Tracts in Mathematics*, No. 15.

In Cauchy's theorem, the paths belonging to one group, all begin at one point and end at a second point. In the corresponding theorem of asymptotic evaluation, it is sufficient to specify a certain *continuum* in which the paths belonging to one group must start and a certain *continuum* in which they must end. The continua in question are of a special type; they are called, in this paper, 'valleys' or 'valley continua.' An 'M-valley' associated with the function $f(z)$ is a continuum in which the real part of $f(z)$ has M as its upper bound.* Under certain circumstances, if m_1 and m_2 are two M-valleys associated with the function $f(z)$, and if C is a path starting in the continuum m_1 and ending in the continuum m_2, the asymptotic value of the integral

$$e^{-nM}\int_C e^{nf(z)}\,F(z)\,dz$$

is the same whichever path C may be. The results are obtained by considering the deductions which can be drawn from the behaviour of the real part of the function $f(z)$ on the path C, as to the asymptotic value, for large values of n, of the integral I taken along the path C. Although, on occasion it has been pointed out † that the integral I taken along a certain path differs from the integral taken along a second path C' by a term of order $e^{-n\delta}$, where δ is some positive number, the author is unaware of any previous attempt to study the general problem of grouping paths in such a way, that the functions defined by the integral I taken along paths belonging to one group admit the same asymptotic expansion for large values of n.

2. The asymptotic evaluation of the integral

$$\int e^{nf(z)}\,F(z)\,dz$$

has been discussed in a number of particular cases. Thus the analysis necessary for the Bessel functions defined by means of the integral

$$\int e^{-iN\sin z}\,e^{ikz}\,dz \qquad (N = ne^{i\nu})$$

taken along certain paths, was developed by Debye ‡ and supplemented by Watson.§ The analysis for the various functions defined by the integral

$$\int e^{i(z^3-Nz)}\,dz \qquad (N = ne^{i\nu})$$

---

* The idea of a region (not necessarily a continuum) in which the real part of $f(z)$ is less than a specific number M has been used by Debye, Brillouin, Watson, and Perron. *V. inf.*

† Cp. Perron, *Münchener Berichte* (1917), pp. 191-220.

‡ *Mathematische Annalen* (1909), Vol. 67, pp. 535-568; *Münchener Berichte* (1910) (5).

§ *Proceedings of the London Mathematical Society* (1916) (2), Vol. 17, pp. 116-148. Cp. also *Quarterly Journal* (1917), Vol. 48, pp. 1-18; *Transactions of the Cambridge Philosophical Society* (1918), Vol. 22, pp. 277-308.

along certain paths, and for various other functions defined by integrals which have a polynomial in the exponent of the integrand (e. g. by the integral

$$\int e^{i(z^4-pz^2-qz)}\,dz)$$

was set up by Brillouin.[*] The analysis for the function $W_{k,m}(N)$ defined in terms of the integral

$$\int (-z)^{-k-\frac{1}{2}+m}(1+z/N)^{k-\frac{1}{2}+m}\,e^{-z}\,dz\ . \qquad (N=ne^{i\nu})$$

was given by Watson.[†] These various writers called the method employed the Method of Steepest Descents,[‡] the Sattelpunkte Methode,[§] and the Méthode du Col.[*] In 1917, Perron's[†] important paper appeared. It dealt with the asymptotic evaluation of the integral

$$\int e^{nf(z)}\,F(z)\,dz$$

along certain types of paths. The asymptotic evaluation of the integral I along the standard paths adopted in this paper is derived from Perron's analysis. Perron's analysis is, however, carried a step further.

1. DEFINITIONS AND PRELIMINARY CONSIDERATIONS.

1.1. Let

$$I_C(n) = \int_C e^{nf(z)}\,F(z)\,dz,$$

C being a path in the z-plane. The absolute value of the function $I_C(n)$ is evidently dependent upon the behaviour of the real part of $f(z)$ on the path C. Suppose that there exists a positive number δ such that

$$\Re f(z) \leq M - \delta$$

on the path C. Then, assuming that the integral $I_C(n)$ exists, at least for sufficiently large values of n, we may deduce that

$$\begin{aligned} |\,I_C(n)\,| &= \Big|\int_C e^{(n-b)f(z)}\,e^{bf(z)}F(z)\,dz\,\Big| \\ &\leq e^{(n-b)(M-\delta)}\,\Big|\int_C e^{bf(z)}F(z)\,dz\,\Big| \qquad (n \geq b) \\ &\leq K\,e^{n(M-\delta)} \end{aligned}$$

where K is independent of n. Thus

$$I_C(n)\,e^{-nM} = O(e^{-n\delta})$$

---

[*] *Annales de l'École Normale Supérieure* (1916) (3), Vol. 33, pp. 17-69.

[†] *Loc. cit.* [‡] See footnote § of p. 270. [§] See footnote ‡ of p. 270.

where $\delta > 0$.

Suppose now that there are two paths C_a and C_b starting at the same point and that a third path C_r exists which joins the end points of C_a and C_b and is specified with respect to the integral I in such a way that

$$\int_{C_b} e^{nf(z)} F(z)\, dz = \left(\int_{C_a} + \int_{C_r}\right) e^{nf(z)} F(z)\, dz;$$

then, if there exists a positive number δ such that on the path C_r

$$\Re f(z) \leq M - \delta$$

it follows that

$$e^{-nM} \int_{C_b} e^{nf(z)} F(z)\, dz - e^{-nM} \int_{C_a} e^{nf(z)} F(z)\, dz = O(e^{-n\delta})$$

it being assumed that the integral

$$\int e^{nf(z)} F(z)\, dz$$

taken along the paths C_a, C_b and C_r exists, at least for sufficiently large values of n.

Again, suppose that there are two paths C_a and C_b and that there exist paths C_r and C_s such that

$$\int_{C_b} e^{nf(z)} F(z)\, dz = \left(\int_{C_r} + \int_{C_a} + \int_{C_s}\right) e^{nf(z)} F(z)\, dz.$$

Then, if on C_r and C_s

$$\Re f(z) \leq M - \delta$$

where δ is some positive number, it follows that

$$e^{-nM} \int_{C_b} e^{nf(z)} F(z)\, dz - e^{-nM} \int_{C_a} e^{nf(z)} F(z)\, dz = O(e^{-n\delta}).$$

And here again it has been assumed that the integral along the various paths exists, at least for sufficiently large values of n.

1.2. It will be convenient to introduce the symbol $\sim_n$ to denote the relation between two functions $\phi(n)$ and $\psi(n)$, when there exists a positive number δ such that, for large values of n

$$\phi(n) - \psi(n) = O(e^{-n\delta}).$$

Suppose that $\psi(n)$ admits the asymptotic development *

$$\sum_{m=0}^{\infty} c_m n^{-(a+m)/p}$$

for large values of n, where p is some positive integer, then if

$$\phi(n) \sim_n \psi(n)$$

---

* A function $\chi(n)$ is said to admit the asymptotic development

$$a_0 + a_1 n^{-1/p} + a_2 n^{-2/p} + \cdots$$

for large values of n,
[which is also written $\chi(n) \backsim a_0 + a_1 n^{-1/p} + a_2 n^{-2/p} + \cdots$]

it follows that $\phi(n)$ admits the same asymptotic development.

Further, we shall say that certain paths are 'equipollent with respect to the integral I,' when the value of the integral I along any of the paths is the same.

1.3. The results obtained above may now be restated in the following way:

1. If C_a and C_b are two paths starting at the same point, and if there exists a third path C_r such that C_b and $C_a + C_r$ are equipollent for the integral

$$I = \int e^{nf(z)} F(z)\, dz,$$

then if on C_r

$$\Re f(z) \leq M - \delta$$

where δ is some positive number, it follows that

$$e^{-nM} \int_{C_b} e^{nf(z)} F(z)\, dz \sim_n e^{-nM} \int_{C_a} e^{nf(z)} F(z)\, dz.$$

2. If C_a and C_b are two paths and there exist two other paths C_r and C_s such that C_b and $C_r + C_a + C_s$ are equipollent for the integral

$$I = \int e^{nf(z)} F(z)\, dz,$$

then, if on C_r and C_s

$$\Re f(z) \leq M - \delta$$

where δ is some positive number, it follows that

$$e^{-nM} \int_{C_b} e^{nf(z)} F(z)\, dz \sim_n e^{-nM} \int_{C_a} e^{nf(z)} F(z)\, dz.$$

1.4. These considerations at once direct attention to the group of paths which have the characteristic that on them, the real part of $f(z)$ is less than some specific number M. We therefore proceed to investigate the regions of the z-plane in which

$$\Re f(z) \leq M - \delta < M$$

for various values of M.

---

if
$$\underset{n \to \infty}{lt} \left| n^{k/p} \{\chi(n) - [a_0 + a_1 n^{-1/p} + \cdots + a_{k-1} n^{-k/p}]\} \right| = 0$$
for all values of k. The expression is also used in an extended sense. A function $\chi(n)$ is said to admit the asymptotic development

$$\theta(n) + \Theta(n)\, \{a_0 + a_1 n^{-1/p} + \cdots\}$$

for large values of n, if the function

$$(\chi(n) - \theta(n))/\Theta(n)$$

admits the asymptotic development

$$a_0 + a_1 n^{-1/p} + \cdots$$

in the sense defined above. Cp. Barnes, 1906, *Philosophical Transactions of the Royal Society of London*, Series A, Vol. 206, pp. 249-297.

2. THE LIE OF THE FUNCTION $\mathfrak{R}f(z)$ ON THE COMPLEX PLANE.

2.1. We will write for convenience

$$R_f(z) = \mathfrak{R}f(z), \quad I_f(z) = \mathfrak{I}f(z)$$

and in general, when it is not necessary to make the reference to the function $f(z)$ explicit, these symbols may be replaced by the symbols $R(z)$, $J(z)$, simply. Now consider the surface consisting of the aggregate of points (x, y, R). By analogy with a relief map, in which at every point R represents the height of the land above sea-level, or some other standard of reference, we may call the lines on the surface on which R is constant the 'level' lines. On these lines J will be changing most rapidly and with the same analogy in view, these lines may be called the 'lines of greatest slope' or the 'lines of steepest descent or steepest ascent.' If at a point the partial differential coefficients of R and J with respect to x and y satisfy the equations

$$\partial R/\partial x = \partial J/\partial y, \quad \partial J/\partial x = -\partial R/\partial y$$

it follows that

$$\nabla^2 R = 0$$

and therefore R cannot at that point have a maximum or mimimum. At a point at which

$$d^s f/dz^s = 0 \quad (s = 1, 2, \cdots, p), \quad d^{p+1}f/dz^{p+1} \neq 0$$

there will be p lines of greatest slope and p level lines. We may say that a point z is a 'col' or 'pass' of order p, if at that point the first p differential coefficients of $f(z)$ with respect to z vanish and the $(p+1)th$ differential coefficient does not vanish.[*]

2.2. We are primarily concerned with the question of paths in the z-plane on which $R(z)$ has a certain upper bound. We will say that a point lies at the level M or on the M-level, if

$$R(z) = M.$$

We will call the aggregate of points which constitute the levels inferior to the level M, the 'underland M,' and the aggregate of points which constitute the levels superior to the M level, the 'overland M.' Thus the underland M is the class of points such that

$$R(z) = M - \delta$$

where δ is any positive number. Similarly the overland M is the class of points such that

$$R(z) = M + \delta$$

---

[*] The language of § 2.1 is taken from the papers cited on p. 4.

where δ is any positive number. Evidently with these definitions, every point in the z-plane belongs to the underland, overland or level associated with each number M. We will assume that the function $R(z)$ is a continuous function of x and y; then it follows that any path joining two points, one of which lies in the underland M and the other in the overland M, will traverse the level M at least once. The region which forms an underland or overland is plainly an open region. The limit of any sequence of points in an overland or underland either lies in the overland or underland in question, or lies on the level associated with the region. In fact, the region consisting of an underland M together with the level M is a closed region. An overland and its associated level, in the same way, form a closed region. With the assumption introduced above as to the continuity of $R(z)$, an underland evidently has one of the two distinguishing marks of a continuum; for if a point z belongs to the underland M, points sufficiently near z will also belong to the underland M. The same is true of an overland. It remains to be discussed whether these regions possess the second characteristic of a continuum, namely, that between any two points belonging to one such region, there is a path lying wholly in the region in question.

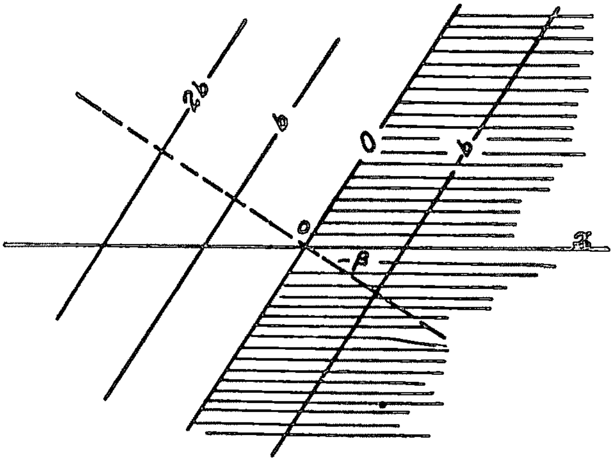


FIG. 2.31.

$f(z) = -B - Bz = -bre^{i\beta+i\theta}$.

o-underland shaded.

2.3. We proceed to discuss the lie of the function $R_f(z)$ in various particular cases.

Let

(2.31) $$f(z) = -Bz = -bre^{i(\beta+\theta)}.$$

The level line through the origin is $\sin(\beta+\theta)=1$. This line divides the plane in the underland $M=0$ and the overland $M=0$. The underlands and overlands $M=b$, $M=2b$, $\cdots$, $M=-b$, $M=-2b$, $\cdots$, abut along the levels $M=b$, $M=2b$, $\cdots$, $M=b$, $M=-2b$, $\cdots$. Any two points belonging to the underland $M=0$, may be joined by a path lying wholly in this region, and therefore, in this case, the underland $M=0$ is a continuum. Similarly every other underland and every overland is a continuum.

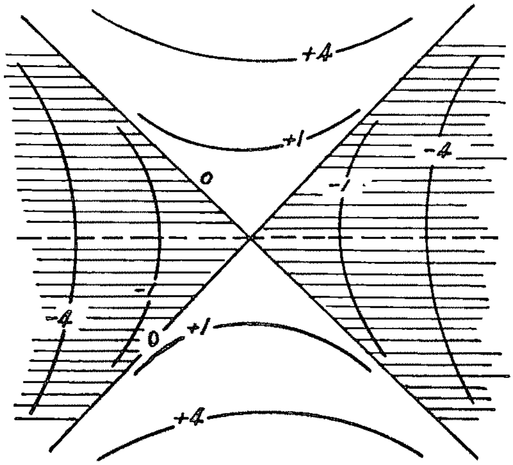


FIG. 2.32.

$f(z)=-z^2$.

o-underland shaded.

Let

(2.32) $$f(z)=-z^2.$$

In this case there is a col of order two at the origin. Further, since

$$R=-x^2+y^2;\ J=-2xy,$$

the levels are hyperbolae with the lines

$$x^2-y^2=0$$

as asymptotes, and the lines of greatest slope are hyperbolae with the lines

$$xy=0$$

as asymptotes. The underland $M=0$ is shaded in the diagram. Evidently, it is not the case that between any two points in this underland, a path can be drawn lying entirely in the underland. To connect the two points $(a, 0)$, $(-a, 0)$ which lie in this underland, a path must enter the overland $M = 0$, or at least traverse the col at the origin. On the other hand, any two points which lie in one of the shaded quadrants can be connected by a path lying wholly in this quadrant. Each underland and each overland in fact forms two continua, in this particular case.

The distinction between an underland which forms one continuum and an underland which consists of more than one continuum is clearly essential for the purpose in hand. Thus take for example the function defined by the integral

$$\int e^{-nz^2} z^2 dz$$

taken along any path running from $(-a, 0)$ to $(a, 0)$. Both points $(a, 0)$, $(-a, 0)$ lie in the underland $M = 0$. But there exists no path between them, on which

$$R(-z^2) = y^2 - x^2 \geq -\delta$$

where δ is some positive number, and therefore it does not follow that

$$\int_{-a}^{+a} e^{-nz^2} z^2 dz \sim_n 0$$

If, however, we consider instead the integral taken along any path running from $(a, 0)$ to $(2a, 0)$, the situation is changed, for we may now select the straight line path from $(a, 0)$ to $(2a, 0)$, on the whole of which

$$R(-z^2) = y^2 - x^2 \leq -a^2$$

and we consequently deduce the result

$$\int_{a}^{2a} e^{-nz^2} z^2 dz \sim_n 0.$$

2.4. In order to draw attention to this important distinction between an underland which is such that any two points belonging to it may be joined by a path lying wholly in this underland, and an underland which is such that in the case of some pairs of points belonging to the underland there is no path joining them and lying wholly in the underland, we introduce the notion of a valley. We call a class of points an 'M-valley,' if the unattained upper bound of $R(z)$ for points in the class is M and if the points form a continuum. It is occasionally convenient to refer to an M-valley as a valley whose index is M. In the same way, we might call a class of points an 'M-mountain continuum' if the unattained lower bound of $R(z)$ for the points in the class is M, and if the points form a continuum.

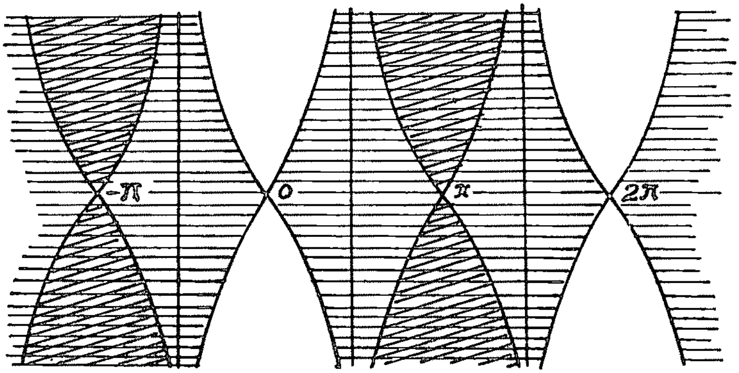


FIG. 2.41.
$(+1)$-underland
(-1)-underland
$f(z) = \cos z$.

We may take as another illustration of the definitions the case when

$$f(z) = \cos z. \tag{2.41}$$

There are cols of order two at the points $(n\pi, 0)$. Those at the points $(2r\pi, 0)$ lie at the level $+1$, those at the points $([2r+1]\pi, 0)$, at the level -1. The underland $+1$, which is shaded in the figure, of course, contains the underland -1, so that the lower cols together with part of the overland associated with the level -1 on which they lie is contained in the underland $+1$. The underland $+1$ consists of an enumerable set of valleys. The question then arises as to the most convenient way of specifying any one valley of the enumerable set which in this case make up an underland. Of the valleys whose index is $+1$, there are two and only two which have the origin on their boundaries; there are, again, two and only two of these valleys which have the point $(2r\pi, 0)$ on their boundaries. It is evidently a simple matter to specify one or other of the two valleys which have the point $(2r\pi, 0)$ on their boundaries, by citing the directions of the lines of steepest descent through this point, which lie in the valleys. Indeed, we may conveniently call the two valleys which abut at the origin the $(\theta = 0)$ valley and the $(\theta = \pi)$ valley. The two valleys with index -1 which abut at the point $(\pi, 0)$ in the same way may be called the $(\theta = \pi/2)$ valley and the $(\theta = -\pi/2)$ valley.

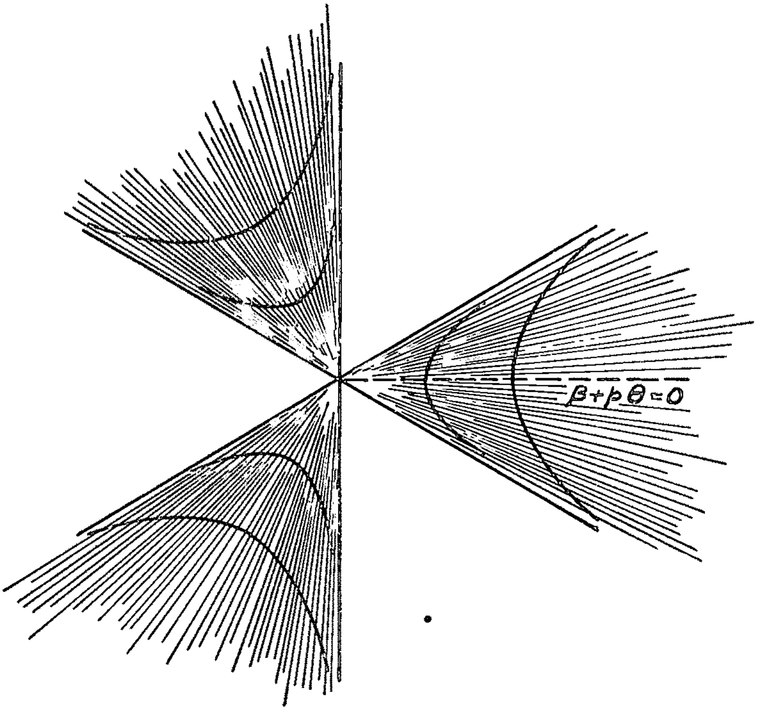


FIG. 2.42.

$f(z) = -Bz^p = -br^p e^{i\beta + ip\theta}$ $(p = 3)$.

o-underland shaded.

Thus to take another example, if

$$f(z) = -Bz^p = -br^p e^{i(\beta + p\theta)} \tag{2.42}$$

the level lines through the origin have the equation

$$\cos(\beta + p\theta) = 0$$

and the lines of greatest slope, the equation

$$\sin(\beta + p\theta) = 0.$$

The underland $M = 0$ falls into p valleys and the overland $M = 0$ into p mountain continua which all abut at the origin which is a col of order p. It will be convenient to specify any one of these valleys whose index is zero, by citing the direction of the line of steepest descent through the origin, which lies in the valley in question. In the diagram (which illustrates the case when $p = 3$), the valley $(\theta = -\beta/p)$ and the valley $[\theta = (\pm 2\pi - \beta)/p]$ are shaded. The remaining valleys are evidently the valleys

$$\theta = (4\pi - \beta)/p,\ (6\pi - \beta)/p,\ \cdots,\ [(2p-1)\pi - \beta]/p.$$

2.5. Suppose now that in the neighborhood of the point z_s, $f(z) - f(z_s)$ admits the convergent expansion

$$\begin{aligned} & -B(z - z_s)^p + \sum_{m=1}^{\infty} B_m (z - z_s)^{m+p} \\ = & -{}_s f_1(z) \qquad + \sum_{m=1}^{\infty} B_m (z - z_s)^{m+p} \\ = & -b r e^{i\beta + ip\theta} \qquad + \sum_{m=1}^{\infty} b_m\, r^{m+p}\, e^{i\beta_n + ip\theta} \end{aligned}$$

where p is a positive integer and ${}_s b \neq 0$, and suppose further that

$$p\alpha_t + \beta = 2t\pi \qquad (t = 0, 1, \cdots, p-1).$$

Then, evidently at the point z_s, the function $f(z)$ has a col of order p. The lines through the point z_s whose equations are

$$\cos(\beta + p_s\theta) = 0, \quad \sin(\beta + p\theta) = 0$$

now represent the tangents to the level lines through the col and the tangents to the lines of greatest slope through the col, and in general not these lines themselves. Consider, however, the arrangement of points into valleys by the function

$${}_s f_1(z) = -B(z - z_s)^p.$$

The lines whose equation are

$$\cos(\beta + p\theta) = 1$$

which may also be written in the form

$$\theta = \alpha_t \qquad (t = 0, 1, \cdots, p-1)$$

are the lines of steepest descent themselves for the function ${}_s f_1(z)$ through the col at the point z_s. Now the underland $M=0$ for the function ${}_sf_1(z)$ is defined by the relation

$$\Re\, {}_sf_1(z) \leqq -\delta < 0$$

and the underland $M=0$ for the function $f(z)-f(z_s)$ by the relation

$$\Re[f(z)-f(z_s)] \leqq -\delta < 0\,.$$

But

$$f(z)-f(z_s) = {}_sf_1(z) + \epsilon(z-z_s)^p$$

where ϵ can be made as small as we like, by taking $|z-z_s|$ sufficiently small. Thus if a positive number δ exists, such that

$$\Re[f(z)-f(z_s)] \leqq -\delta < 0$$

it follows that a positive number δ' also exists, such that

$$\Re\, {}_sf_1(z) \leqq -\delta' < 0.$$

In fact the points which form the underland $M=0$ for the function ${}_sf_1(z)$ and are sufficiently near z_s also form the underland $M=0$ for the function $f(z)-f(z_s)$. And further, the points sufficiently near the point z_s which form one valley for the one function also form one valley for the other function. Thus points sufficiently near the point z_s are segregated into valleys by the function ${}_sf_1(z)$ in the same way as they would be by the function $f(z)-f(z_s)$.[*]

2.6. We shall therefore adopt the following procedure in specifying valleys associated with the function $f(z)$. We shall say that the points (z_t) form a valley 'associated with the function $f(z)$ at the point z_s,' if the points form a valley and if there are points of this class in the neighbourhood of the point z_s. Further, we shall call a valley, 'the α_t-valley associated with the function $f(z)$ at the point z_s,' if it is the valley associated with the function $f(z)$ which contains as its line of greatest slope the line whose tangent at the point z_s is given by

$$z-z_s = re^{i\alpha_t}.$$

Thus if in the neighbourhood of the point z_s the function $f(z)-f(z_s)$ admits the convergent expansion

---

[*] It is plain that the fact that valleys are open regions, whatever the function with which they are associated, is the crux of this argument and that nothing of the kind is true with respect to the level lines through the points z_s, which are associated with the function ${}_sf_1(z)$ and the function $f(z)-f(z_s)$ respectively.

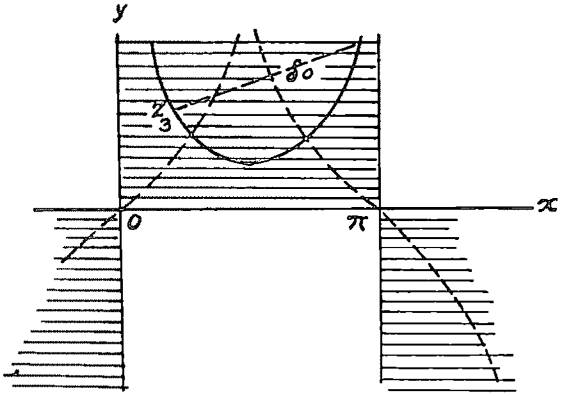
$$-B(z-z_s)^p + \text{higher powers,}$$
$$= -be^{i\beta}(z-z_s)^p + \text{higher powers,}$$

there are evidently p valleys which abut at the point z_s, and with these definitions they will be called the valleys

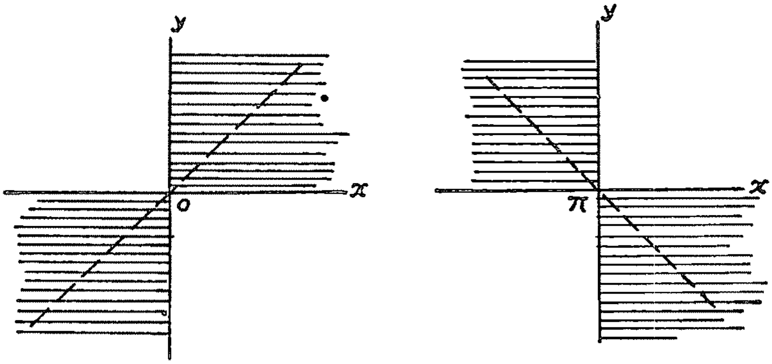
$$\theta = \alpha_t \qquad (t = 0, 1, \cdots, p-1).$$

where

$$\beta + p\alpha_t = 2t\pi.$$



$f(z) - f(0) = -i(\cos z - 1)$: o-underland shaded.



$_0f_1(z) = iz^2/2!$.
o-underland shaded.

$_\pi f_1(z) = -(i/2!)(z-\pi)^2$.
o-underland shaded.

FIG. 2.61.

In particular, we may remark that associated with each valley of the function at the point z_s, there exists a positive number δ (which may be infinite) which depends on $f(z)$ and on z_s, and is such that the path

$$z - z_s = re^{i\alpha_t} \qquad 0 < r \leq \delta$$

consisting of a segment of the tangent to the line of steepest descent in the α_t-valley, lies wholly in this α_t-valley associated with the function $f(z)$ at the point z_s.

Consider the case of the function

$$f(z) = -i \cos z. \tag{2.61}$$

In the neighborhood of the origin, and indeed for all finite values of z,

$$f(z) = -i(1 - z^2/2! + \cdots).$$

Now compare the valleys associated with the function $f(z)$ at the origin and those associated with the function

$${}_0f_1(z) = iz^2/2! = r^2/2 \ \ (\cos(\pi/2 + 2\theta) + i \sin(\pi/2 + 2\theta)).$$

Evidently the lines

$$z = re^{i\pi/4}, \quad z = re^{-3i\pi/4}$$

which are the lines of steepest descent for the function ${}_0f_1(z)$ at the origin lie within the valleys $(\theta = \pi/4)$ and $(\theta = -3\pi/4)$ associated with the function $f(z)$ at the origin so long as

$$r \leq \delta < \pi/2^{1/2}.$$

Again, the lines

$$z - \pi = re^{-i\pi/4}, \quad z - \pi = re^{3i\pi/4}$$

which are the lines of steepest descent for the function

$${}_\pi f_1(z) = -(i/2)(z - \pi)^2$$

at the point $(\pi, 0)$ lie in the valleys $(\theta = -\pi/4)$ and $(\theta = 3\pi/4)$ associated with the function $f(z)$ at the point $(\pi, 0)$, so long as

$$r \leq \delta < \pi/2^{1/2}.$$

All these considerations plainly apply also to the case of the valley associated with a point z_s, in the neighbourhood of which $f(z) - f(z_s)$ admits the convergent expansion

$$-B(z - z_s) + \sum_{m=1}^{\infty} B(z - z_s)^{m+1}.$$

Thus in the particular case under consideration, if

$$R(z_s) = M$$

$$\begin{aligned} f(z) - f(z) &= -i(\cos z - \cos z_s) \\ &= i \sin z_s (z - z_s) + \text{higher powers}, \end{aligned}$$

the tangent to the line of steepest descent at z_s, which is the normal to the level line at the point z_s, has equation

$$z - z_s = re^{ia}$$

where

$$ie^{ia} \sin z_s$$

is real and negative. The segment of this line given by

$$0 < r \geq \delta$$

will lie in the valley of $f(z)$ at this point, if $\delta < \delta_0$ where δ_0 is the length of the chord of the level line

$$R(z_s) = -M_s$$

through the point z_s.

NOTE.—The theory of underlands and valleys associated with a function of a complex variable may be developed step by step by the methods of Principia Mathematica. The primitive idea of the system is that of the class of points which form the M-level for the function $f(z)$. We will write

$$\text{lev}_f \text{‘} M$$

for the M-level so that the symbol ‘lev_f’ denotes the relation between a class of points (z_s) and an ordinal number M, when the class (z_s) constitutes the level M for the function $f(z)$. The various levels associated with the various ordinal numbers may be ordered by the relation ‘inferior to’ which may be denoted by the symbol I. We shall say that one level is inferior to another level, when the ordinal number associated with the one level is less than the ordinal number associated with the other. Thus [*]

$$I = \text{lev}_f ; <$$

To define the underland M we consider the levels inferior to the level M, and take the logical sum of all these classes. Thus the underland M, which may be written ‘under ‘M,’ is given by

$$\text{under} \text{‘} M = s \text{‘} \overrightarrow{I} \text{‘} \text{lev}_f \text{‘} M.$$

Now consider k the class of levels inferior to the level M. Evidently

$$k = \overrightarrow{I} \text{‘} \text{lev} \text{‘} M.$$

Now k forms a segment of the relation I and the series of segments of the relation I, namely $s \text{‘} I$, where [†]

$$s \text{‘} I = I_{lc} \sqsubset D \text{‘} I_\epsilon$$

is, plainly, ordinally similar to the series of underlands. The theory of the series of segments associated with various kinds of serial relations has been extensively discussed by Whitehead and Russell in Principia Mathematica,[†] 210-217. It is easily deducible from the theorems already known about series of segments that if a relation P is

---

[*] *Principia Mathematica*, 150.

[†] *Loc. cit.*, 170, 212.

Dedekindian, the relation P and the relation s‘P are ordinally similar. If therefore, we assume that the levels form a Dedekindian series, it at once follows that the underlands form an ordinally similar series. This fact is of importance in the question of the boundary of an underland. For, take the class of levels inferior to the M-level. Together with the M-level, the class forms a class which is closed * with respect to the relation I. We may then at once deduce that the underland M, which is an open region, forms with the level M a closed region. Other properties of underlands may be deduced from the definitions given above. In particular, properties of the various valleys into which underlands may be resolved may be deduced from assumptions as to the various simple curves into which levels may be resolved.

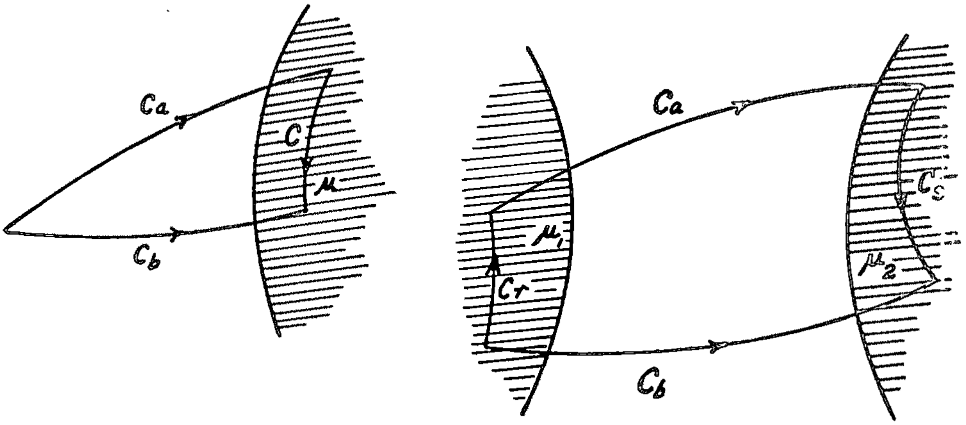


FIG. 3.1.

3. THE EXISTENCE THEOREMS.

3.1. The results of § 1.3 may now be stated in the following way:

1. If C_a and C_b are two paths starting at the same point and ending in m, an M-valley associated with the function $f(z)$,

$$e^{-nM}\int_{C_b} e^{nf(z)}F(z)\,dz \sim_n e^{-nM}\int_{C_a} e^{nf(z)}F(z)\,dz.$$

provided that there exists a third path C which lies in this valley and is such that the paths C_b and $C_a + C$ are equipollent for the integral

$$I = \int e^{nf(z)}F(z)\,dz.$$

2. If C_a and C_b are two paths starting in m_1, an M-valley and ending in m_2 a second M-valley associated with the function $f(z)$,

---

* *Loc. cit.*, 216.

$$e^{-nM}\int_{C_b} e^{nf(z)}F(z)\,dz \sim_n e^{-nM}\int_{C_a} e^{nf(z)}F(z)\,dz$$

provided that there exist paths C_r and C_s which lie in the first and second valleys respectively and are such that the paths C_b and $C_r + C_a + C_s$ are equipollent for the integral

$$I = \int e^{nf(z)}\,F(z)\,dz.$$

These theorems are existence theorems in the sense that they assert of a certain class of paths associated with certain M-valley continua that whichever paths of the class C_a and C_b may be

$$e^{-nM}\int_{C_b} e^{nf(z)}F(z)\,dz \sim_n e^{-nM}\int_{C_a} e^{nf(z)}F(z)\,dz$$

just as the first existence theorem of complex integration, Cauchy's theorem, asserts of a certain class of paths, that whichever paths of the class C_a, C_b may be

$$\int_{C_b} F(z)\,dz = \int_{C_a} F(z)\,dz.$$

On the one hand, Cauchy's theorem is restricted to classes of paths which start at one point z_1 and end at another point z_2, while the present theorem deals with classes of paths which end in an M-valley and start either at the one and the same point or in another M-valley. On the other hand the present theorem is restricted to asserting of any two paths C_a and C_b belonging to certain groups defined with reference to M-valleys, that

$$e^{-nM}\int_{C_b} e^{nf(z)}F(z)\,dz \sim_n e^{-nM}\int_{C_a} e^{nf(z)}F(z)\,dz$$

whereas Cauchy's theorem asserts of any two paths C_a and C_b belonging to certain groups that

$$\int_{C_b} F(z)\,dz = \int_{C_a} F(z)\,dz.$$

3.2. Many particular cases of these existence theorems occur in the theory of asymptotic expansions. To take the simplest case of all, consider the integral

$$I(n) = \int e^{-Nz}\,z^{k-1}\,dz = \int e^{-nze^{i\nu}}\,z^{k-1}\,dz$$

taken along various paths in the z-plane. Let

$$\Re\,(N) > 0, \quad k > 0.$$

If C be any path starting at the origin and ending anywhere in the C-valley associated with the function

$$f(z) = -ze^{i\nu} \tag{3.21}$$

(the half plane shaded in the figure)

$$\int e^{-nze^{i\nu}} z^{k-1}\,dz \sim_n \int e^{-nze^{i\nu}} z^{k-1}\,dz = [\Gamma(k)/n^k]e^{-i\nu k}.$$

This fact is evidently the reason for the importance of the Gamma function in the theory of asymptotic expansions. Whatever the path C may be, it is in general a simple matter to relate the function

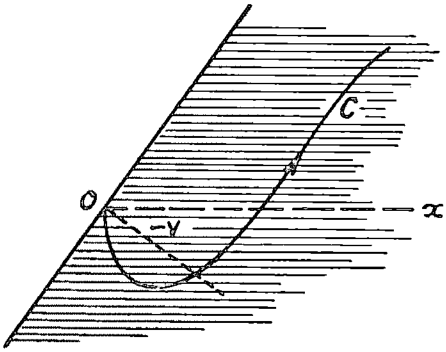


FIG. 3.21.
$f(z) = -ze^{i\nu}$.

$$I_C(n) = \int_C e^{-nze^{i\nu}} F(z)\,dz$$

to a set of integrals which may be expressed in terms of Gamma functions.*

3.3. The proviso in the existence theorems as to the equipollence of certain paths for the integral I raises an interesting point. For suppose

$$I = \int [e^{-nz}/(z-a)]\,dz, \tag{3.31}$$

and consider the integral taken along the set of paths which start at the origin and end at points in the half plane

$$x > 0$$

---

* Cp. Barnes, *loc. cit.*, p. 254. Watson, *loc. cit.*, *Proceedings of the London Mathematical Society*, p. 133.

which is the underland 0 for the function

$$f(z) = -z$$

and indeed the one and only 0-valley for this function.

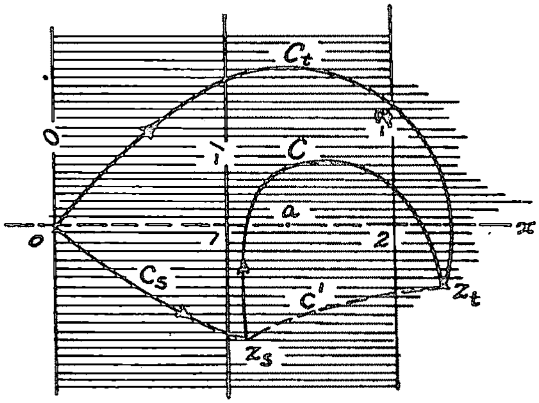


FIG. 3.31.

$f(z) = -z$.

Two paths C_s and C_t ending at the points z_s and z_t are shewn in the diagram. In spite of the existence of the pole of the function

$$F(z) = 1/(z-a)$$

at the point $(a, 0)$, there exists a path C from z_s to z_t which is such that the

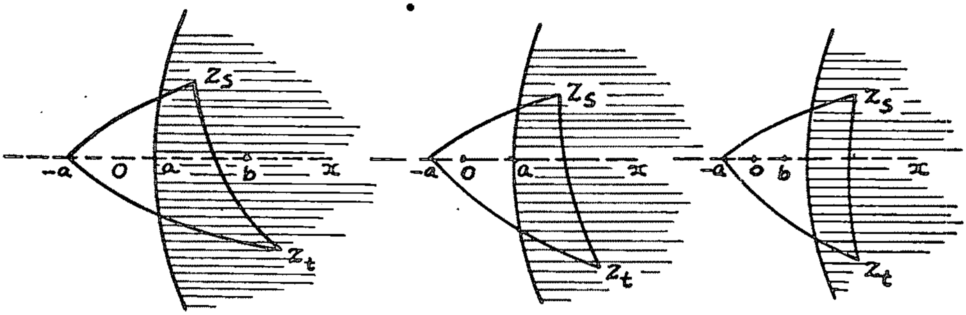


FIG. 3.32. $f(z) = -z^2$.

$b > a$ $b = a$ $b < a$

paths C_t and $C_s + C$ are equipollent for the integral I. Therefore evidently

$$\int_{C_s} [e^{-nz}/(z-a)]\, dz \sim_n \int_{C_t} [e^{-nz}/(z-a)]\, dz.$$

Now the residue of the integrand at the pole $z = a$, is e^{-na}, so that if a path C' had been selected, which with the paths C_s and C_t enclosed this point, we should have had the result

$$\int_{C_s} [e^{-nz}/(z-a)]\, dz - \int_{C_t} [e^{-nz}/(z-a)]\, dz = 2\pi i\, e^{-na}$$

so that as before

$$\int_{C_s} [e^{-nz}/(z-a)]\, dz \sim_n \int_{C_t} [e^{-nz}/(z-a)]\, dz.$$

On the other hand consider the integral

$$\int [e^{-nz^2}/(z-b)]\, dz \tag{3.32}$$

along a set of paths starting at the point $(a, 0)$ and ending an M-valley where

$$M = \Re(-z^2)_{z=-a} = -a^2.$$

We will select as the M-valley to be considered the continuum which contains the portion of the real axis on which x is positive and sufficiently large. The boundary of this valley is evidently one of the branches of the hyperbola

$$x^2 - y^2 = a^2.$$

Evidently, if $b = b_1 > a$, the pole of the integrand lies in the M-valley in question. We therefore have as in the previous case

$$e^{-nM} \int_{C_s} [e^{-nz^2}/(z-b_1)]\, dz \sim_n e^{-nM} \int_{C_t} [e^{-nz^2}/(z-b_1)]\, dz$$

whichever paths of the set of paths starting at $(a, 0)$ and ending in the M-valley, C_s and C_t may be. If however $b = b_2 \leqq a$, this result does not persist, and in particular we do not have the result that

$$\int_{C_s} [e^{-nz^2}/(z-b_2)]\, dz + \int_{C} [e^{-nz^2}/(z-b_2)]\, dz = \int_{C_t} [e^{-nz^2}/(z-b_2)]\, dz$$

in the case of the paths shewn in the diagram. This conclusion would carry the consequence that

$$e^{na^2} \int_{C_s} [e^{-nz^2}/(z-b_2)]\, dz - e^{na^2} \int_{C_t} [e^{-nz^2}/(z-b_2)]\, dz = O(e^{-n\delta})$$

where δ is some positive number, whereas in fact

$$\left(\int_{C_s} + \int_{C}\right) [e^{-n(z^2-a^2)}/(z-b_2)]\, dz - \int_{C_t} [e^{-n(z^2-a^2)}/(z-b_2)]\, dz$$
$$= 2\pi i\, e^{n(a^2-b_2^2)}.$$

and

$$\int [e^{-n(z^2-a^2)}/(z-b_2)]\, dz = O(e^{-n\delta}) \qquad (\delta > 0)$$

so that

$$\left(\int_{C_s} - \int_{C_t}\right) [e^{-n(z^2-a^2)}/(z-b_2)]\, dz \sim_n 2\pi i\, e^{n(a^2-b_2^2)}.$$

4. THE STANDARD PATHS ASSOCIATED WITH THE FUNCTION $f(z)$.

4.1. In the theory of a function defined by the integral

$$I = \int F(z)\, dz$$

taken along a path C, the first problem is the specification of a class of paths which is such that the integral along all these paths is the same. Cauchy's theorem is concerned with this problem. The next problem is the exaluation of the function

$$I_C = \int_C F(z)\, dz$$

for as wide a class of paths as possible. Suppose that C_s is the path consisting of the circular arc

$$z - z_s = \rho e^{i\theta} \qquad -\pi \leqq \theta \leqq \pi$$

and evaluate first

$$\int_{C_s} (z-z_s)^{A-1}\, dz$$

where the integrand is the one-valued function

$$(z-z_s)^{A-1} = e^{(A-1)\log(z-z_s)}$$

and $\log(z-z_s)$ is purely real on the line on which $(z-z_s)$ is real and positive. We easily obtain the result that

$$\int_{C_s} (z-z_s)^{A-1}\, dz = 2i\rho^A \sin \pi A/A.$$

If now we consider the integral

$$I = \int F(z)\, dz$$

along the same path C_s, when, in the neighbourhood of the point z_s, $F(z)$ admits the convergent development

$$\sum_{m=0}^{\infty} A_m (z-z_s)^{A+m-1},$$

for sufficiently small values of ρ we may integrate the series for $F(z)$ term-by-term and we obtain the result

$$I_{C_s} = 2i\{(\sin \pi A/A)\rho^A + [\sin \pi(A+1)/(A+1)]\rho^{A+1} + \cdots\}$$

the series on the right being evidently convergent. This result enables us to evaluate a function defined by the integral I taken along any path C which is equipollent for the integral I to a standard path consisting of the circular arc

$$z - z_s = \rho e^{i\theta} \qquad -\pi \leq \theta \leq \pi$$

provided that in the neighbourhood of the point z_s, $F(z)$ admits the convergent expansion

$$\sum_{m=0}^{\infty} A_m(z - z_s)^{A+m-1}$$

and that ρ is sufficiently small. If $A = -p$, where p is a positive integer or zero,

$$I_{C_s} = 2iA_p \lim_{A+p\to 0} [\sin \pi(A+p)/(A+p)]\rho^{A+p} = 2\pi i A_p$$

and we obtain the theorem of residues, which is indeed a particular case of integration along the standard path C_s.

4.2. The procedure in the theory of the asymptotic evaluation of a function defined by an integral taken along a certain path is entirely similar. We take two types of standard paths.

Suppose z_s is a point in the neighbourhood of which $f(z) - f(z_s)$ admits the convergent expansion

$$-B(z - z_s)^p + \sum_{m=0}^{\infty} B_m(z - z_s)^{p+m}$$

where p is a positive integer. Let $\theta = \alpha$ be one of the p valleys associated with the function $f(z)$ at the point, and consider the straight line path

$$z - z_s = re^{i\alpha} \qquad 0 \leq r \leq \delta$$

We will choose δ sufficiently small to ensure that this path shall end at a point in this α-valley. It will be convenient to denote this path by the symbol

$$z_s(\alpha)_\delta.$$

If now C is a path which starts at the point z_s and ends in this α-valley which is a valley associated with the function at the point z_s, then it follows, in general, that*

$$\int_C e^{nf(z)-nf(z_s)} F(z)\, dz \sim_n \int_{z_s(\alpha)\delta} e^{nf(z)-nf(z_s)} F(z)\, dz.$$

We will call a path which starts at a point z_s and ends in a valley associated with $f(z)$ *at that point,* 'a path of the first type'; in particular, we will call the path $z_s(\alpha)_\delta$ which is itself a path of the first type, 'a standard path of the first type.' Then evidently, the asymptotic evaluation of the integral

$$\int e^{nf(z)-nf(z_s)} F(z)\, dz$$

along any path of the first type starting at z_s can in general be derived from the asymptotic evaluation of this integral along the appropriate standard path of the first type.

4.3. The second existence theorem suggests a second type of standard path. Suppose that z_s is a point in the neighbourhood of which the function $f(z)$ admits the convergent expansion

$$-B(z-z_s)^p + \sum_{m=1}^{\infty} B_m (z-z_s)^{p+m}$$

where p is a positive integer greater than unity. Then there will be at least two valleys associated with the function $f(z)$ which abut at this point. Of these valleys, let

$$\theta = \alpha_1, \qquad \theta = \alpha_2$$

be two. Consider now the path consisting of the circular arc

$$z - z_s = \delta e^{i\theta} \qquad\qquad \alpha_1 \leq \theta \leq \alpha_2$$

We will choose δ sufficiently small to ensure that this path shall begin in the α_1-valley associated with the function $f(z)$ at the point z_s and shall end in the α_2-valley associated with the function $f(z)$ at the same point. It will be convenient to denote this path by the symbol

$$z_s(\alpha_1, \alpha_2)\delta$$

---

* It is evidently necessary to assume that there exists a path C_α which lies in this valley $\theta = \alpha$ and is such that the paths $C + C_\alpha$ and $z_0(\alpha)_\delta$ are equipollent for the integral in question. Cp. § 3.1.

If now C is a path starting in this α_1-valley and ending in this α_2-valley (both the valleys being associated with the function $f(z)$ at one point), it follows, in general,[*] that

$$\int_C e^{nf(z)-nf(z_s)}\,F(z)\,dz \sim_n \int_{z_s(\alpha_1,\alpha_2)\delta} e^{\,f(z)-nf(z_s)}\,F(z)\,dz.$$

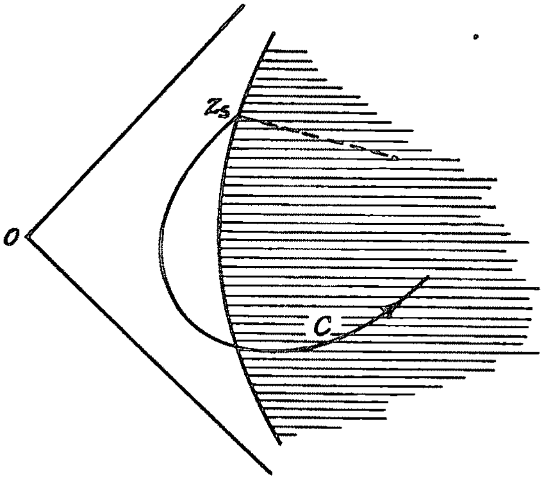


FIG. 4.41.
$f(z) = -z^2$.

We will call a path which starts in one valley associated with the function $f(z)$ at a point z_s and ends in a second valley associated with the function $f(z)$ at the same point, 'a path of the second type'; in particular, we will call the path $z_s(\alpha_1, \alpha_2)_\delta$ which is itself a path of the second type, 'a standard path of the second type.' Then evidently the asymptotic evaluation of the integral

$$\int e^{nf(z)-nf(z_s)}\;F(z)\,dz$$

along any path of the second type associated with the point z_s can, in general, be derived from the asymptotic evaluation of this integral along the appropriate standard path of the second type.

4.4. The diagram shows the standard path of the first type associated with the function

$$f(z) = -z^2 \tag{4.41}$$

---

[*] It is evidently necessary to assume that there exist paths C_{α_1} and C_{α_2} which lie in the valleys α_1 and α_2 respectively, and which are such that the paths $C_{\alpha_1} + C + C_{\alpha_2}$ and $z_s(\alpha_1, \alpha_2)_\delta$ are equipollent for the integral in question. Cp. § 3.1.

at a point z_s. Evidently the asymptotic evaluation of the integral

$$\int e^{nz^2-nz_s^2}\, F(z)\, dz$$

along this standard path would yield the asymptotic evaluation of the integral along any path C whatever which starts at the point z_s and ends within the section of the plane bounded by the branch of the hyperbola which passes through the point z_s.

The diagram shows a standard path of the second type associated with the function

$$f(z) = i(z - \sin z) \tag{4.42}$$

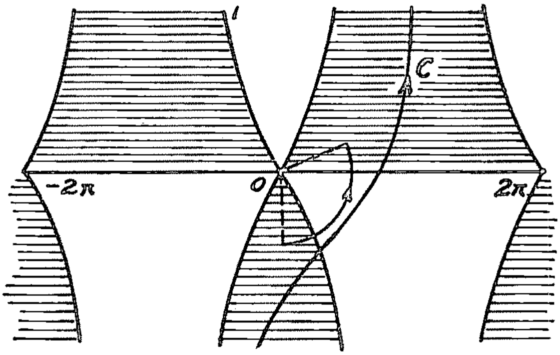


FIG. 4.42.
$f(z) = i(z - \sin z)$.
o-underland shaded.

at the origin. If C is any path running from $(a - i\infty)$ to $(b + i\infty)$ where

$$-\pi/2 < a < \pi/2, \qquad \pi/2 < b < 3\pi/2$$

evidently, in general,

$$\int_C e^{in(z-\sin z)}\, F(z)\, dz \sim_n \int_{O(-\pi/2,\, \pi/6)\delta} e^{in(z-\sin z)}\, F(z)\, dz.$$

5. THE ASYMPTOTIC EVALUATION OF THE INTEGRAL ALONG THE STANDARD PATHS.

5.1. In evaluating the integral

$$\int e^{nf(z)-nf(z_s)}\, F(z)\, dz$$

along the standard paths of the first and second type, we may evidently strict ourselves to the case when the point z_s is at the origin without any ss in generality.

In the course of the discussion of the integral

$$I = \int e^{nf(z)} z^{A-1} \phi(z)\, dz$$

along a certain class of paths, Perron[*] has obtained the asymptotic value of the integral I taken along a straight line path $O(\alpha)_\delta$. If we assume that the real part of A is positive and that in the neighbourhood of the origin the functions $f(z)$ and $\phi(z)$ admit the convergent developments

$$f(z) - f(0) = -Bz^p + \sum_{m=1}^{\infty} B_m z^{p+m} = -br^p e^{i\beta + ip\theta} + \sum_{m=1}^{\infty} B_m z^{p+m}$$

$$\phi(z) = \sum_{m=0}^{\infty} A_m z^m$$

where p is a positive integer and

$$p\alpha_t + \beta = 2t\pi$$

it follows, from Perron's analysis, that for sufficiently small values of δ,

$$\int_{O(\alpha_t)\delta} e^{nf(z)-nf(0)} z^{A-1} \phi(z)\, dz \sim 1/p \sum_{k=0}^{\infty} P_k \Gamma(A+k/p)\, e^{i\alpha_t(A+k)} (n^{1/p}\, b)^{-(A+k)}$$

where P_k is the coefficient of z^k in the expansion in powers of z of the function

$$\sum_{m=0}^{\infty} A_m z^m \times [1 - (B_1 z + B_2 z^2 \cdots)/B_0]^{-(A+k)/p}.\dagger$$

5.2. This result may be extended. Suppose that in the neighbourhood of the origin, $\phi(z)$ admits the *asymptotic* development

$$A_0 + A_1 z + A_2 z^2 + \cdots.$$

for values of arg z between $\alpha_t \pm \gamma$ where $\gamma \geqq \gamma_0 > 0$.

Let

$$\phi_k(z) = \sum_{m=0}^{k-1} A_m z^m.$$

Then the condition satisfied by $\phi(z)$ in this case implies that

$$|\, \phi(z) - \phi_k(z) \,| \leqq K \,|\, z \,|^k$$

---

[*] *Loc. cit.*

[†] A theorem which is substantially the particular case of this theorem which results when

$$f(z) = -Bz^p$$

was proved independently by Watson, 1916, *Proceedings of the London Mathematical Society*, Series 2, 17, p. 133. See also *Bessel Functions* by the same author, § 8.3.

for small values of z. Thus, if δ be chosen sufficiently small,

$$\left| \int_{O(a_t)\delta} e^{nf(z)-nf(0)A-1} \left(\phi(z) - \phi_k(z)\right) dz \right| \leq K \left| \int_{O(a_t)\delta} e^{nf(z)-nf(0)} z^{A+k-1}\, dz \right|$$

and by the foregoing result

$$\int_{O(a_t)\delta} e^{nf(z)-nf(0)}\, z^{A+k-1}\, dz = 1/p\ \Gamma\left(\frac{A+k}{p}\right) \frac{e^{ia_t(A+k)}}{(n^{1/p}b)^{A+k}} \left(1 + O(1/n)^{1/p}\right)$$

Thus

$$\int_{O(a_t)\delta} e^{nf(z)-nf(0)}\, z^{A-1}\, \phi(z)\, dz = \int_{O(a_t)\delta} e^{nf(z)-nf(0)}\, z^{A-1}\, \phi_k(z) + O(1/n)^{[\Re(A)+k]/p}.$$

But consider now the integral

$$I_k = \int_{O(a_t)\delta} e^{nf(z)-nf(0)A-1}\ \phi_k(z)\, dz.$$

The function $\phi_k(z)$ in the integrand is merely a polynomial in z of order $k-1$. The original theorem therefore again applies and we may at once assert that for sufficiently small values of δ,

$$I_k \sim 1/p \sum_{\nu=0}^{\infty} \Gamma(A + \nu/p)\ {}_kP_\nu\, e^{ia_t(A+\nu)}\ (n^{1/p}b)^{-(A+\nu)}$$

where ${}_kP_\nu$ is the coefficient of z^ν in the expansion in powers of z of the function

$$(A_0 + A_1 z + \cdots + A_{k-1}z^{k-1}) \times \left[1 - (B_1 z + B_2 z^2 + \cdots)/B\right]^{-(A+\nu)/p}.$$

With this definition of the ${}_kP_\nu$-coefficients, we have for sufficiently small values of δ the result that

$$\int_{O(a_t)\delta} e^{nf(z)-nf(z0)}\, z^{A-1}\, \phi(z) = 1/p \sum_{\nu=0}^{k-1} \Gamma\left(\frac{A+\nu}{p}\right) {}_kP_\nu\ \frac{e^{ia_t(A+\nu)}}{(n^{1/p}b)^{A+\nu}} + O(1/n)^{[\Re(A)+k]/p}.$$

Now when $\nu \leq k-1$ evidently

$${}_kP_\nu = P_\nu$$

where P_ν is the coefficient of z^ν in the expansion in powers of z of the function

$$\sum_{m=0}^{\infty} A_m z^m \times \left[1 - (B_1 z + B_2 z^2 + \cdots)/B\right]^{-(A+\nu)/p}.$$

We may therefore write the result in the more convenient form: if δ be sufficiently small

$$\int_{O(a_t)\delta} e^{nf(z)-nf(0)}\ z^{A-1}\, \phi(z)\, dz \sim 1/p \sum_{k=0}^{\infty} \Gamma\left(\frac{A+k}{p}\right) P_k\ \frac{e^{ia_t(A+k)}}{(n^{1/p}b)^{A+k}}.$$

of §5.1 is therefore true even if $\phi(z)$ satisfies the milder condition of having an asymptotic development in the neighbourhood of the origin which is valid at least for a small sector enclosing the straight line path of integration.

5.3. Perron also obtains the asymptotic value of the integral

$$\int e^{nf(z)-nf(0)} z^{A-1} \phi(z)\, dz.$$

along the standard path of the second type $O(\alpha_1, \alpha_2)\delta$. If[*] we assume that in the neighbourhood of the origin, the functions $f(z)$ and $\phi(z)$ admit the same convergent developments as before, it follows that, if δ be chosen sufficiently small

$$\int_{O(\alpha_1,\alpha_2)\delta} e^{nf(z)-nf(0)} z^{A-1} \phi(z)\, dz \sim 1/p \sum_{k=0}^{\infty} \Gamma\left(\frac{A+k}{p}\right) P_k \frac{e^{i\alpha_2(A+k)} - e^{i\alpha_1(A+k)}}{(n^{1/p}b)^{A+k}}$$

where P_k is the coefficient of z^k in the expansion in powers of z of the function

$$\sum_{m=0}^{\infty} A_m z^m \times [1-(B_1 z + B_2 z^2 + \cdots)/B]^{-(A+k)/p}.$$

5.4. It is evidently possible to extend this result as in the case of the previous result. If instead of having a convergent development in the neighbourhood of the origin, $\phi(z)$ admits the asymptotic development

$$A_0 + A_1 z + \cdots$$

for values of arg z between γ_1 and γ_2 where

$$\gamma_1 < \alpha_1 < \alpha_2 < \gamma_2,$$

similar arguments to those used above will enable us to prove that it is still the case that

$$\int_{O(\alpha_1,\alpha_2)\delta} e^{nf(z)-nf(0)} z^{A-1}\phi(z)\, dz \sim 1/p \sum_{k=0}^{\infty} \Gamma\left(\frac{A+k}{p}\right) P_k \frac{e^{i\alpha_2 i(A+k)} - e^{1\alpha_1(A+k)}}{(n^{1/p}b)^{A+k}}$$

---

[*] A theorem which is substantially this theorem in the particular case when

$$f(z) = -Bz$$

was proved by Barnes, *loc. cit.*, p. 254.

where P_k is the coefficient of z^k in the expansion in powers of z of the functi

$$\sum_{m=0}^{\infty} A_m z^m \times [1-(B_1 z + B_2 z^2 + \cdots)/B]^{-(A+k)/p}$$

provided only that δ be chosen sufficiently small.

5.5. The asymptotic evaluation of the integral

$$\int e^{nf(z)-nf(0)}\, F(z)\, dz$$

taken along standard paths of the first and second types associated with the origin has now been carried out for the case when $f(z)$ and $F(z)$ satisfy certain conditions. It is therefore possible to write down the asymptotic value of the integral

$$\int e^{nf(z)-nf(0)}\, F(z)\, dz$$

taken along any path of the first or second type associated with the origin, when $f(z)$ and $F(z)$ satisfy certain conditions in the neighbourhood of the origin and by a change of origin to write down the asymptotic value of the integral

$$\int e^{nf(z)-nf(z_s)}\, F(z)\, dz$$

taken along any path of the first or second types associated with any point z_s, given that the functions $f(z)$ and $F(z)$ satisfy corresponding conditions in the neighbourhood of the point z_s.[*]

---

[*] Certain writers (*loc. cit.*, p. 4) using the so-called method of steepest descents (Méthode du Col, Methode der Sattelpunkte), have discussed the integral

$$\int e^{nf(z)-nf(z_s)}\ F(z)\ dz$$

along a path C which passes through a point z_s at which $f'(z)$ vanishes and has the equation

$$I f(z) = I f(z_s)$$

when $f(z)$ and $F(z)$ are various special functions. (Cp. Watson, "Bessel Functions," § 8.3 for an account of the method.) This procedure of integrating along the line of steepest descent through a point at which two or more valleys abut, is, of course, suggested by the fundamental idea of the relation between the behaviour of the real part of $f(z)$ on a path and the asymptotic value for large values of n, of the integral

$$\int e^{nf(z)}\ F(z)\ dz$$

along that path. It is, however, more convenient to adopt as standard paths for the asymptotic evaluation of functions defined by the integral I along various pathe associated with a point z_s, not the line of steepest descent through z_s, but the small segments of the tangents to the lines of steepest descent through the point, and the circular arcs of small radius bounded by the tangents to lines of steepest descent lying in different valleys associated with the function $f(z)$ at the point z_s. For, we then require infor-

5.6. An interesting fact will be observed in connexion with the asymptotic evaluation of the integral

$$\chi(n) = \int_{O(a_1,a_2)\delta} e^{nf(z)-nf(0)}\,\phi(z)\,z^{A-1}\,dz.$$

It has been proved that, whether the real part of A is positive or not,

$$\chi(n) \sim 1/p \sum_{k=0}^{\infty} P_k\,\Gamma\left(\frac{A+k}{p}\right).\frac{e^{ia_2(A+k)} - e^{ia_1(A+k)}}{(n^{1/p}b)^{A+k}},$$

so that whether the real part of A is positive or not

$$\int_{O(a_1,a_2)\delta} e^{nf(z)}\,\phi(z)\,z^{A-1}\,dz = Ke^{nf(0)}\,n^{-A/p}\,\{1+O(1/n^{1/p})\}.$$

---

mation as to the behaviour of $f(z)$ and $F(z)$ in the neighbourhood of the point z_s, simply, instead of requiring information as to their behaviour all the way along the line of steepest descent in question. Further, there are cases (as given above) when the integral along the lines of steepest descent from one valley associated with the function at the point z_s into a second valley associated with the function as that point does not exist, owing to an infinity of $F(z)$ at the point z_s, when the integral along the standard path of the second type consisting of a circular arc about the point z_s, exists. The notion of integrating along lines of steepest descent is therefore not essential to the development of the idea of the relation between the behaviour of the real part of the function $f(z)$ on a path C and the asymptotic value for large values of n of the integral I along that path. No prominence is given, therefore, to the name 'method of steepest descents.'

We are more concerned, however, to emphasize the fact that nothing whatever depends upon the point z_s being a point at which $f'(z)$ vanishes. (This fact is implicit in the papers of Brillouin and Perron already cited.) The functions defined by the integral

$$\int e^{nf(z)-nf(z_s)}\,F(z)\,dz$$

along paths from z_s which end in a valley associated with the function $f(z)$ at the point z_s, admit the same asymptotic development for large values of n, whether z_s is a point at which $f'(z)$ vanishes or not. In the latter case there will be only one valley associated with the function $f(z)$ at the point z_s instead of several. We adopt as a standard path the small segment of the tangent to the line of steepest descent through the point z_s. Analytically this case presents no special features with respect to the asymptotic evaluation of the integral

$$\int e^{nf(z)-nf(z_s)}\,F(z)\,dz$$

along paths which start at z_s and end in the a-valley (the one and only valley) associated with the function $f(z)$ at the point z_s. We have in fact that

$$\int e^{nf(z)-nf(z_s)}\,(z-z_s)^{A-1}\phi(z)\,dz$$

along such a path is, in general, asymptotically equivalent to

$$\sum_{\nu=0}^{\infty} \Gamma\left(\frac{A+\nu}{p}\right) P_\nu\,\frac{e^{ia(A+\nu)}}{(n^{1/p}b)^{A+\nu}}$$

and $p=1$. With points at which two, three, . . . valleys abut, p has other special values. For this reason, no prominence is given to the names 'Méthode du col,' 'Methode der Sattelpunkte.'

where k is independent of n. Now if there is a path running from $\delta e^{i\alpha_1}$ to $\delta e^{i\alpha_2}$ on which the upper bound of the real part of $f(z)$ is M, it follows that

$$\int_{O(\alpha_1,\alpha_2)\delta} e^{nf(z)}\, \phi(z)\, z^{A-1}\, dz = O(e^{nM}).$$

When the real part of A is positive, we may choose as a path running from $\delta e^{i\alpha_1}$ to $\delta e^{i\alpha_2}$, the rays

$$z = re^{i\alpha_1}, \qquad z = re^{i\alpha_2} \qquad\qquad 0 \leq r \leq \delta$$

and on these rays the upper bound of the real part of $f(z)$ is evidently equal to the value of the real part of $f(z)$ at the origin, and we have the result

$$\int_{O(\alpha_1,\alpha_2)\delta} e^{nf(z)}\, \phi(z)\, z^{A-1}\, dz = O(e^{n\Re f(0)}),$$

a result which agrees with the statement

$$\int_{O(\alpha_1,\alpha_2)\delta} e^{nf(z)}\, \phi(z)\, z^{A-1}\, dz = Ke^{nf(0)}\, n^{-A/p} \{1 + O(1/n^{1/p})\}$$

in the case in question, namely when the real part of A is positive.

However, when the real part of A is no longer positive, the fact that the statements

$$\int_{O(\alpha_1,\alpha_2)\delta} e^{nf(z)}\, z^{A-1}\, \phi(z)\, dz = O(e^{n\Re f(0)}),$$

$$\int_{O(\alpha_1,\alpha_2)\delta} e^{nf(z)}\, z^{A-1}\, \phi(z)\, dz = Ke^{nf(0)}\, n^{-A/p} \{1 + O(1/n^{1/p})\}$$

are no longer in accord, is reflected in the lie of the function on the plane of z. For since the real part of A is no longer positive, any path from $\delta e^{i\alpha_1}$ to $\delta e^{i\alpha_2}$ must avoid the origin. But any path which starts in one valley associated with the function $f(z)$ at the origin and ends in another valley associated with the function $f(z)$ must either cross the origin at least once—which is impossible in this case though perfectly possible in the previous case when the real part of A was positive—or pass some distance over a mountain region associated with the function $f(z)$ at the origin. Thus on any path from $\delta e^{i\alpha_1}$ to $\delta e^{i\alpha_2}$ the upper bound of the real part of $f(z)$ is greater than the real part of $f(z)$ at the origin. However small η may be there exists a path from $\delta e^{i\alpha_1}$ to $\delta e^{i\alpha_2}$ on which the upper bound of the real part of $f(z)$ is equal to

$$\Re f(0) + \eta.$$

Thus we may assert that

$$\int_{\delta e^{i\alpha_1}}^{\delta e^{i\alpha_2}} e^{nf(z)}\, F(z)\, dz = O(e^{n[\Re f(0)+\eta]})$$

where η is any positive number however small. But we cannot assert the limit of this set of assertions, namely that

$$\int_{\delta e^{i\alpha_1}}^{\delta e^{i\alpha_2}} e^{nf(z)} F(z)\, dz = O(e^{n\Re f(0)}).$$

The order of magnitude of the integral is, in fact, given by the result

$$\int_{\delta e^{i\alpha_1}}^{\delta e^{i\alpha_2}} e^{nf(z)} F(z)\, dz = 1/p \ e^{nf(0)}\, \Gamma\left(\frac{A}{p}\right) \frac{e^{i\alpha_2 A} - e^{i\alpha_1 A}}{(n^{1/p} b)^A} \ [1 + O(1/n)^{1/p}]$$

a result which is in complete accord with the fact that whatever the value of η

$$\int_{\delta e^{i\alpha_1}}^{\delta e^{i\alpha_2}} e^{nf(z)} F(z)\, dz = O(e^{n(\Re f(0)+\eta)})$$

and that it is not the case that

$$\int_{\delta e^{i\alpha_1}}^{\delta e^{i\alpha_2}} e^{nf(z)} F(z)\, dz = O(e^{n\Re f(0)}).$$

6.1. It has proved possible to obtain the asymptotic evaluation of the integral

$$\int_C e^{nf(z)-nf(z_s)} F(z)\, dz$$

when C is a path of the first or second type associated with a point z_s, under certain conditions.

Suppose now that C is a path which can be subdivided into a finite or enumerable set of paths of the first or second type associated with a sequence of points (z_t). If C_t be the path associated with the point z_t, then

$$\int_{C_t} e^{nf(z)-nf(z_t)} F(z)\, dz \sim_n \int_{\gamma_t} e^{nf(z)-nf(z_t)} F(z)\, dz$$

where γ_t is the appropriate standard path. Now suppose that of the various points (z_t) with which the paths are associated, the subset (z_T) are at the same level M, which is superior to the levels of the points of the set not belonging to the subset. Then, evidently

$$e^{-nM} \int_{C_t} e^{nf(z)} F(z)\, dz \sim_n 0$$

except in the case of the paths (C_τ) associated with the subset (z_T). Therefore

$$e^{-nM} \int_C e^{nf(z)} F(z)\, dz \sim_n e^{-nM} \sum{}' \int_{C_\tau} e^{nf(z)} F(z)\, dz$$

$$\sim_n e^{-nM} \sum{}' \int_{\gamma_\tau} e^{nf(z)} F(z)\, dz$$

the prime denoting that the summation is taken only over the subset of paths associated with the subset (z_τ). The asymptotic evaluation is therefore complete.

6.2. It is of interest to consider the case of the asymptotic evaluation is therefore the integral

$$\int_{-\pi}^{\infty} e^{2n \cos z}\, e^{-z^3}\, dz$$

for the integration has to be carried out over an enumerable infinity of cols. We subdivide the path into the enumerable infinity of paths

$$(-\pi, \pi), \quad (\pi, 3\pi), \cdots, \quad ([2r-1]\pi,\ [2r+1]\pi), \cdots$$

and evidently the path from $-\pi$ to $+\pi$ is a path of the second type associated with the origin.* Thus

$$\int_{-\pi}^{+\pi} e^{2n(\cos z-1)}\, e^{-z^3}\, dz \sim \tfrac{1}{2} \sum_{\nu=0}^{\infty} {}_0P_\nu\, \Gamma \left(\frac{1+\nu}{2}\right) \frac{(1-e^{\pi i\nu})}{(n^{1/2})^{1+\nu}}$$

$$= \sum_{\nu=0}^{\infty} \frac{\Gamma(\nu+\frac{1}{2})\, {}_0P_{2\nu}}{n^{\nu+1/2}} \quad ,$$

where ${}_0P_\nu$ is the coefficient of z^ν in the expansion in powers of z of the function

$$\left(\frac{1-\cos z}{z^2/2}\right)^{-(1+\nu)/2} e^{-z^3}.$$

Similarly

$$\int_{(2r-1)\pi}^{(2r+1)\pi} e^{2n(\cos z-1)}\, e^{-z^3} \sim e^{-(2rn)^3} \sum_{\nu=0}^{\infty} \frac{\Gamma(\nu+\frac{1}{2})\, {}_rP_{2\nu}}{n^{\nu+1/2}}$$

where ${}_rP_\nu$ is the coefficient of t^ν in the expansion in powers of t of the function

$$\left(\frac{1-\cos t}{t^2/2}\right)^{-(1+\nu)/2} \exp - [3(2r\pi)^2 t + 3(2r\pi) t^2 + t^3].$$

Thus

$$\int_{-\pi}^{(2r-1)\pi} e^{2n(\cos z-1)}\, e^{-z^3} \sim \sum_{\nu=0}^{k} e^{-(2r\pi)^3} \sum_{\nu=0}^{\infty} \frac{\Gamma(\nu+\frac{1}{2})\, {}_rP_{2\nu}}{n^{\nu+1/2}} \quad :$$

and we could readily justify the passage to the limit and assert that

$$\int_{-\pi}^{\infty} e^{2n(\cos z-1)}\, e^{-z^3}\, dz \sim \sum_{\nu=0}^{\infty} \frac{\Gamma(\nu+\frac{1}{2}) P_{2\nu}}{n^{\nu+1/2}}$$

where

$$P_{2\nu} = {}_0P_{2\nu} + e^{-(2\pi)^3}\ {}_1P_{2\nu} + e^{-(4\pi)^3}\ {}_2P_{2\nu} + \cdots.$$

---

* Cp. figure 2.41.

Generalized Quaternion Algebras and the Theory of Numbers.

By Lois W. Griffiths.

1. *Introduction.* The unified treatment of arithmetics of certain generalized quaternion algebras presented in this paper yields important theorems on representation of integers in quaternary quadratic forms, on the number of representations, and on the solution of Diophantine equations. The methods are essentially those applied by Dickson[*] to four algebras, two of which are instances in this paper.

2. *Generalized Quaternion Algebras.* A generalized quaternion algebra over the field of rational numbers has a modulus 1 and consists of all quantities $r = \sigma + \xi i + \eta j + \zeta k$, in which the *coordinates* σ, ξ, η, ζ are rational numbers, while

$$i^2 = \alpha, \quad j^2 = \tau, \quad k^2 = -\alpha\tau,$$
$$ij = -ji = k, \quad -jk = kj = \tau i, \quad -ki = ik = \alpha j,$$

in which α and τ are rational. The quantity r has the conjugate

$$\bar{r} = \sigma - \xi i - \eta j - \zeta k,$$

and norm

$$Nr = \sigma^2 - \alpha\xi^2 - \tau\eta^2 + \alpha\tau\zeta^2. \tag{1}$$

Hence for any two quantities r and s

$$Nr = r\bar{r} = \bar{r}r = N\bar{r}, \quad N(rs) = NrNs. \tag{2}$$

3. *Representation of Integers.* If α and τ are non-zero integers, as always henceforth, then the representation of an integer μ in the quaternary quadratic form

$$\sigma^2 - \alpha\xi^2 - \tau\eta^2 + \alpha\tau\zeta^2 \tag{3}$$

is equivalent to the existence in the algebra of a quantity whose coordinates are integers and norm is μ. For the algebras with parameters $(\alpha, \tau) = (-1, -1)$, $(-1, -3)$, $(-1, 3)$, $(-1, -7)$, such quantities are integral elements of the algebra, and the discussion of the arithmetics of these algebras given by Dickson * includes complete results of the type indicated in § 1 not only for the forms (3) but for allied forms in which cross product terms appear. The present paper derives similar results for the fifteen algebras with parameters

---

[*] *American Journal of Mathematics*, Vol. 46 (1924), pp. 1-16; *Algebren und Ihre Zahlentheorie*, Orell Füssli, Zurich (1927), Chapter IX; *Algebras and Their Arithmetics*, Chapter X, University of Chicago Press.

(4) $(-1,-3), (-2,-3), (-1,-7), (2,-3), (4,-3), (2,5), (-2,5), (3,5), (-3,5), (-4,5), (3,-7), (2,-11), (4,-11), (-2,13), (-4,13)$,

by considering a set S which has all the properties of a set of integral elements except that it is not necessarily maximal.

The fundamental set S consists of all quantities

$$q = (\sigma + \eta/2) + (\xi + \zeta/2)i + j\eta/2 + k\zeta/2 = \sigma + \xi i + \eta H + \zeta L, \tag{5}$$

in which $H = (1 + j)/2$, $L = (i + k)/2$, and the *coefficients* σ, ξ, η, ζ are integers. Hence S is closed under addition and multiplication, and contains the modulus. Furthermore, since

$$Nq = \sigma^2 - \alpha\xi^2 + \epsilon\eta^2 - \alpha\epsilon\zeta^2 + \sigma\eta - \alpha\xi\zeta, \quad \epsilon = (1 - \tau)/4, \tag{6}$$

is an integer, q satisfies the quadratic equation $x^2 - (2\sigma + \eta)x + Nq = 0$ with integer coefficients and leading coefficient unity. Hence S is a set of integral quantities if it is maximal, as in the two instances $(-1, -3)$, $(-1, -7)$. Henceforth all quantities considered are in S. If p, q, g are quantities of S such that $g = pq$, then g has a *left divisor* (factor) p and a *right divisor* q. A quantity of S which divides the modulus is called a *unit*. If u is a unit, gu is a *right associate* of g. A quantity which is not a unit is *prime in* S if it is not the product of two quantities of S each of which is not a unit.

The representation of an integer μ in the quaternary quadratic form

$$\sigma^2 - \alpha\xi^2 + \epsilon\eta^2 - \alpha\epsilon\zeta^2 + \sigma\eta - \alpha\xi\zeta, \quad \epsilon = (1 - \tau)/4, \tag{7}$$

is equivalent to the existence in S of a quantity whose coefficients are integers and norm is μ. Hence by (5) the representation of μ in (3) is equivalent to the representation of μ in (7) with η and ζ both even, that is, to the existence in S of a quantity whose coordinates are integers and norm is μ.

THEOREM 1. *Every positive and negative integer is represented in* (7)

---

* *Algebren und Ihre Zahlentheorie*, pp. 162-197; *Algebras and Their Arithmetics*, § 91, § 105.

with (α, τ) *of* (4), *except that, if* α *and* τ *are both negative, no negative integer, and, if* $|\alpha| = 4$, *no integer* $4^a 2\mu$ $(a \geqq 0,\ \mu$ *odd*$)$ *is represented.*

It is easily verified that

$$(8) \qquad H + \mathrm{L},\ -1 - i + 2H + L,\ H,\ H,\ H,\ H,\ H,\ i + H,\\ 1 + H + L,\ -1 + 2H + L,\ 1 + H,\ 1 + H$$

are units of negative norm for the algebras (4) respectively. Hence -1 and, by (2) and (7), $|\alpha|^a$ $(a \geqq 0)$ are represented. By (5) and (6), with $|\alpha| = 4$, $Nq \equiv 0 \pmod 2$ implies $\sigma \equiv \eta \equiv 0 \pmod 2$ and hence $Nq \equiv 0 \pmod 4$. Therefore it remains to prove the theorem for primes not dividing α. This is done by lemmas 1, 2, 3, whose proof is an extension of that by Dickson[*] with $(\alpha, \tau) = (-1, -7)$.

LEMMA 1. *If* π *is a prime which does not divide* α, *there are integers* λ, μ, ω *such that* $1 + \mu + \epsilon\mu^2 - \alpha\lambda^2 = \pi\omega$, $0 \neq |\omega| < \pi$, $0 \leqq |\lambda|, |\mu| \leqq \pi/2$.

LEMMA 2. *If* π *is an integer different from* $0, 1, -1$, *and if* ω, λ, μ *are integers such that* $1 + \mu + \epsilon\mu^2 - \alpha\lambda^2 = \pi\omega$, *then* $0 \neq |\omega| < |\pi|$ *if if* $0 \leqq |\lambda|, |\mu| \leqq |\pi|/2$ *with* π *odd and if* $-(|\pi| - 2)/2 \leqq \lambda, \mu \leqq |\pi|/2$ *with* π *even.*

The proof consists in solving the inequality $-\pi^2 < 1+\mu+\epsilon\mu^2-\alpha\lambda^2 < \pi^2$ for the pairs of integers α, ϵ, subject to these conditions on λ, μ.

LEMMA 3. *If* π *is a prime which does not divide* α, *there is a quantity* g *of* S *whose norm is* π.

By lemma 1 there are integers λ, μ, ω such that $1 + \mu + \epsilon\mu^2 - \alpha\lambda^2 = \pi\omega$, $0 \neq |\omega| < \pi$. Give g the notation $A + Bi + CH + DL$, where

$$(9) \qquad A = \pi\sigma - \alpha\lambda\xi + \epsilon\mu\eta, \qquad B = \xi,\\ E = \pi\zeta + \mu\xi - \lambda\eta, \qquad C = \eta.$$

Hence by (6) it is sufficient to prove the existence of integers σ, ξ, η, ζ such that

$$(10) \qquad 4\pi = 4\pi^2\sigma^2 - 4\alpha\pi\omega\xi^2 + 4\epsilon\pi\omega\eta^2 - 4\alpha\epsilon\pi^2\zeta^2\\ + 8\alpha\pi\lambda(-\sigma\xi + \epsilon\eta\zeta) + 4\pi(1 + 2\epsilon\mu)(\sigma\eta - \alpha\xi\zeta).$$

Since $\alpha\omega\pi \neq 0$, multiply (10) by $-\alpha\omega/\pi$ and write

$$(11) \qquad \sigma' = \alpha\xi, \quad \xi' = -\sigma, \quad \eta' = \alpha\zeta, \quad \zeta' = -\eta,$$

---

[*] *American Journal of Mathematics*, Vol. 46 (1924), pp. 11-13; *Algebren und Ihre Zahlentheorie*, § 101.

to obtain

$$(12)\qquad -4\alpha\omega = 4\omega^2\sigma'^2 - 4\alpha\omega\pi\xi'^2 + 4\epsilon\omega\pi\eta'^2 - 4\alpha\epsilon\omega^2\zeta'^2 + 8\alpha\omega\lambda(-\sigma'\xi' + \epsilon\eta'\zeta') + 4\omega(1 + 2\epsilon\mu)(\sigma'\eta' - \alpha\xi'\zeta').$$

This, by comparison with (9) and (10), expresses the fact that

$$N(A' + B'i + C'H + D'L) = -\alpha\omega, \text{ where}$$

$$(13)\qquad A' = \omega\sigma' - \alpha\lambda\xi' + \epsilon\mu\eta', \qquad B' = \xi',$$
$$E' = \omega\zeta' + \mu\xi' - \lambda\eta', \qquad C' = \eta'.$$

Now if $|\omega| \neq 1$ the least residues κ, ν of λ, μ modulo ω determine integers γ, β, π' such that $\lambda = \kappa + \omega\gamma$, $\mu = \nu + \omega\beta$, $1 + \nu + \epsilon\nu^2 - \alpha\kappa^2 = \omega\pi'$; furthermore, by lemma 2, $0 \neq |\pi'| < |\omega| < \pi$. Hence (13) become

$$(14)\qquad A' = \omega\sigma_1 - \alpha\kappa\xi_1 + \epsilon\nu\eta_1, \qquad B' = \xi_1,$$
$$E' = \omega\zeta_1 + \nu\xi_1 - \kappa\eta_1, \qquad C' = \eta_1,$$

where

$$(15)\qquad \sigma_1 = \sigma' - \alpha\gamma\xi' + \epsilon\beta\eta', \qquad \xi_1 = \xi',$$
$$\zeta_1 = \zeta' + \beta\xi' - \gamma\eta', \qquad \eta_1 = \eta',$$

so that (12) becomes

$$(16)\qquad -4\alpha\omega = 4\omega^2\sigma_1{}^2 - 4\alpha\omega\pi'\xi_1{}^2 + 4\epsilon\omega\pi'\eta_1{}^2 - 4\alpha\epsilon\omega^2\zeta_1{}^2 + 8\alpha\omega\kappa(-\sigma_1\xi_1 + \epsilon\eta_1\zeta_1) + 4\omega(1 + 2\epsilon\nu)(\sigma_1\eta_1 - \alpha\xi_1\zeta_1).$$

Substitute for $\sigma_1, \cdots, \zeta_1$ from (15) and (11), multiply by $-\pi'/\alpha\omega$, and write

$$(17)\qquad X = \xi + \gamma\sigma + \epsilon\beta\zeta, \qquad Y = \eta + \beta\sigma + \alpha\gamma\zeta$$

to obtain

$$(18)\qquad 4\pi' = 4\pi'^2\sigma^2 - 4\alpha\pi'\omega X^2 + 4\epsilon\pi'\omega Y^2 - 4\alpha\epsilon\pi'^2\zeta^2 + 8\alpha\pi'\kappa(-\sigma X + \epsilon Y\zeta) + 4\pi'(1 + 2\epsilon\nu)(\sigma Y - \alpha X\zeta).$$

Hence by (17) the solution of (18) in integers is equivalent to that of (10) in integers.

If $|\pi'| \neq 1 \neq |\omega|$, by the process applied to (12) to obtain (16), (18) yields an equation in π', ω', λ', μ' such that $0 \neq |\omega'| < |\pi'| < |\omega| < \pi$, which is precisely similar to (10) in π, ω, λ, μ, whose solution is equivalent to that of (10). If $|\omega| = 1$, $\gamma = \omega\lambda$, $\beta = \omega\mu$ implies $\kappa = \nu = 0$, $|\pi'| = |\omega| = 1$. If $\pi' = 1$, (18) has the solution $\sigma = 1$, $X = Y = \zeta = 0$; if $\pi' = -1$, -4π is to be used on the left of (10) initially and one of (8) at this point. Hence the proof by descent is complete.

Thus finally theorem 1 is proved.

LEMMA 4. *Except with* $(-1, -7)$, *there are in* S *three units* $u_1 = 1$, u_2, u_3 *of positive norm such that for every quantity* g *in* S *at least one of* $u_h g u_f$ $(h, f = 1, 2, 3)$ *has integral coordinates. If* $Ng \equiv \alpha \pmod{2}$, *at least one of* gu_h $(h = 1, 2, 3)$ *has integral coordinates.*

Give the notations $g = \sigma + \xi i + \eta H + \zeta L$; $u = \kappa + \lambda i + \mu H + \nu L$. Then $ug = A + Bi + CH + DL$, where

$$
\begin{aligned}
(19)\quad A &= \kappa\sigma + \alpha(\lambda+\nu)\xi - \epsilon\mu\eta + \alpha\epsilon\nu\zeta, & C &= \mu\sigma - \alpha\nu\xi + (\kappa+\mu)\eta + \alpha\lambda\zeta,\\
B &= \lambda\sigma + (\kappa+\mu)\xi - \epsilon\nu\eta + \epsilon\mu\zeta, & D &= \nu\sigma - \mu\xi + (\lambda+\nu)\eta + \kappa\zeta,
\end{aligned}
$$

and similarly for gu. Then it is verified directly with $\sigma, \xi, \eta, \zeta \equiv 0, 1 \pmod{2}$ in all ways, that the units $-1 + H$, H satisfy the requirements with the algebras $(-1, -3)$ $(2, -3)$, $(-2, -3)$, $(4, -3)$, that is, in abbreviated notation, with the algebras $(-1, \pm 2, 4; -3)$. Similarly $1+H$, $2-H$ with $(\pm 2, \pm 3, -4; 5)$; $-1 + i + H$, $i + H$ with $(2, -11)$, $(-4, 13)$; $2 + 3i + 3H$, $-5 + 3i + 3H$ with $(3, -7)$, $(4, -11)$, $(-2, 13)$. On the other hand, with $(-1, -7)$ the units are $1, -1, i, -i$, none of which satisfies for $g = H$.

THEOREM 2. *Every positive and negative integer represented in the form* (7) *of theorem* 1 *is represented in the form* (3), *except some integers not divisible by* 4 *with* $(\alpha, \tau) = (-1, -7)$.

This follows from (2) and lemma 4. Clearly 3 is not represented in the form (3) with $(-1, -7)$; whereas for multiples of 4 with $(-1, -7)$, the representation of μ in (7) is equivalent to that of 4μ in (3).

4. *Solution of Diophantine Equations.*

LEMMA 5. *A quantity* g *of* S *whose norm is divisible by a prime* π, $\neq 2$ *with* $|\alpha| = 4$, *has a right (left) divisor of norm* π.

By theorem 1 it suffices to consider $g \not\equiv 0 \pmod{\pi}$. Give the notation $g = \kappa + \lambda i + \mu H + \nu L$. If π divides α, $|\alpha| \neq 1, 4$. Then $Ng \equiv 0 \pmod{\alpha}$ implies $\kappa \equiv 0 \equiv \mu \equiv gi \pmod{\alpha}$, and since $Ni = -\alpha$, there is a quantity q such that $g = q\bar{i}$. Hence, if u is a unit whose norm is of the same sign as Ni, ui satisfies as a right divisor. If $\alpha \not\equiv 0 \pmod{\pi}$, the method of proof is that by Dickson (*ibid.*) with $(-1, -7)$. The three cases have hypotheses respectively $\kappa^2 - \alpha\epsilon\nu^2 \not\equiv 0 \pmod{\pi}$, $\epsilon\mu^2 - \alpha\lambda^2 \not\equiv 0 \pmod{\pi}$, $\kappa^2 - \alpha\epsilon\nu^2 \equiv 0 \equiv \epsilon\mu^2 - \alpha\lambda^2 \pmod{\pi}$. Except with $(-2, -3)$, $|\alpha| = 4$, there is a contradiction which makes the third case impossible as in Dickson's proof; otherwise, in the third case, $-1 + 2H$ satisfies as a divisor. This completes the proof of the lemma, since a right divisor p of $\bar{g}$ determines a left divisor $\bar{p}$ of g, and $Np = N\bar{p}$.

LEMMA 6. *With* $|\alpha| = 4$, *there are in* S *no quantities of norm* 2 *or* -2 *and there are quantities of even norm which have neither a left nor a right divisor of norm* 4 *or* -4.

The first statement is a corollary of theorem 1. It is verified by (19) and the conditions that $q = \kappa + \lambda i + \mu H + \nu L$ be such that $Nq = \pm 4$, $qg \equiv 0 \pmod{4}$ or $gq \equiv 0 \pmod{4}$, that $g = 4 - i + 2H + L$ is a quantity satisfying the second statement.

Lemmas 5 and 6 imply lemma 7 and theorem 3 on the solution of Diophantine equations.

LEMMA 7. *With* $|\alpha| \neq 4$, *if* g *is in* S *and* $Ng = \delta_1 \cdots \delta_n$ *there are quantities* $d_1, \cdots, d_n$ *in* S *such that* $g = d_1 \cdots d_n$ *and* $\delta_h = Nd_h$. *With* $|\alpha| = 4$, *a similar statement holds if not more than one of* $\delta_1, \cdots, \delta_n$ *is divisible by* 2.

THEOREM 3. *All integral solutions* $\sigma, \xi, \eta, \zeta, \delta_1, \cdots, \delta_n$ *of*

$$\sigma^2 - \alpha\xi^2 + \epsilon\eta^2 - \alpha\epsilon\zeta^2 + \sigma\eta - \alpha\xi\zeta = \delta_1 \cdots \delta_n, \quad \epsilon = (1 - \tau)/4, \tag{20}$$

with (α, τ) *of* (4) *and* $|\alpha| \neq 4$, *are obtained from* $\sigma + \xi i + \eta H + \zeta L = d_1 \cdots d_n$, $\delta_h = Nd_h$, *where* $d_1, \cdots, d_n$ *range independently over* S. *With* $|\alpha| = 4$, *precisely all integral solutions in which at most one of* $\delta_1, \cdots, \delta_n$ *is even, and that divisible by an even power of* 4, *are so obtained with at most one of* $d_1, \cdots, d_n$ *of even norm.*

It is easily seen by lemma 4 that, with α odd and $(\alpha, \tau) \neq (-1, -7)$, if g has integral coordinates and π is an odd prime dividing Ng then there are quantities p, q in S, each having integral coordinates, such that $g = pq$ and $Np = \pi$. Hence there follows a lemma analogous to lemma 7, and

THEOREM 4. *All integral solutions, in which at most one of* $\delta_1, \cdots, \delta_n$ *is even, of*

$$\sigma^2 - \alpha\xi^2 - \tau\eta^2 + \alpha\tau\zeta^2 = \delta_1 \cdots \delta_n, \tag{21}$$

with $(\alpha, \tau) = (-1, -3), (3, 5), (-3, 5), (3, -7)$, *are obtained from* $\sigma + \xi i + \eta H + \zeta L = d_1 \cdots d_n$, $\delta_h = Nd_h$, *where* $d_1, \cdots, d_n$ *range independently over* S *but are such that each has integral coordinates and at most one is of even norm.*

The remaining lemmas of § 4 add the uniqueness feature to the factorization of quantities in S. This is essential to the enumerations of § 5.

Lemma 8. *If* $|\alpha| \neq 4$ *there are in* S *quantities* p_h ($h = 1$ *if* α *is even,* $h = 1, 2, 3$ *if* α *is odd*) *such that* $|Np_h| = 2$ *and that, if* g *is a quantity of* S *having* $Ng = 2^a\mu$ ($a > 0$, μ *odd*), *then there are unique quantities* m, q *in* S *such that* $g = mq2^b$, *where* 2^b *is the highest power of* 2 *dividing* g, $|Nm| = |\mu|$, $|Nq| = 2^{a-2b}$, *and* q *is a product of factors* p_h. *Furthermore* q *is a product of factors* p_h *in only one way.*

The proof for α even is in that of lemma 5. For α odd, by (6) and (19), it is verified that the following quantities $\bar{p}_h$ are of norm 2 or -2 and such that for any g one and only one of $\bar{p}_h g$ is divisible by 2: with $(-1, 3; -7)$ $H, -1 + H, 1 + i$; with $(-1, -3), (3, 5)$ $1 + i, H + L, -1 - i + H + L$; *with* $(-3, 5)$ $i + H, -1 + i + H, i + H + L$. Then $g = p_h q$ as in lemma 5.

Lemma 9. *If* π *is a prime and* m, p, q *are quantities of* S *such that* $Np = \pi$ *and* π *divides* mpq, *then* π *divides* mp *or* pq.

If α is divisible by π and $|\alpha| = 4$, the lemma is vacuously true. If α is divisible by π and $|\alpha| \neq 4$, then $\pi = |\alpha|$ and $mpq = \alpha r$. Hence α divides Nm or Nq, and by the proof of lemma 5 α divides mp or pq. The proof when α is not divisible by π follows that of Dickson (*ibid.*) with lemma 8 for the case $\pi = 2$.

Lemma 10. *If* π *is a prime,* $\neq 2$ *with* $|\alpha| = 4$, *dividing* Ng *but not* g, *then* g *and* π *have a common left (right) divisor of norm* π, *determined uniquely up to a unit right (left) factor.*

This follows for $\pi = 2$ by lemma 8. For $\pi > 2$, by lemma 5 there are quantities p, q such that $g = pq$ and $Np = \pi$, and by lemma 9, if P is any common divisor of norm π, $0 \equiv \bar{P}g \equiv \bar{P}pq \pmod{\pi}$, π divides $\bar{P}p$, and P is a right associate of P.

A quantity is *proper* when and only when the greatest common divisor of its coefficients is unity. Hence by lemma 10 the factorizations of lemma 7 and preceding theorem 4 are, for the prime factors of Ng in fixed but arbitrary order, unique up to unit factors if g is proper. Hence finally, using lemma 10 and the method of Dickson, any two quantities g and q of S have a greatest common left (right) divisor determined uniquely up to a unit right (left) factor with $|\alpha| \neq 4$, and the same statement holds with $|\alpha| = 4$ if not both Ng and Nq are divisible by 2. For the algebra $(-1, -3)$ Dickson* established the existence of this greatest common divisor directly by the process of division with remainders of decreasing norms as in the theory of ordinary integers, and then proved the foregoing theorems on factorization of integral quantities, representation of integers and solution of Diophantine equations.

5. *Number of Representations.* If ν is any integer not excepted in lemma 16, there will be found finite numbers, $F(\nu)$ and $G(\nu)$ respectively, of sets of all and all proper quantities in S such that every quantity in S of norm ν or $-\nu$ is in one and only one set, and such that a given set consists of the right associates of any one of its members. Then the numbers of fundamental representations in the form (7), from which all and all proper representations are derived by unimodular transformations, are determined as $F(\nu)$ and $G(\nu)$.

If ν is prime to α and τ, $F(\nu)$ and $G(\nu)$ are found by the method of Dickson.†

LEMMA 11. *If π is a prime which divides neither α nor τ, $F(\pi) = \pi + 1$.*

For $\pi = 2$ this is lemma 8. For $\pi > 2$ the proof is Dickson's, with $\rho = \kappa\kappa_1 - \alpha\lambda\lambda_1 - \tau\mu\mu_1 + \alpha\tau\nu\nu_1$ here.

It is easily verified that if π is a rational prime and $Np = \pi$ then p is a prime quantity, and conversely. Hence lemma 10 implies lemma 12 and they in turn lemmas 13 and 14.

LEMMA 12. *A product of prime quantities is proper if and only if no factor is a right associate of the conjugate of its predecessor.*

LEMMA 13. *If μ is an integer prime to α and to τ and $\pi^a\kappa^b \cdots$ its prime factorization, then $G(\mu)$ equals $Q(\mu) = \mu(1 + 1/\pi)(1 + 1/\kappa) \cdots$.*

LEMMA 14. *If μ is an integer prime to α and to τ, $F(\mu)$ is the sum $f(\mu)$ of all the positive divisors of μ.*

---

[*] *Algebras and Their Arithmetics*, p. 150; *Algebren und Ihre Zahlentheorie*, p. 163.
[†] *Algebren und Ihre Zahlentheorie*, pp. 175-179.

Lemmas 13 and 14 are extended to all integers, when possible, by enumerations for α and τ similar to those in lemmas 8 and 11.

LEMMA 15. *There are in S quantities p_h $[h=0, 1, \cdots, 2|\tau|$ with $(-2, -3)$, $|\alpha|=4$, $h=0$ otherwise$]$ such that $|Np_h|=|\tau|$ and that, if g is a quantity of S having $Ng=\tau^a\mu$ $[a>0,\ \mu \not\equiv 0 \pmod{\tau}]$, then there are in S unique quantities m, q, s such that $g=\tau^b qm$, τ^b is the highest power of τ dividing g, $|Nm|=|\mu|$, $|Nq|=|\tau^{a-2b}|$, and q is a power of p_0 if $h=0$, otherwise $q=p_0{}^c s$, $0 \leqq c \leqq a-2b$, and s is a product of the p_h with $h=1, \cdots, 2|\tau|$. Furthermore s is a product of these p_h in only one way.*

Give the notation $g=\sigma+\xi i+\eta H+\zeta L$. Then by (6)

$$0 \equiv Ng \equiv 4Ng \equiv (2\sigma+\eta)^2-\alpha(2\xi+\zeta)^2 \pmod{\tau}. \tag{22}$$

Hence, except with $(-2, -3)$, $|\alpha|=4$, $2\sigma+\eta \equiv 0 \equiv 2\xi+\zeta \equiv (-1+2H)g \pmod{\tau}$, so that $p_0=1-2H$, of norm $-\tau$, satisfies for every g. The proof with $(-2, -3)$, $|\alpha|=4$ is long. Thus, by (19), for $\sigma, \xi, \eta, \zeta \equiv 0, 1, \cdots, |\tau|-1 \pmod{\tau}$ in all ways consistent with (22), $\bar{p}_0 g \equiv 0 \pmod{\tau}$ and hence p_0 is a left divisor of g if and only if $2\sigma+\eta \equiv 0 \pmod{\tau}$, and otherwise $\bar{p}_h$ as follows are satisfactory (upper signs taken together):

with $(-2, 4; -3)$ $H \mp L,\ -1 \pm i,\ -1 \pm i+H \mp L;$

with $(-4, 5)$ $-3-H \pm 2L,\ 7 \pm 2i-5H,\ \pm i-3H \pm L,$
$-1 \mp 2i-H \mp 3L,\ -2 \mp 2i+H \mp 3L;$

with $(4, -11)$ $5 \pm 3i,\ \pm 3-2i+L,\ -4 \mp i+H,\ -3 \mp 2i+3H,$
$4+H \pm L,\ 2 \pm i+H \pm L,\ -3 \mp 2i+H \pm L,$
$-5 \mp i+H \pm L,\ 2i \pm 3H,\ 3 \pm i+H,$
$\mp 3+i+L;$

with $(-4, 13)$ $3 \mp i,\ -5 \mp i+3H,\ 2i \mp H,\ 1 \mp i+H \pm L,$
$3 \pm i+H,\ \mp 7+3i+3L,\ -1 \pm 2i+3H,$
$-2 \mp 2i+3H,\ \pm 7-6i+3L,\ -4 \mp i+H,$
$-2+H \pm L,\ -1 \mp 2i+H,\ 2 \pm i+3H.$

Furthermore only one of p_h $(h=1, \cdots, 2|\tau|)$ is a left divisor unless $2\sigma+\eta \equiv 0 \pmod{\tau}$.

LEMMA 16. *If* $\nu = \alpha^a \tau^b \mu$ ($a = 0$ *with* $|\alpha| = 4$ *but otherwise* $a \geqq 0$, $b \geqq 0$, μ *relatively prime to* α *and to* τ), $F(\nu) = \beta f(\mu)$, *where* $f(\mu)$ *is the sum of all the positive divisors of* μ, $\beta = 1$ *except with* $(-2, -3)$, $|\alpha| = 4$, *and otherwise* $\beta = 1 + 2 |\tau| (|\tau|^b - 1)/(|\tau| - 1)$.

By lemmas 5 and 14 it is sufficient to prove $F(\nu/\mu) = \beta$. If $a > 0$ with $|\alpha| = 4$ this would not be sufficient, by lemma 6. If $|\alpha| \neq 4$, α is a prime and, for every positive integer a, $F(\alpha^a) = 1$ by the proof of lemma 5. By lemma 15 $F(\tau^b) = 1$ except with $(-2, -3)$, $|\alpha| = 4$. With $(-2, -3)$, $|\alpha| = 4$ a product $p_h p_f$ has p_0 as a left divisor if and only if $h = 0$ or $f = 0$ or h and f are not both from the set $1, \cdots, |\tau|$ (here the quantities with upper signs have subscripts $1, \cdots, |\tau|$). Hence for any positive integer b each of the $F(\tau^b)$ sets contains one and only one of the quantities: all products of b factors with subscripts from $1, \cdots, |\tau|$, all products of b factors with subscripts from $|\tau| + 1, \cdots, 2|\tau|$, and all products $p_0 q$ where q is in turn one quantity from each of the $F(\tau^{b-1})$ sets. Hence by induction $F(\tau^b) = 1 + 2|\tau|(|\tau|^b - 1)/(|\tau| - 1)$. Finally $F(\nu/\mu) = F(\alpha^a)F(\tau^b) = \beta$, as stated.

THEOREM 5. *All representations of* ν, *positive with* $(-1, -3)$, $(-2, -3)$, $(-1, -7)$, *in the form* (7) *are obtained from* $F(\nu) = \beta f(\mu)$ *fundamental representations by all transformations of determinant*

$$\begin{vmatrix} \kappa' & \alpha\lambda' & -\epsilon\mu' & \alpha(\lambda' + \epsilon\nu') \\ \lambda' & \kappa' & \lambda' + \epsilon\nu' & -\epsilon\mu' \\ \mu' & \alpha\nu' & \kappa' + \mu' & -\alpha\lambda' \\ \nu' & \mu' & -\lambda' & \kappa' + \mu' \end{vmatrix} = \kappa'^2 - \alpha\lambda'^2 + \epsilon\mu'^2 - \alpha\epsilon\nu'^2 + \kappa'\mu' - \alpha\lambda'\nu' = 1. \tag{23}$$

The notations are those of lemma 16. Each of the $F(\nu)$ fundamental sets contains a quantity of norm ν by (8), and every quantity qv, where $Nq = -\nu$ and $Nv = -1$, is in one of these fundamental sets. Hence the quantities of norm ν are right associates of $\beta f(\mu)$ quantities. Write $g = \sigma + \xi i + \eta H + \zeta L$, $u = \kappa' + \lambda' i + \mu' H + \nu' L$. Then by (19) the coefficients of gu are linear homogeneous functions of $\sigma, \cdots, \zeta$ with determinant (23). Hence by (2) the theorem follows.

THEOREM 6. *Exept with* $(-1, -7)$, *if* $\nu \equiv \alpha$ (*mod* 2), *and if* $\nu > 0$ *with* $(-1, -3)$, $(-2, -3)$, *all representations of* ν *in the form* (3) *are obtained from* $\beta f(\mu)$ *fundamental representations by transformations of determinant* (23), *but some quadruples so obtained may not represent* ν *in* (3).

This follows from theorem 5 by lemma 4.

LEMMA 17. *If* $\nu = \alpha^a\tau^b\mu$ $(a = 0$ *with* $|\alpha| = 4$ *but otherwise* $a \geqq 0$, $b \geqq 0$, μ *relatively prime to* α *and to* $\tau)$, $G(\nu) = \rho Q(\mu)$, *where* $\rho = 0$ *if* $a > 1$ *or* $b > 1$ *and otherwise* $\rho = 1$ *with* $(\alpha, \tau) \neq (-2, -3)$, $|\alpha| = 4$; *where* $\rho = 0$ *if* $a > 1$ *and otherwise* $\rho = 1$ *if* $b = 0$, $\rho = 2|\tau| + 1$ *if* $b = 1$, $\rho = 2|\tau|^{b-1}(|\tau| + 1)$ *if* $b > 1$ *with* $(-2, -3)$, $|\alpha| = 4$.

By lemmas 13 and 5 it is sufficient to prove $G(\nu/\mu) = G(\alpha^a)G(\tau^b) = \rho$. With $|\alpha| \neq 4$, $(\alpha, \tau) \neq (-2, -3)$, by the proof of lemma 16, $G(\alpha^a)G(\tau^b) = 0$ if $a > 1$ or $b > 1$ and otherwise $G(\alpha^a\tau^b) = 1$. With $(-2, -3)$ $\rho = 0$ if $a > 1$. If $a = 0$ or 1 with $(-2, -3)$ and if $a = 0$ with $|\alpha| = 4$, $\rho = 1$ if $b = 0$ and $\rho = 2|\tau| + 1$ if $b = 1$. Finally if $b > 1$ each of the $F(\tau^b) - G(\tau^b)$ sets of quantities of norms $\pm\tau^b$ which are not proper contains one and only one of the quantities p_0p_0q, where q is in turn one quantity from each of the $F(\tau^{b-2})$ fundamental sets. Hence $G(\tau^b) = F(\tau^b) - F(\tau^{b-2}) = 2|\tau|^{b-1}(|\tau| + 1)$.

THEOREM 7. *All proper representations of* ν, *positive with* $(-1, -3)$, $(-2, -3)$, $(-1, -7)$, *in the form* (7) *are obtained from* $G(\nu) = \rho Q(\mu)$ *fundamental representations by transformations of determinant* (23).

The notations are of lemma 17. This theorem is not extended to the form (3) since there are quantities in S for which the greatest common divisor of the coefficients is different from unity while that of the coordinates is unity.

With $(-1, -3)$, $(-2, -3)$, $(-1, -7)$ there are precisely 12, 6, 4 units respectively since the forms (7) and (3) are positive. Then theorems 5, 6, 7 are theorem 8 and the classic theorem 9.

THEOREM 8. *All* {*all proper*} *representations of* $|\alpha^a\tau^bN|$ $(a, b \geqq 0$, N *prime to* α *and* $\tau)$ *in* (7) *with* $(\alpha, \tau) = (-1, -3)$, $(-2, -3)$, $(-1, -7)$, *are in number* $12f(N)$, $6[1 + 3(3^b - 1)]f(N)$, $4f(N)$ $\{12\rho Q(N)$, $6\rho Q(N)$, $4\rho Q(N)\}$.

THEOREM 9. *All representations of* 2^a3^bN $(a > 0$, $b \geqq 0$, N positive *and prime to* 6) *in the form* $\sigma^2 + 2\xi^2 + 3\eta^2 + 6\zeta^2$ *are in number* 2 *or* 6 *times* $1 + 3(3^b - 1)$ *times the sum of all the positive divisors of* N *according as* $a = 1$ *or* $a > 1$.

This result was obtained otherwise by Liouville.[*] Similar theorems for the form (3) with $(\alpha, \tau) = (-1, -3), (-1, -7)$ have been obtained by the present method by Dickson.[†]

The methods of this paper have been applied to the algebras $(2, -15)$, $(3, -15)$, $(-3, 17)$, but the results are not included here since certain exceptions would recur in the theorems. The results of § 3 have been established for four infinite systems of algebras by a modification of that method which does not, however, permit of application to obtain the results of § 4, 5. These algebras seem to be the only ones to which the method of § 3 is applicable.

---

[*] Dickson, *History of the Theory of Numbers*, Vol. 3, p. 229.

[†] *Algebren und Ihre Zahlentheorie*, Chapter IX.

On Hyperelliptic θ-Functions with Rational Characteristics.

By Oscar Zariski.

1. *Preliminaries.* A θ-function of the arguments $v_1, \ldots, v_p$, attached to a definite normalized period matrix

$$\| 0, \cdots, 0, \pi i, 0, \cdots, 0; a_{j1}, \cdots, a_{jp} \|; \quad a_{ij} = a_{ji},$$

and of characteristic $[g_1, \ldots, g_p; h_1, \ldots, h_p]$, where the g's and the h's are real numbers, is designated by the symbol $\theta[g;h]((v_\sigma))$, and is defined by the following series:

$$\theta[g;h]((v_\sigma)) = \sum_{(m)\,-\infty}^{\infty} \exp\Big[\sum_{1}^{p}{}_{i,j}\, a_{ij}(m_i+g_i)(m_j+g_j) + 2\sum_{1}^{p}{}_i\,(m_i+g_i)(v_i+h_i\pi i)\Big],$$

where in the first summation each of the p letters m_i runs through all integral values from $-\infty$ to $+\infty$. If ρ_{ij} denotes the real part of a_{ij}, the series is convergent for all finite values of the v's, as soon as $\sum_{1}^{p}{}_{i,j}\,\rho_{ij}x_i x_j$ is a definite negative quadratic form. If the elements g and h of the characteristic are rational numbers with the common denominator r, $g_i = \epsilon_i/r$, $h_i = \epsilon_i'/r$, then we will also designate the characteristic $[g;h]$ and the function $\theta[g;h]((v_\sigma))$ by the symbols $[\epsilon]_r$, $\theta[\epsilon]_r((v_\sigma))$.

A system of simultaneous linear combinations of the periods will be denoted by $\{g;h\}$:

$$\{g;h\} = \sum_{1}^{p}{}_i\, g_i a_{1i} + h_1\pi i, \cdots, \sum_{1}^{p}{}_i\, g_i a_{pi} + h_p \pi i,$$

and the corresponding period characteristic will be designated by $(g;h)$. In the case of rational period characteristics we will also use respectively the notations $\{\epsilon\}_r, (\epsilon)_r$.*

The θ-function $\theta[0;0]((v_\sigma))$ of characteristic zero will be simply designated by $\theta((v_\sigma))$. If the a_{ij}'s are the periods of a set of p normal abelian integrals u_i of the first kind, attached to an algebraic curve C of genus p, the function $\theta[g;h]((u_\sigma))$ of the arguments $u_1, \ldots, u_p$ is a Riemannian θ-function. The case which interests us is the one in which C is an hyperelliptic curve:

---

* We use here the notations of A. Krazer, *Lehrbuch der Thetafunktionen*, Leipzig, Teubner, 1903. This treatise will be referred to in the sequel as (Kr.).

$$(1) \qquad y^2 = f(x) = (x-k)(x-k_1) \ldots (x-k_{2p+1}).$$

We construct the two-sheeted Riemann surface R of C by joining the sheets along $p+1$ non-intersecting cuts: kk_1; k_2k_3; $\ldots$; k_{2p-2}, k_{2p-1}; k_{2p}, k_{2p+1}. We define on R a set of $2p$ retrosections b_i, a_i $(i=1, \ldots, p)$ in the usual manner, namely: let b_i be a circuit in the first (upper) sheet surrounding the cut $k_{2i-2}k_{2i-1}$ (we put $k_0 = k$), and let a_i be a circuit which crosses the only two cuts $k_{2i-2}k_{2i-1}$, $k_{2p}k_{2p+1}$. Let $u_1, \ldots, u_p$ be a set of p abelian integrals of the first kind, normal with respect to the chosen set of retrosections, the period of u_i relative to b_j being πi if $i=j$, and 0 if $i \neq j$. Let a_{ij} denote the period of u_i relative to the retrosection a_j. Then $a_{ij} = a_{ji}$. With these conditions the u's are determined to within an additive constant, which we fix by assigning the branch point k_{2p+1} as common lower limit of the integrals.

If we put all the arguments u_σ equal to zero, the result $\theta[g;h]((0))$ is a function of the periods only, hence a function of the moduli of the curve C. In the hyperelliptic case $\theta[g;h]((0))$ is a function of the branch points $k, k_1, \ldots, k_{2p+1}$. These functions were first considered by Riemann,* who pointed out that every abelian integral of the second kind can be expressed by means of the logarithmic derivative of the function $\theta((u))$ and by means of $\theta((0))$, which appears in the additive constant of the integral. This remark of Riemann was the point of departure of two papers by Thomae, dedicated to the determination of $\theta((0))$ as a function of the moduli, or rather as a function of the branch points of the given algebraic function.† The same argument is treated by Fuchs.‡

Thomae finds a system of equations on partial derivatives which satisfies $\theta((0))$. In the hyperelliptic case Thomae was able to integrate completely these equations (see the second of the cited papers) and gave thus a remarkable expression not only of $\theta((0))$, but of any $\theta[\epsilon]_2((0))$. While the problem of expressing explicitly any hyperelliptic $\theta[\epsilon]_2((0))$ was thus completely solved by Thomae, the same problem for any rational characteristic $[\epsilon]_r (r > 2)$ presents extreme difficulties. In another paper* Thomae treated and solved this problem for the functions $\theta[\epsilon]_3((0))$ in the elliptic case.

---

* "Theorie der Abelschen Funktionen," *Crelle's Journal*, Vol. 54 (1857), p. 151.

† "Bestimmung von $d \log \theta(0, 0, \cdot \cdot \cdot, 0)$ durch die Klassenmoduln, *Crelle's Journal*, Vol. 66 (1866), pp. 92-96; "Beitrag zur Bestimmung von $\theta(0, 0, \cdot \cdot \cdot, 0)$ durch die Klassenmoduln algebraischer Functionen," *Crelle's Journal*, Vol. 71 (1870), pp. 201-222.

‡ "Ueber die Form der Argumente der Thetafunctionen und über die Bestimmung von $\theta(0, 0, \cdot \cdot \cdot, 0)$ als Function der Klassenmoduln," *Crelle's Journal*, Vol. 73 (1871), pp. 305-323; "Ueber die linearen Differentialgleichungen, welchen die Periodicitätsmoduln der Abelschen Integrale genügen, und über verschiedene Arten von Differentialgleichungen für $\theta(0, 0, \cdot \cdot \cdot, 0)$" *Ibid.*, pp. 324-339.

Without trying to write explicitly the expression of $\theta[\epsilon]_r((0))$, we want to investigate in this paper the nature of these functions limiting the argument to the hyperelliptic case. Especially we want to connect these functions with certain invariants of the polynomial $f(x)$ in equation (1), the vanishing of which admits an immediate geometric and algebraic interpretation. We prefer to explain our argument with a concrete instructive example:

Given a quartic plane curve C_4 without double points, one can draw 12 tangents to the curve from a generic point of the plane. However, if 12 lines of a pencil are taken in an arbitrary way, and if we look for a curve C_4 which should touch these lines (we put 12 conditions, while the quartics depend upon 14 arbitrary constants), no *irreducible* quartic will satisfy the required conditions. All solutions of the problem are furnished by degenerate C_4's which contain a double component. Therefore the 12 tangents to a C_4 constitute a particular projective set Γ_{12}. Indeed, the projectively distinct C_4's are ∞^6, hence the projectively distinct Γ_{12}'s of tangents are ∞^8. We deduce that these Γ_{12}'s have to satisfy *one* condition. From an algebraic point of view the peculiarity of the set of 12 tangents to a C_4 depends upon the fact that the discriminant $D(x)$ of the quartic in y is of the form

$$D(x) = i^3 - \lambda j^2, \tag{2}$$

where i and j are polynomials of degree 4 and 6 respectively, and λ is a constant factor. Now, the right-hand member of (2) contains 11 arbitrary constants; hence a generic polynomial of degree 12 cannot be represented as a sum of a square and a cube.

An intrinsic geometric explanation of the impossibility of constructing an irreducible C_4 which should touch 12 arbitrary lines of a pencil, is the following: The lines of a pencil cut out on the C_4 a linear involution g_4^1, which is not complete, because it is contained in the g_4^2 cut out by the lines of the plane. On the other hand, if we assign in advance the 12 branch points of a 4-valued algebraic function of genus 3, the corresponding $g_4{}^1$ on the quartic curve will be complete in general; i. e. for an arbitrary position of the branch points. Hence this $g_4{}^1$ will not be cut out by the lines of a pencil, but by conics through 4 points on the curve.

---

* "Darstellung des Quotienten zweier Thetafunctionen, deren Argumente um Drittel ganzer Periodicitätsmoduln unterscheiden, durch algebraische Functionen," *Mathematische Annalen*, Vol. 6 (1873), pp. 603-612.

We observe that the particular Γ_{12}'s considered above, are also furnished by the 12 cubics with a double point belonging to a pencil of cubics, and also by the 12 tangents drawn to an hyperelliptic C_6 with a triple point from a point P on the curve. Indeed, the consideration of the cubic surface represented on a doubly-covered plane with a branch quartic curve C_4 leads to the obvious conclusion that to the 12 tangents to the C_4 correspond, on the surface, 12 cubics with a double point belonging to a pencil of plane sections. To prove the second remark we suppose that the point P is at infinity on the y-axis, and that the triple point is at the origin $x=y=0$. The equation of C_5 is of the fourth degree in y, and the discriminant $D(x)$, of degree 18, is still of the form $D(x)=i^3-\lambda j^2$. It is easily seen that $x=0$ is a 6-fold root of $D(x)=0$, and a double and triple root of $i=0$, $j=0$ respectively. Putting $D(x)=x^6D_1(x)$, $i=x^2i_1$, $j=x^3j_1$, we have $D_1(x)=i_1{}^3-\lambda j_1{}^3$, where the 12 roots of $D_1(x)$ correspond to the 12 tangents parallel to the y-axis.

The algebraic condition that a polynomial $f(x)$ should be of the form $i^3-\lambda j^2$ is expressed by the vanishing of a certain invariant, which, however, it would not be easy to write explicitly. But a transcendental relation can be obtained by the following considerations:

We consider the hyperelliptic curve C of genus 5

$$x_3{}^2\phi(x_1,x_2)=\psi(x_1,x_2), \tag{3}$$

where $\phi(x_1,x_2)$, $\psi(x_1,x_2)$ are binary forms of degree 5 and 7 respectively, and $\phi\psi=x_2{}^{12}f(x_1/x_2)$. The curve C possesses a 5-fold $x_1=x_2=0$, the principal tangents at this point being given by the equation $\phi(x_1,x_2)=0$. Let us consider the linear system ∞^7 of curves

$$x_3\phi(x_1,x_2)=u(x_1,x_2), \tag{4}$$

where $u(x_1,x_2)$ is an arbitrary binary form of degree 6. The curves of this system cut out on C a complete series $g_{12}{}^7$, which, it may be observed, is the 6-fold of the hyperelliptic series $g_2{}^1$ cut out on C by the lines through the point $x_1=x_2=0$. Eliminating x between (3) and (4) we obtain:

$$\phi(x_1,x_2)\psi(x_1,x_2)-u^2(x_1,x_2)=0.$$

If, then, there exists a form u such that the first member of this equation becomes a perfect cube, the corresponding curve of the system (4) will have 4 contacts of 2nd order with the curve C. In other words, there will be a group of the $g_{12}{}^7$ composed of 4 triple points, and this set of 4 points will not be made up of two pairs belonging to the $g_2{}^1$. Conversely, if such a trisection of the $g_{12}{}^7$ is possible, then the form $\phi(x_1, x_2)\psi(x_1, x_2)$, or the polynomial $f(x)$, can be put into the form of the sum of a square and a cube.

Let us consider now the normal abelian integrals $u_\sigma(\sigma = 1, 2, \ldots, 5)$ of the curve C. The sum of the values of u_σ is constant for all groups of the $g_{12}{}^7$. If, according to the convention made above, the integrals vanish at the branch point k_{11}, this constant sum is equal to zero. If the series $g_{12}{}^7$ contains a group made up of 4 triple points $\eta_1, \eta_2, \eta_3, \eta_4$, we will have

$$3\{u_\sigma(\eta_1) + u_\sigma(\eta_2) + u_\sigma(\eta_3) + u_\sigma(\eta_4)\} \equiv 0 \ (\operatorname{mod} a_{\sigma j}).$$

Hence

$$u_\sigma(\eta_1) + u_\sigma(\eta_2) + u_\sigma(\eta_2) + u_\sigma(\eta_4) \equiv \{\epsilon\}_3. \tag{5}$$

By a well-known theorem of Riemann,* $\theta((u_\sigma(\eta_1) + u_\sigma(\eta_2) + \cdots + u_\sigma(\eta_{p-1}) + K_\sigma))$ vanishes for every position of the $p - 1$ points η_i. Here the K_σ's are the Riemannian constants. In the hyperelliptic case, the retrosections a_i, b_i being chosen as before, we have †

$$K_\sigma \equiv \{1, 1, \cdots, 1; p, p-1, \cdots, 1\}_2 = \{1^p; p, p-1, \cdots, 1\} = \{\kappa\}_2.$$

From (5) we deduce then, for $p = 5$, $\theta((\{\epsilon\}_3 + \{\kappa\}_2)) = 0$.

Since $\theta[g; h]((u_\sigma))$ differs from $\theta((u_\sigma + \{g; h\}))$ by a non-vanishing factor only, we can write the last relation thus

$$\theta[\{\kappa\} + \{\epsilon\}_3]((0)) = 0, \tag{6}$$

where by $[\kappa + \{\epsilon\}_3]$ we denote the sum of the characteristics (κ), $(\epsilon)_3$. The relation (6) expresses in transcendental form the required condition. The left-hand member is a function of the branch points k_i. We obtain a symmetric rational function of the k_i's by taking the product of all functions $\theta[\kappa + \{\epsilon\}_3]((0))$ relative to the $3^{10} - 1$ different characteristics $[\epsilon]_3$. This product will contain the desired invariant of the polynomial $f(x)$ as a factor. Starting from these facts which will be developed in the sequel, we will be able to calculate the degree of this invariant.

2. *Interpretation of the relation $\theta[\kappa + \{\epsilon\}_r]((0)) = 0$ for any value of p and r.* Suppose that the moduli of the curve of equation (1) satisfy the relation

$$\theta[\kappa + \{\epsilon\}_r]((0)) = 0, \tag{7}$$

---

* See, for instance (Kr.), p. 427.

† (Kr.), p. 459.

where $\kappa = [\kappa]_2$ is the characteristic $[1^p;\ p,\ p-1, \cdots, 1]$, and $[\epsilon]_r$ is a rational characteristic. We deduce then that there exist $p-1$ points $\eta_1, \ldots, \eta_{p-1}$ on C such that

$$\sum_{1}^{p-1}{}_{\nu}\, u_\sigma(\eta_\nu) \equiv \sum_{1}^{p-1}{}_{\mu}\, (\epsilon_\mu/r)a_{\sigma\mu} + (\epsilon_\sigma'/r)\pi i, \quad (\sigma = 1, 2, \cdots, p).$$

Hence $r\sum_{1}^{p-1}{}_{\nu}\, u_\sigma(\eta_\nu) \equiv 0.$

We consider two cases: $r(p-1)$ is 1) even, 2) odd.

1) $r(p-1) = 2m$. In this case the complete series $|g_{2m}|$ determined by the set of the $p-1$ r-fold points η_i is the m-fold of the hyperelliptic series g_2^1. We transform C into a curve $\bar{C}$ of order $p+2$ with a p-fold point at the origin. The equation of $\bar{C}$ in homogeneous coordinates will be the following:

$$x_3^2\phi_p(x_1, x_2) = \psi_{p+2}(x_1, x_2), \tag{8}$$

where ϕ_p and ψ_{p+2} are binary forms of degree p and $p+2$ respectively, and

$$\phi_p(x_1, x_2)\,\psi_{p+2}(x_1, x_2) = x_2^{2p+2}f(x_1/x_2) = f(x_1, x_2).$$

The complete series mentioned is a g_{2m}^{2m-p} and is cut out on $\bar{C}$ by the curves of the system

$$x_3\phi_p(x_1, x_2)\,\phi_{m-p-1}(x_1, x_2) = \psi_m(x_1, x_2), \tag{9}$$

where ϕ_{m-p-1} and ψ_m are arbitrary binary forms of degree $m-p-1$, m respectively. Indeed, the system (9) contains degenerate curves $\psi_m = 0$, composed of m arbitrary lines through the origin $x_1 = x_2 = 0$. Eliminating x_3 between (8) and (9) we obtain an equation of degree $2m$:

$$\phi_p\psi_{p+2}\cdot\phi_{m-p-1} - \psi^2{}_m = 0.$$

Since the series contains a set made up of $p-1$ r-fold points, the left-hand member of the last equation becomes a perfect r-power, when ϕ_{m-p-1}, ψ_m are conveniently chosen. We have then the following result:

If relation (7) *holds, and if* $r(p-1) = 2m$, *then it is possible to find* 3 *binary forms* $\phi_{m-p-1}, \psi_m, \psi_{p-1}$, *the degrees of which are indicated by the subscripts, such that*

$$\phi^2{}_{m-p-1}f = \psi^2{}_m + \psi^r{}_{p-1}. \tag{10}$$

A direct computation of constants shows that (10) implies effectively *one* condition on the coefficients of f. If $r = 3$ and $p = 5$, we have the case of the 12 tangents to a C_4, considered above. An analogous geometric interpretation can be given for any odd p and $r = 3$. Consider a curve L of order $(p + 3)/2$ possessing a $(p - 5)/2$-fold point P and $(p - 5)/2$ double points Q_i. The genus of L is $p - 2$. The lines through P cut out a $g_4{}^1$ on L, whence $2p + 2$ tangents can be drawn to the curve through P. The set of these tangents satisfy one condition, given by (10). Indeed, we can suppose that the equation of L is of degree 4 in one of the variables, say y. The discriminant $D(x)$ is the sum of a square and a cube, and at the same time $D(x)$ is the product of the square of a polynomial of degree $m - p - 1 = (p - 5)/2$ (which corresponds to the $(p - 5)/2$ lines joining P to the double points Q_i) and the polynomial of degree $2p + 2$.

2) Suppose now that $r(p - 1)$ is odd and put $r(p - 1) = 2m + 1$. In this case we obtain the complete series $| g_{2m+1} |$ determined by the $p - 1$ r-fold points η_i, by adding to the m-fold of the $g_2{}^1$ the common lower limit k_{2p+1} of the integrals u_σ. We consider again the transformed curve $\bar{C}$ of equation (8), and we suppose that $\phi_p(x_1, x_2)$ contains the factor $x_1 - k_{2p+1}x_2$.

The complete series mentioned above is a g_{2m+1}^{2m+1-p} and is cut out on $\bar{C}$ by the curves of the system

$$(9') \qquad [x_3\phi_p(x_1, x_2)/(x_1 - k_{2p+1}x_2)] \cdot \phi_{m-p}(x_1, x_2) = \psi_m(x_1, x_2),$$

where ϕ_{m-p} and ψ_m are arbitrary binary forms of degree $m - p$ and m respectively. Eliminating x_3 between (8) and (9′) we obtain an equation of degree $2m + 1$:

$$\phi^2{}_{m-p}\phi_p\psi_{p+2}/(x_1 - k_{2p+1}x_2) - (x_1 - k_{2p+1}x_2)\psi_m{}^2 = 0.$$

We deduce, as before, that it is possible to find 3 binary forms $\phi_{m-p}, \psi_m, \psi_{p-1}$ such that

$$\phi^2{}_{m-p}f/(x_1 - k_{2p+1}x_2) = (x_1 - k_{2p+1}x_2)\psi^2{}_m + \psi^r{}_{p-1}.$$

In this relation the roots of the polynomial f do not appear symmetrically. Hence we can draw the conclusion in advance that if $r(p - 1)$ is an odd number, the $r^{2p} - 1$ functions

$$\theta[\kappa + \{\epsilon\}_r]((0))$$

are permuted among themselves only by those substitutions of the branch points which leave fixed the point k_{2p+1}. This statement will be proved later in a direct way.

3. *The formulas of Thomae.* In view of later applications we develop here briefly the main result stated by Thomae in the second of his cited papers.

The values of the integrals u_σ at the branch points $k, k_1, \cdots, k_i, \cdots,$

k_{2p+1} constitute systems of half-periods $\{a_i\}_2$, the characteristics of which are. indicated by the following table: [*]

$$(11) \quad \begin{cases} (a) = (1, 0^{p-1}; 0^p); & (a_1) = (1, 0^{p-1}; 1, 0^{p-1}); \\ (a_2) = (0, 1, 0^{p-2}; 1, 0^{p-1}); & (a_3) = (0, 1, 0^{p-2}; 1, 1, 0^{p-2}); \\ \quad \vdots \qquad\qquad \vdots & \quad \vdots \qquad\qquad \vdots \\ (a_{2\rho-2}) = (0^{\rho-1}, 1, 0^{p-\rho}; 1^{\rho-1}, 0^{p-\rho+1}); & (a_{2\rho-1}) = (0^{\rho-1}, 1, 0^{p-\rho}; 1^{\rho}, 0^{p-\rho}); \\ \quad \vdots \qquad\qquad \vdots & \quad \vdots \qquad\qquad \vdots \\ (a_{2p-2}) = (0^{p-1}, 1; 1^{p-1}, 0); & (a_{2p-1}) = (0^{p-1}, 1; 1^p); \\ (a_{2p}) = (0^p; 1^p); & (a_{2p+1}) = (0^p; 0^p). \end{cases}$$

Here, for instance, $(a_{2\rho-2}) = (0^{\rho-1}, 1, 0^{p-\rho}; 1^{\rho-1}, 0^{p-\rho+1})$ means that

$$u_\sigma(k_{2\rho-2}) \equiv a_{\sigma\rho}/2 + \mu_\sigma \pi i/2,$$

where $\mu_\sigma = 1$, if $\sigma < \rho$, and $\mu_\sigma = 0$, if $\sigma \geqq \rho$.

We point out that the $2p + 1$ characteristics (a_i) $(i = 0, 1, \cdots, 2p)$ constitute a fundamental system and that the $(a_{2\nu})$ are even characteristics, while the $(a_{2\nu-1})$ are odd. Moreover we have

$$(11') \qquad (a) + (a_2) + \cdots + (a_{2p}) = (a_1) + (a_3) + \cdots + (a_{2p-1}) \\ = (1^p; p, p-1, \cdots, 1) = (\kappa).$$

Let us consider the following p integrals of the first kind

$$W_\sigma = \int (x^{\sigma-1}/y)\, dx,$$

and let

$$A_{\sigma\mu} = \int_{b\mu} (x^{\sigma-1}/y)\, dx,$$

be the period of W_σ relative to the retrosection b_μ.

We denote by $D(k, k_2, \cdots, k_{2p})$ the product of all differences $(k_{2\rho} - k_{2\rho'})$, where $\rho < \rho'$, and likewise by $D(k_1, k_3, \cdots, k_{2p+1})$ the product of all differences $(k_{2\rho+1} - k_{2\rho'+1})$, where $\rho < \rho'$. Thomae proves the following formula:

$$(12) \qquad \theta((0)) = (|A_{\sigma\mu}|/(2\pi i)^p)^{1/2} \\ \times [D(k, k_2, \ldots, k_{2p}) \cdot D(k_1, k_3, \ldots, k_{2p+1})]^{1/4},$$

---

[*] See (Kr.), p. 448. In this table the denominator 2 is omitted.

where $|A_{\sigma\mu}|$ denotes the determinant of the periods $A_{\sigma\mu}$. Let the branch points be distributed in two sets

$$k_{i_1}, k_{i_2}, \cdots, k_{i_{p+1}};$$

$$k_{j_1}, k_{j_2}, \cdots, k_{j_{p+1}};$$

where $i_1 < i_2 < \ldots < i_{p+1}$; $j_1 < j_2 < \ldots . < j_{p+1}$. We have

$$\sum_{1}^{p+1}{}_{\nu}\, u_\sigma(k_{i_\nu}) \equiv \sum_{1}^{p+1}{}_{\nu}\, u_\sigma(k_{j_\nu}) \equiv \sum_{1}^{p+1}{}_{\nu}\, \{a_{i_\nu}\} = \{\epsilon\}_2.$$

Thomae finds the following formula, analogous to (12):

$$(12') \qquad \theta[\epsilon]_2((0)) = (|A_{\sigma\mu}|/(2\pi i)^p)^{1/2} \times [D(k_{i_1}, k_{i_2}, \cdots, k_{i_{p+1}}) \cdot D(k_{j_1}, k_{j_2}, \cdots, k_{j_{p+1}})]^{1/4}.$$

The formulas (12) and (12′) express as functions of the branch points all those functions $\theta[\epsilon]_2((0))$, which do not vanish identically. Their number is, as known, equal precisely to $\binom{2p+1}{p}$.

4. $\theta[\epsilon]_r((0))$ *as function of the branch points.* Let $\theta[\epsilon]_r((0))$ be any θ-function with a rational characteristic, and let us consider the ratio

$$\theta[\epsilon]_r((0))/\theta((0)).$$

We prove that it is an *algebraic* function of the branch points. Indeed, if $\delta, \delta_1, \delta_2, \ldots, \delta_p$ are $p+1$ arbitrary points of the curve C, and if we have at $\delta_i : x = x_i, y = y_i$, the ratio

$$\theta[\epsilon]_r((u_\sigma(\delta) - \sum_{1}^{p}{}_i\, u_\sigma(\delta_i) - K_\sigma))/\theta((u_\sigma(\delta) - \sum_{1}^{p}{}_i\, u_\sigma(\delta_i) - K_\sigma))$$

is the rth root of a rational function Q of each of the points (x, y); (x_i, y_i), which, as function of δ, becomes ∞ of order r at the p points $\delta_1, \ldots, \delta_p$, and 0 of order r at the p zero's $\eta_1, \ldots, \eta_p$ of the numerator. The construction of such a function requires algebraic operations only. Let a function $R(x, y; x_1, y_1; \ldots; x_p, y_p)$ which is to satisfy the same conditions as Q, be constructed algebraically. Then R will depend algebraically upon the branch points, and will differ from Q by a factor independent of x and y:

$$Q = B(x_1, y_1; \ldots; x_p, y_p) R(x, y; x_1, y_1; \ldots; x_p, y_p).$$

We have only to prove that B, which is a rational function of (x_i, y_i), depends algebraically upon the branch points. This is shown by the following method of calculating B, indicated by Riemann.[*] We interchange, in the expression of Q, δ and δ_1, and substitute instead of $\delta_1, \ldots, \delta_p$ the point η_1 and the points $\bar{\eta}_2, \ldots, \bar{\eta}_p$, which are the conjugates of $\eta_2, \ldots, \eta_p$ in the hyperelliptic $g_2{}^1$. We have then, assuming that at $\eta_i : x = x_i', y = y_i'$:

$$Q(x_1', y_1'; x, y; x_2', -y_2'; \cdots; x_p', -y_p')$$

$$= \frac{\theta[\epsilon]_r((u_\sigma(\eta_1) - u_\sigma(\delta) - \sum_2^p{}_i\, u_\sigma(\bar{\eta}_i) - K_\sigma))}{\theta((u_\sigma(\eta_1) - u_\sigma(\delta) - \sum_2^p{}_i\, u_\sigma(\bar{\eta}_i) - K_\sigma))}.$$

Due to the relations

$$\sum_2^p{}_\nu\, u_\sigma(\bar{\eta}_\nu) + K_\sigma \equiv - \sum_2^p{}_\nu\, u_\sigma(\eta_\nu) - K_\sigma,$$

$$\sum_1^p{}_\nu\, u_\sigma(\eta_\nu) \equiv \sum_1^p{}_\nu\, u_\sigma(\delta_\nu) - \sum_1^p{}_\mu (\epsilon_\mu/r) a_{\mu\sigma} - (\epsilon_\sigma'/r)\pi i,$$

the last ratio can be transformed, and the result is that

$$Q(x_1', y_1'; x, y; x_2', -y_2'; \cdots; x_p', -y_p')$$

$$= \omega / Q(x, y; x_1, y_1; \cdots; x_p, y_p),$$

where ω is a root of unity. Hence

$$\omega/B^2 = R(x, y; x_1, y_1; \cdots; x_p, y_p)\; R(x_1', y_1'; x, y; x_2', -y_2'; \cdots; x_p', -y_p'),$$

and therefore B depends algebraically upon the branch points.

Suppose now that the points δ, δ_i coincide with the branch points $k_{2p+1}, k_1, k_3, \ldots, k_{2p-1}$ respectively. We obtain then, by means of relation (11′) of the preceding paragraph,

$$\theta[\epsilon]_r((0))/\theta((0))$$

$$= \omega'[B(k_1, 0; k_3, 0; \cdots; k_{2p-1}, 0) \cdot R(k_{2p+2}, 0; k_1, 0; \cdots; k_{2p-1}, 0)]^{1/r},$$

where ω' is a root of unity, and this proves that the left-hand member is an algebraic function of the branch points.

Hence, from the formula of Thomae (12) we deduce that any $\theta[\epsilon]_r((0))$ *is the product of the transcendental factor* $|A_{\sigma\mu}|^{1/2}$ *and of an algebraic function of the branch points.*

The number of all functions $\theta[\epsilon]_r((0))$, if the characteristic is reduced modulo r, is equal to $r^{2p}-1$, if r is a prime number. If r is composite, and $q_1, q_2, \ldots, q_\nu$ are its distinct prime factors, then the said number, which we denote by N_r, is given by the formula

---

[*] See the cited paper, "Theorie der Abelschen Functionen," pp. 154-155.

$$(13) \qquad N_r = \phi(r) - \sum \phi(r/q_1) + \sum \phi(r/q_1q_2) \\ - \cdots + (-1)^\nu \phi(r/q_1q_2\cdots q_\nu),$$

where $\phi(\alpha) = \alpha^{2p}-1$.

Suppose now that the set of the branch points varies continuously, starting from an initial position and returning to it, the result being a permutation (perhaps the identity) of the branch points. What is the effect on a given $\theta[\epsilon]_r((0))$ of such a cyclic variation of the branch points? Are the N_r functions $\theta[\epsilon]_r((0))$, or rather their algebraic factors, permuted among themselves? The answer is negative. A continuous cyclic variation of the set of the branch points defines a transformation of the retrosections a_i, b_i, and hence a transformation of the periods. Any $\theta[\epsilon]_r((0))$ is transformed, to within a transcendental factor *independent of the characteristic* (and therefore due to the presence of the factor $|A_{\sigma\mu}|^{1/2}$ in the expression of $\theta[\epsilon]_r((0))$), into a new function $\theta[\tilde{\epsilon}]_{\bar{r}}((0))$. The characteristics are subjected to a linear *non-homogeneous* transformation: there is an additive term, the effect of which is the addition of a characteristic $[\eta]_2$.* Hence generally $\bar{r} \neq r$.

One may have the impression that in order to obtain a closed field of θ-functions, it is necessary to consider all functions $\theta[\{\eta\}_2 + \{\epsilon\}_r]((0))$, where $\{\eta\}_2$ runs through all 4^p characteristics $[\eta]_2$. However, a transformation of the periods defined by a continuous cyclic variation of the set of the branch points, is not the most general of its kind. As a matter of fact a closed field of θ-functions is obtained by considering a smaller number of them. It is convenient for our purpose to represent any characteristic in the form $[\kappa + \{\epsilon\}_r]$, where $[\kappa]$ is the fixed characteristic $[1^p; p, p-1, \ldots, 1]_2$.

The function $\theta[\kappa + \{\epsilon\}_r]((u_\sigma(\eta)))$ vanishes at p points $\eta_1, \ldots, \eta_p$ such that

$$\sum_1^p {}_i\, u_\sigma(\eta_i) \equiv -\sum_1^p {}_\nu\, (\epsilon_\nu/r)\cdot a_{\sigma\nu} - (\epsilon_\sigma'/r)\cdot \pi i.$$

Hence

$$(14) \qquad r\sum_1^p {}_i\, u_\sigma(\eta_i) \equiv 0 \equiv rp\cdot u_\sigma(k_{2p+1}).$$

We consider first a cyclic variation of the set of the branch points, which leaves fixed the lower limit k_{2p+1} of the integrals u_σ. Then the transformed integrals $\bar{u}_\sigma$ are homogeneous linear combinations of the old integrals u_σ. Let $\theta[\kappa+\{\bar{\epsilon}\}_r]((u_\sigma(\eta)))_{(a)}=\theta[\kappa+\{\epsilon\}_r]((\bar{u}_\sigma(\eta)))_{(\bar{a})}$ be the transformed θ-function. Here the subscript $(\bar{a})$ indicates that the θ-function is defined by the new periods $\bar{a}_{ij}$. Then $\theta[\kappa+\{\epsilon\}_r]((0))_{(a)}$ is transformed into

---

* See (Kr.), chap. 5, especially p. 166, formula (XXVIII), and p. 181. Compare also Krazer and Prym, *Neue Grundlagen einer Theorie der allgemeinen Thetafunctionen*, Leipzig (1892), 2nd part, chap. 6.

$$\theta[\kappa+\{\bar{\epsilon}\}_{\bar{r}}]((0))_{(a)}.$$

The p zeros, say $\bar{\eta}_1, \ldots, \bar{\eta}_p$, of the transformed function

$$\theta[\kappa+\{\bar{\epsilon}\}_{\bar{r}}]((u_\sigma(\eta)))_{(a)}$$

satisfy the relations

$$\sum_1^p{}_i\, u_\sigma(\bar{\eta}_i) \equiv -\sum_1^p{}_\nu\, (\bar{\epsilon}_\nu/\bar{r})\, a_{\sigma\nu} - (\bar{\epsilon}_\sigma'/\bar{r})\pi i.$$

On the other hand, after the cyclic continuous variation of the branch points is completed, (14) becomes

$$r\sum_1^p{}_i\, u_\sigma(\bar{\eta}_i) \equiv rp\cdot u_\sigma(k_{2p+1}) \equiv 0.$$

Hence

$$r\,\{\bar{\epsilon}\}_{\bar{r}} \equiv \{0^p;\ 0^p\},$$

and we deduce $r=\bar{r}$.

Let us consider now a cyclic variation of the set of the branch points which carries k_{2p+1} into k_e. We have

$$u_\sigma(k_e) \equiv \sum_1^p{}_i\, (a_i/2)a_{\sigma i} + (a_\sigma'/2)\pi i,$$

where the integers a_i, a_i' are the elements of the characteristic (a_e) of the table (11). The transformed integrals $\bar{u}_\sigma$ are homogeneous linear combinations of the differences

$$u_\sigma - \sum_1^p{}_i\, (a_i/2)a_{\sigma i} - (a_\sigma'/2)\pi i.$$

The transformed function

$$\theta[\kappa+\{\epsilon\}_r]((\bar{u}_\sigma))_{(\bar{a})}$$

is, to within a factor, of the form

$$\theta[\kappa+\{\bar{\epsilon}\}_{\bar{r}}]((u_\sigma - \sum_1^p{}_i\, (a_i/2)a_{\sigma i} - (a_\sigma'/2)\pi i)), \tag{15}$$

and $\theta[\kappa+\{\epsilon\}_r]((0))$ is transformed into

$$\theta[\kappa+\{\bar{\epsilon}\}_{\bar{r}}]((0)).$$

The zero's $\bar{\eta}_i$ of the transformed function (15) satisfy the relations

$$\sum_1^p{}_i\, u_\sigma(\bar{\eta}_i) \equiv -\Big(\sum_1^p{}_\nu\, (\bar{\epsilon}_\nu/\bar{r}) a_{\sigma\nu} + (\epsilon_\sigma'/\bar{r})\pi i \Big) + \Big(\sum_1^p{}_\nu\, (a_\nu/2) a_{\sigma\nu} + (a_\sigma'/2)\pi i \Big).$$

On the other hand we have by means of (14),

$$r \sum_1^p{}_i\, u_\sigma(\bar{\eta}_i) \equiv rp \cdot u_\sigma(k_e).$$

Hence we deduce

$$r\,\{\bar{\epsilon}\}_r \equiv -r(p-1)\{a_e\}_2.$$

If $r(p-1)$ is even, we have $r\{\bar{\epsilon}\}_r \equiv \{0^p; 0^p\}$, and hence $\bar{r} = r$.

If $r(p-1)$ is odd, we have $r\{\bar{\epsilon}\}_{\bar{r}} \equiv \{a\}_2$, hence $\{\bar{\epsilon}\}_{\bar{r}}$ is of the form $\{\eta\}_r + \{a\}_2$.

We have thus the following result:

If $r(p-1)$ is even, the N_r functions $\theta[\kappa + \{\epsilon\}_r]((0))$ are permuted among themselves, to within factors, for any cyclic variation of the set of the branch points.

If $r(p-1)$ is odd, a closed field of θ-functions is given by the following $2p+2$ sets of functions:

$$(16)\quad \theta[\kappa + \{\epsilon\}_r]((0)),\ \theta[\kappa + a + \{\epsilon\}_r]((0)),\ \theta[\kappa + a_1 + \{\epsilon\}_r]((0)), \cdots, \theta[\kappa + a_{2p} + \{\epsilon\}_r]((0)).$$

Here $a, a_1, \cdots, a_{2p}$ are the characteristics indicated in the table (11). We foresaw this result above (§ 2) from algebraic considerations.

The number of the functions, contained in the sets (16), is $(2p+2)N_r$.

The number of distinct functions in the two cases is $N_r/2$, $(p+1)N_r$, respectively, since $\theta[g;h]((0)) = \theta[-g;-h]((0))$.

The functions $\theta[\kappa + a_i + \{\epsilon\}_r]((0))$ of each set are permuted among themselves by any permutation of the branch points, which leaves fix the branch point k_i. In particular, we deduce that in any transformation of the periods, defined by a permutation of the branch points, the linear transformation of the characteristics, reduced modulo 2,

1) leaves unaltered the characteristic $[\kappa]$, if p is odd;
2) permutes the $2p+2$ characteristics

$$[\kappa],\ [\kappa + a],\quad [\kappa + a_1], \cdots, [\kappa + a_{2p}],$$

among themselves, if p is even.

This is in accordance with the fact, that, if p is odd, $p = 2s - 1$, the function $\theta[\kappa]((v))$ and all its partial derivatives of order $1, 2, \cdots, s-1$ vanish for $v = 0$, while the partial derivatives of order $s - 1$ of any of the functions $\theta[\kappa + a_i]((0))$ do not vanish for $v = 0$. However, if p is even, $p = 2s$, the highest partial derivatives, which vanish, are in both cases of the same order, namely of order $s - 1$.

We put $N_r' = N_r$, if $r(p-1)$ is even, and $N_r' = (2p+2)N_r$, if $r(p-1)$ is odd, and we consider the product of all N_r' θ-functions considered above. We suppose that the characteristics are all reduced in such a way that their elements are numbers $\geqq 0$ and < 1. Denoting this product by $\Pi_r\theta$ we have

$$\prod\nolimits_r \theta = |A_{\sigma\mu}|^{N_r'/2} F_r(k, k_1, \cdots, k_{2p+1}), \tag{17}$$

where, due to the preceding considerations, F_r is an algebraic function of the k's, which is reproduced, to within a factor (a root of unity) by any permutation of the branch points. Hence, a convenient power of F_r is a rational symmetric function of the k's. We will prove later *that F_r itself is an integral symmetric function of the k's.* Denoting by I_r the invariant of the polynomial $f(x)$, the vanishing of which implies the vanishing of one of the θ-functions of the product, F_r will contain as factor a certain power of I_r. Moreover, F_r contains as factor a certain power of the discriminant D of the polynomial $f(x)$, because several θ-functions vanish also when two branch points come together. It is easy to calculate the degree of F_r and also to prove that F_r contains as factor only the square of I_r. It is more difficult to calculate the power of D which appears in F_r, in order to determine the degree of I_r. For this purpose we study in the next section the behavior of a generic function $\theta[g; h]((0))$, of any characteristic, when two branch points come together.

5. *The behavior of $\theta[g; h]((0))$ when two branch points come together.* Since F_r is a symmetric function of the branch points, it is sufficient to restrict our consideration to one pair of branch points k_i, k_j. We choose the pair k, k_1, and while we fix the position of $k_1, k_2, \cdots, k_{2p+1}$, we conceive $\theta[g; h]((0))$ as a function of k only, and make k approach k_1. The behavior of $\theta[g; h]((0))$ in the case of coincidence of any other pair k_i, k_j can be determined by considering the transformation of the periods implied by the permutation $(k\, k_i)(k_1 k_j)$.

We look first at the behavior of the periods a_{ij}, when $k \to k_1$. The curve C

$$y^2 = (x-k)(x-k_1) \cdots (x-k_{2p+1})$$

tends to the curve $\bar{C}$,

$$\bar{y}^2 = (x-k_1)^2(x-k_2)\cdots(x-k_{2p+1}),$$

of genus $p-1$. The $2p-2$ retrosections $b_2, \cdots, b_p$; $a_2, \cdots, a_p$ of the Riemann surface R of C remain the retrosections of the Riemann surface $\bar{R}$ of $\bar{C}$. We denote the two overlapping points of $\bar{R}$ at $x = k_1$, $\bar{y} = 0$ by δ_1, δ_2; δ_1 being on the upper sheet. Denoting by $\bar{u}_\sigma$ the abelian integrals of $\bar{C}$ of the first kind normal with respect to the retrosections b_i, a_i $(i = 2, \cdots, p)$ and which vanish at k_{2p+1}, it is seen at once that

$$\lim_{k\to k_1} u_\sigma = \bar{u}_\sigma \qquad (\sigma = 2, \cdots, p). \tag{18}$$

As far as u_1 is concerned, we have

$$\lim_{k\to k_1} u_1 = \tfrac{1}{2}\bar{\omega}_{\delta_1,\delta_2}, \tag{18'}$$

where $\bar{\omega}_{\delta_1,\delta_2}$ is the normal abelian integral of the third kind on $\bar{C}$, having δ_1, δ_2 as singular points. Inded we have

$$u_\sigma = \int \sum_1^p{}_i\, \alpha_i^{(\sigma)} x^{i-1} dx/y,$$

where $\alpha_i^{(\sigma)}$ is reciprocal of $A_{i\sigma}$ in the determinant $|A_{i\sigma}|$. If $\lim\limits_{k\to k_1} \alpha_i^{(\sigma)} = \bar{\alpha}_i^{(\sigma)}$, then

$$\lim_{k\to k_1} u_\sigma = \int \sum_1^p{}_i\, \bar{\alpha}_i^{(\sigma)} x^{i-1} dx/\bar{y}.$$

It can be proved directly that, if $\sigma \geqq 2$, the polynomial $\sum\limits_1^p{}_i\, \bar{\alpha}_i^{(\sigma)} x^{i-1}$ is divisible by $x - k_1$, and, hence, that the limit considered is an integral of the first kind. However it is sufficient to note that, since any integral u_σ $(\sigma \geqq 2)$ satisfies the relation $\int_{b_1} du_\sigma = 0$, the same relation holds for their limits, when b_1 is replaced by a small circle of center δ_1. Hence the integral limits cannot have logarithmic singularities at δ_1 or δ_2. Since, moreover, the period of $\lim\limits_{k\to k_1} u_\sigma$ relative to the retrosection b_ν is evidently πi, if $\nu = \sigma$, and 0 if $\nu \neq \sigma$, (18) follows. As far as (18′) is concerned, we observe that $\int_{b_1} du_1 = \pi i$, and passing to the limit, we deduce that $\lim\limits_{k\to k_1} u_1$ differs from $\omega_{\delta_1,\delta_2}$ only by the factor 1/2. Denoting by $\bar{a}_{\sigma r}$ the periods of $\bar{u}_\sigma$, we have then

$$\lim_{k\to k_1} a_{\sigma r} = \bar{a}_{\sigma r} \qquad (\sigma, r = 2, 3, \cdots, p) \tag{19}$$

By a well-known theorem on the periods of the normal integral of the third kind we deduce

$$(19') \qquad \lim_{k\to k_1} a_{1r} = \bar{u}_r(\delta_2) - \bar{u}_r(\delta_1) = 2\bar{u}_r(\delta_2) \qquad (r = 2, 3, \cdots, p).$$

As far as a_{11} is concerned, it assumes infinitely great values when $k \to k_1$ and it must be pointed out that the real part of a_{11} tends to $-\infty$.

It is easily seen that all periods $a_{\sigma r}$, except a_{11}, are monodromic functions of k in the neighborhood of k_1, and that a_{11} increases by $2\pi i$, when k turns around k_1. Indeed, when k turns around k_1, the retrosections are transformed as follows:

$$b_i' = b_i \ (i = 1, 2, \cdots, p); \quad a_1' = a_1 + 2b_1; \quad a_j' = a_j \ (j = 2, \cdots, p).$$

We have then the following developments in the neighborhood of k_1:

$$\begin{aligned} a_{\sigma r} &= \bar{a}_{\sigma r} + \alpha_1^{(\sigma r)}(k - k_1) + \alpha_2^{(\sigma r)}(k - k_1)^2 + \cdots \\ &\qquad\qquad\qquad\qquad\qquad\qquad (\sigma, r = 2, \cdots, p) \\ a_{1r} &= 2\bar{u}_r(\delta_2) + \alpha_1^{(1r)}(k - k_1) + \cdots \qquad\qquad (r = 2, \cdots, p) \\ a_{11} &= \log(k - k_1) + \alpha_0^{(11)} + \alpha_1^{(11)}(k - k_1) + \cdots \end{aligned}$$

It may be useful to calculate some more coefficients of these developments. We prove:

$$\left.\begin{aligned} &(20) \qquad \alpha_1^{(\sigma r)} = 0 \\ &(20') \qquad \alpha_2^{(\sigma r)} = \tfrac{1}{2}[d\bar{u}_\sigma/dz \cdot d\bar{u}_r/dz]_{z=\delta_2} \end{aligned}\right\} (\sigma, r = 2, \cdots, p)$$

$$(20'') \qquad \alpha_1^{(1r)} = [d\bar{u}_r/dz]_{z=\delta_2} \qquad (r = 2, \cdots, p)$$

Let us differentiate u_σ with respect to k. We have

$$\frac{du_\sigma}{dk} = \tfrac{1}{2}\int \sum_1^p{}_i\, \alpha_i^{(\sigma)}\, x^{i-1}dx/(x-k)y + \int \sum_1^p{}_i\, \frac{d\alpha_i^{(\sigma)}}{dk}\, x^{i-1}dx/y,$$

whence du_σ/dk is an integral of the second kind, having a pole of the first order at the branch point k. Moreover, du_σ/dk possesses zero differences along the retrosections a_r, and the differences $da_{\sigma r}/dk$ along the retrosections b_r $(r = 1, 2, \cdots, p)$.*

From this we draw the following conclusion, which will be the base of the following considerations, namely, that the *limit of* du_σ/dk $(\sigma = 1, 2, \cdots, p)$ *for* $k \to k_1$ *cannot possess logarithmic singularities at the points* δ_1, δ_2. The same conclusion holds for all derivatives of higher order.

---

* See Thomae, second of the cited papers, p. 211.

Put, for brevity

$$\sum_{1}^{p}{}_{i}\,\alpha_i^{(\sigma)}x^{i-1}=\phi_\sigma(x) \qquad \lim_{k\to k_1}\left(d^\lambda\phi_\sigma(x)/dk^\lambda\right)=\bar{\phi}_\sigma^{(\lambda)}(x)$$

$$\lim_{k\to k_1}\phi_\sigma(x)=\bar{\phi}_\sigma(x) \qquad \lim_{k\to k_1}\left(d^\lambda u_\sigma/dk^\lambda\right)=\bar{u}_\sigma^{(\lambda)}.$$

We have then

$$\bar{u}_\sigma^{(1)}=\tfrac{1}{2}\int\bar{\phi}_\sigma(x)\,dx/(x-k_1)\bar{y}+\int\bar{\phi}_\sigma^{(1)}(x)\,dx/\bar{y}. \tag{21}$$

We saw before that $\bar{\phi}_\sigma(x)$ is divisible by $x-k_1$, if $\sigma\geqq 2$. Hence, the points δ_1, δ_2 are not poles of $\bar{u}_\sigma^{(1)}$. Since, on the other hand, $\bar{u}_\sigma^{(1)}$ cannot possess logarithmic singularities, it follows that $\bar{u}_\sigma^{(1)}$ is everywhere regular on $\bar{R}$. Since $\bar{u}_\sigma^{(1)}$ possesses zero differences along the retrosections a_σ $(\sigma=2,\cdots,p)$, it follows that it is identically zero, and this proves the relation (20).

Consider now $\bar{u}_1^{(1)}$, which is an integral of the second kind, having two poles δ_1, δ_2. Since

$$\int\bar{\phi}_1(x)\,dx/\bar{y}=\tfrac{1}{2}\bar{\omega}_{\delta_1,\delta_2},$$

we deduce that the residues of $\bar{u}_1^{(1)}$ at δ_1 and δ_2 are $\frac{1}{4}$, $-\frac{1}{4}$ respectively. Hence, denoting by $\bar{t}_\delta$ a normal integral of the second kind, relative to the pole δ, we have

$$\bar{u}_1^{(1)}=\tfrac{1}{4}\bar{t}_{\delta_1}-\tfrac{1}{4}\bar{t}_{\delta_2}.$$

Since the constant difference of $\bar{t}_\delta$ along b_σ is $-2(d\bar{u}_\sigma/dz)_{z=\delta}$, we deduce

$$\lim_{k\to k_1}(da_{1\sigma}/dk)=-\tfrac{1}{2}(d\bar{u}_\sigma/dz)_{z=\delta_1}+\tfrac{1}{2}(d\bar{u}_\sigma/dz)_{z=\delta_2}=(d\bar{u}_\sigma/dz)_{z=\delta_2},$$
$$(\sigma=2,\cdots,p),$$

and this proves (20″).

Consider now $\bar{u}_\sigma^{(2)}$ for $\sigma\geqq 2$. We have

$$\bar{u}_\sigma^{(2)}=\tfrac{3}{4}\int\bar{\phi}_\sigma(x)\,dx/(x-k_1)^2\bar{y}+\int\bar{\phi}_\sigma^{(1)}(x)\,dx/(x-k_1)\bar{y}$$
$$+\int\bar{\phi}_\sigma^{(2)}(x)\,dx/\bar{y}.$$

Taking into consideration that $\bar{u}_\sigma^{(1)}$, given by (21), is identically zero, we transform $\bar{u}_\sigma^{(2)}$ and obtain

$$\bar{u}_\sigma^{(2)}=\tfrac{1}{4}\int\bar{\phi}_\sigma(x)\,dx/(x-k_1)^2\bar{y}+\int\bar{\phi}_\sigma^{(2)}(x)\,dx/\bar{y}.$$

We know in advance that the logarithmic terms in these two integrals eliminate each another, and hence $\bar{u}_\sigma^{(2)}$ possesses two poles of the first order at δ_1 and δ_2.

Since.

$$\bar{u}_\sigma = \int \bar{\phi}_\sigma(x)\,dx/\bar{y},$$

we deduce that the residues of $\bar{u}_\sigma^{(2)}$ at δ_1 and δ_2 are

$$-\tfrac{1}{4}(d\bar{u}_\sigma/dz)_{z=\delta_1},\ -\tfrac{1}{4}(d\bar{u}_\sigma/dz)_{z=\delta_2},$$

respectively. Hence

$$\bar{u}_\sigma^{(2)} \doteq -\tfrac{1}{4}(d\bar{u}_\sigma/dz)_{z=\delta_1}\cdot \bar{t}_{\delta_1} - \tfrac{1}{4}(d\bar{u}_\sigma/dz)_{z=\delta_2}\cdot \bar{t}_{\delta_2},$$

and consequently

$$\begin{aligned}\lim_{k\to k_1}\ (d^2 a_{\sigma r}/dk^2) &= \tfrac{1}{2}(d\bar{u}_\sigma/dz\cdot d\bar{u}_r/dz)_{z=\delta_1} + \tfrac{1}{2}(d\bar{u}_\sigma/dz\cdot d\bar{u}_r/dz)_{z=\delta_2}\\ &= (d\bar{u}_\sigma/dz\cdot d\bar{u}_r/dz)_{z=\delta_2} \qquad (\sigma, r = 2, \cdots, p);\end{aligned}$$

which proves (20′).

One can prove in the same way

$$\lim_{k\to k_1}\ (d^3 a_{\sigma r}/dk^3) = \tfrac{3}{2}[d/dz(d\bar{u}_\sigma/dz\cdot d\bar{u}_r/dz)]_{z=\delta_2}, \qquad (\sigma, r = 2, \cdots, p).$$

We observe, without insisting upon it, that while the calculated coefficients are expressed by means of the values which the integrals $\bar{u}_\sigma$ and their derivatives assume at the point δ_2, the coefficients of the higher order depend also upon the coefficients of the development of the integral of the second kind t_{δ_2} at the point δ_2.

We investigate now the behavior of any $\theta[g;h]((0))$ for $k\to k_1$.

In order not to interrupt the later exposition we prove two preliminary lemmas, referred to a curve C, hyperelliptic or not.

LEMMA I. *If C is of general moduli and if $\theta((mu_\sigma + e_\sigma))$, where m and e_σ are constants, is identically zero, then m is necessarily a rational number.*

Indeed, suppose that we have identically, i. e. for any point ξ on C,

$$\theta((mu_\sigma(\xi) + e_\sigma)) = 0. \tag{22}$$

Then we may write

$$mu_\sigma(\xi) + e_\sigma \equiv \sum_{1}^{p-1} u_\sigma(\eta_i) + K_\sigma,$$

where the K_σ's are the Riemannian constants. By considering a cyclic path for the point ξ, we deduce from (22)

$$\theta((mu_\sigma(\xi) + m\Omega_\sigma + e_\sigma)) = 0, \tag{22′}$$

where the Ω_σ's constitute an arbitrary system of simultaneous periods, and hence

$$(23) \qquad mu_\sigma(\xi) + m\Omega_\sigma + e_\sigma \equiv \sum_1^{p-1}{}_i \, u_\sigma(\eta_i') + K_\sigma.$$

Now, the infinite sets

$$(23') \qquad m\Omega_1, m\Omega_2, \ldots, m\Omega_p,$$

cannot be all distinct, modulo the periods. In fact, if they were, two systems of periods Ω, Ω' could be found, such that the numbers of the set

$$m\Omega_1 + \Omega_1', \; m\Omega_2 + \Omega_2', \ldots, \; m\Omega_p + \Omega_p'$$

would be infinitesimals, or else differ by infinitesimals from any set of numbers given in advance. Hence, by (22′), $\theta((v))$ would vanish identically, as function of the arguments v, which is impossible.

Suppose that between the different sets (23′) there are q distinct sets, modulo the periods. Then (23), for different systems Ω, and for a given ξ, leads to only q distinct sets of $p-1$ points η_i'. Denoting these $q(p-1)$ points by $\eta_i^{(j)}$ $(i = 1, 2, \ldots, p-1; j = 1, 2, \ldots, q)$, we have

$$\sum_1^q{}_j \sum_1^{p-1}{}_i \, u_\sigma(\eta_i^{(j)}) \equiv qmu_\sigma(\xi) + c_\sigma,$$

where the c_σ's are constants. We define thus an algebraic correspondence on C, on which to a point ξ correspond $q(p-1)$ points η. If C is of general moduli, this correspondence cannot be singular, and hence qm is an integer, i. e. m is a rational number.

A slight modification of this proof becomes necessary, if it happens that for any ξ and Ω the equation (23) does not determine uniquely the set of $p-1$ points η_i'. In this case, if s is the dimension of the complete series $|g_{p-1}|$ the sets of which satisfy equation (23), we have only to fix s of the points η_i': the remaining $p-s-1$ points are thus uniquely determined. One of the indices of the correspondence defined above becomes then $q(p-s-1)$.

We observe that the only condition required in the proof is that C should not possess singular correspondences. Since the general hyperelliptic curve satisfies this condition,[*] the lemma is valid also in the hyperelliptic case.

LEMMA II. *If m is a rational number and if $\theta[\epsilon]_r((mu_\sigma))$ is identically zero, then m is necessarily an integer and $r = 2$.*

---

[*] See A. Lefschetz, "A Theorem on Correspondences on Algebraic Curves," *American Journal of Mathematics*, Vol. 50, No. 2 (April, 1928), pp. 159-166.

It is enough to prove this statement for hyperelliptic curves. We suppose that the integrals u_σ have a common lower limit. Then, if $\theta[\epsilon]_r((mu_\sigma))$ is identically zero, we have also

$$\theta[\epsilon]_r((0))=0;$$

and this relation implies, as we saw above, a condition on the branch points, if $r>2$. Hence $r=2$. Now, let $m=q_1/q$. Then we deduce, as in (22′), that

$$\theta[\epsilon]_2((\{\eta\}_q))=0,$$

or,

$$\theta[\{\epsilon\}_2+\{\eta\}_q]((0))=0,$$

where $[\eta]_q$ is any arbitrary characteristic relative to the denominator q. Hence, reasoning as above, we deduce $q\leqq 2$. If $q=2$, then $[\{\epsilon\}_2+\{\eta\}_2]$ is substantially any characteristic $[\eta]_2$, and it is impossible that all 4^p functions $\theta[\eta]_2((0))$ should vanish. Hence $q=1$, and m is an integer.

We consider now a rational characteristic $[g;h]$ and study the behavior of the function $\theta[g;h]((0))$ when $k\rightarrow k_1$. Since the real part of a_{11} tends to $-\infty$, it follows that every term

$$\exp\,[\,\sum_1^p{}_{i,j}\,a_{ij}(m_i+g_i)(m_j+g_j)+2\sum_1^p{}_i\,(m_i+g_i)h_i\pi i]$$

tends to zero, unless $g_1=m_1=0$. Hence, if $g_1\neq 0$, we have

$$\lim_{k\rightarrow k_1}\theta[g;h]((0))=0.$$

But the functions $\theta[g;h]((0))$, where $g_1\neq 0$, are not the only ones which vanish for $k\rightarrow k_1$. Indeed, if $g_1=0$, we will have, by means of (19),

$$\lim_{k\rightarrow k_1}\theta[g;h]((0))=\bar{\theta}[g_2,\cdots,g_p;\,h_2,\cdots,h_p]((0)),$$

where $\bar{\theta}$ denotes functions θ relative to the limit curve $\bar{C}$ of genus $p-1$. If $[g_2,\cdots,g_p;\,h_2,\cdots,h_p]$ is a conveniently chosen characteristic $[\epsilon]_2$ (for instance, an odd characteristic), the second term of the last relation vanishes. In any case, if $\theta[g;h]((0))$ possesses a zero at $k=k_1$, we wish to determine the order of this zero. We adopt, for brevity, the notation

$$\phi\leq\psi,$$

to indicate that for $k\rightarrow k_1$ the limit of the ratio ϕ/ψ is a finite number. If this limit is $\neq 0$, we will write

$$\phi\sim\psi.$$

We saw above that

$$e^{a_{11}} \sim k - k_1.$$

We arrange all terms in the expression of $\theta[g;h]((0))$ with respect to m_1, and put

$$\sum_{-\infty}^{+\infty}{}_{(m_2,\ldots,m_p)} \exp\,[\,\sum_1^p{}_{ij}\, a_{ij}(m_i+g_i)(m_j+g_j) + 2\sum_1^p{}_i\,(m_i+g_i)h_i\pi i$$
$$= A_{m_1}\,\exp\,[a_{11}(m_1+g_1)^2 + 2(m_1+g_1)h_1\pi i].$$

We have then

$$\theta[g;h]((0)) = \sum_{-\infty}^{+\infty}{}_{m_1} A_{m_1}\,\exp\,[a_{11}(m_1+g_1)^2 + 2(m_1+g_1)h_1\pi i]. \tag{24}$$

Due to (19) and (19′) the limit of the general term of A_{m_1} for $k \to k_1$ is

$$\exp\,[\,\sum_2^p{}_{i,j}\,\bar{a}_{ij}(m_i+g_i)(m_j+g_j) + 2\sum_2^p{}_i\,(m_i+g_i)\,[2(m_1+g_1)\bar{u}_i(\delta_2) + h_i\pi i].$$

Hence

$$\lim_{k\to k_1} A_{m_1} = \bar{\theta}[g_2,\cdots,g_p;\,h_2,\cdots,h_p]((2(m_1+g_1)\bar{u}_\sigma(\delta_2))). \tag{25}$$

The general term of (24) has, at $k = k_1$, a zero of order $(m_1+g_1)^2$ at least. We obtain the smallest value of $(m_1+g_1)^2$ by putting $m_1 = 0$, if $g_1 < \frac{1}{2}$, and $m_1 = -1$, if $g_1 > \frac{1}{2}$. If $g_1 = \frac{1}{2}$, this minimum corresponds to both the values $0, -1$ of m_1.

Let $g_1 < \frac{1}{2}$. Then, due to (25),

$$\lim_{k\to k_1} \exp\,[-a_{11}g_1^2]\cdot\theta[g;h]((0))$$
$$= \exp\,[2g_1h_1\pi i]\cdot\bar{\theta}[g_2,\cdots,g_p;\,h_2,\cdots,h_p]((2g_1\bar{u}_\sigma(\delta_2)));$$

hence

$$\theta[g;h]((0)) \sim (k-k_1)^{g_1^2}, \tag{26}$$

unless

$$\bar{\theta}[g_2,\cdots,g_p;\,h_2,\cdots,h_p]((2g_1\bar{u}_\sigma(\delta_2))) = 0. \tag{27}$$

Likewise, we have for $g_1 > \frac{1}{2}$:

$$\lim_{k\to k_1} \exp\,[-a_{11}(1-g_1)^2]\cdot\theta[g;h]((0))$$
$$= \exp\,[-2(1-g_1)h_1\pi i]\cdot\bar{\theta}[g_2,\cdots,g_p;\,h_2,\cdots,h_p]$$
$$((-2(1-g_1)\bar{u}_\sigma(\delta_2)));$$

hence

$$\theta[g;h]((0)) \sim (k-k_1)^{(1-g_1)^2}, \tag{26′}$$

unless

$$(27') \qquad \bar\theta[g_2, \cdots, g_p; h_2, \cdots, h_p]((-2(1-g_1)\bar u_\sigma(\delta_2))) = 0.$$

Finally, if $g_1 = \frac{1}{2}$, we have

$$\begin{aligned}\lim_{k\to k_1} \exp[-a_{11}/4]\cdot\theta[g;h]((0)) &= \exp[h_1\pi i]\bar\theta[g_2, \cdots, g_p; h_2, \cdots, h_p]((\bar u_\sigma(\delta_2))) + \\ &+ \exp[-h_1\pi i]\bar\theta[g_2, \cdots, g_p; h_2, \cdots, h_p]((-\bar u_\sigma(\delta)));\end{aligned}$$

hence

$$(26'') \qquad \theta[g;h]((0)) \sim (k-k_1)^{1/4},$$

unless

$$\begin{aligned}(27'') \qquad &\exp[h_1\pi i]\cdot\bar\theta[g_2, \cdots, g_p; h_2, \cdots, h_p]((\bar u_\sigma(\delta_2))) + \\ &+ \exp[-h_1\pi i]\cdot\bar\theta[g_2, \cdots, g_p; h_2, \cdots, h_p]((-\bar u_\sigma(\delta_2))) = 0.\end{aligned}$$

Since the branch points k_i are independent of each other, the equalities (27), (27′), (27″) have to be considered as identities, i. e. as valid for any point δ_2 on $\bar R$. Hence, from the Lemma II we deduce, that (27) and (27′) are possible only if $[g_2, \cdots, g_p; h_2, \cdots, h_p]$ is a $[\epsilon]_2$ and $g_1 = \frac{1}{2}$ or 0.

We prove that (27″) implies also that $[g_2, \cdots, g_p; h_2, \cdots, h_p]$ should be a $[\epsilon]_2$ and that $\theta[g_2, \cdots, g_p; h_2, \cdots, h_p]((\bar u_\sigma(\delta)))$ itself should vanish identically. In fact, suppose that $[g_2, \cdots, g_p; h_2, \cdots, h_p] = [\epsilon]_r$, and $r > 2$. The function $\bar\theta[g; h]((\bar u_\sigma(\delta)))$ cannot vanish identically, if the branch points of $\bar C$ are in arbitrary position (Lemma II). For the same reason $\bar\theta[g;h]((-\bar u_\sigma(\delta))) = \bar\theta[-g;-h]((\bar u_\sigma(\delta)))$ does not vanish identically. Each one of them possesses therefore $p-1$ zeros, say $\delta_1, \cdots, \delta_{p-1}$; $\delta_1', \cdots, \delta_{p-1}'$ respectively. We have

$$\sum_{1}^{p-1}{}_i\ \bar u_\sigma(\delta_i) \equiv -\sum_{2}^{p}{}_j\ g_j\bar a_{\sigma j} - h_\sigma\pi i + \bar K_\sigma,$$

$$\sum_{1}^{p-1}{}_i\ \bar u_\sigma(\delta_i') \equiv \sum_{2}^{p}{}_j\ g_j\bar a_{\sigma j} + h_\sigma\pi i + \bar K_\sigma,$$

where the $\bar K_\sigma$'s are the Riemannian constants, i. e. a system of half periods of characteristic $[1^{p-1}; p-1, p-2, \cdots, 1]$. If now (27″) holds, the zeros of both functions coincide, and therefore

$$2[\sum_{2}^{p}{}_j\ g_j\bar a_{\sigma j} + h_\sigma\pi i] \equiv 0, \qquad (\sigma = 2, 3, \cdots, p).$$

Hence $\{g_2, \cdots, g_p; h_2, \cdots, h_p\}$ is a half period, which contradicts our hypothesis. Hence $r = 2$.

We prove now that $\bar{\theta}[g;h]((\bar{u}_\sigma(\delta)))$ vanishes identically. In fact, let us suppose in the first place, that $[g_2, \cdots, g_p; h_2, \cdots, h_p]$ is an even characteristic. Then, if $\bar{\theta}[g;h]((\bar{u}_\sigma(\delta))) \neq 0$, we deduce from (27″)

$$\exp[h_1\pi i] + \exp[-h_1\pi i] = 0;$$

hence $h_1 = \frac{1}{2}$. It follows that $[g_1, \cdots, g_p; h_1, \cdots, h_p] = [\frac{1}{2}, g_2, \cdots, g_p; \frac{1}{2}, h_2, \cdots, h_p]$ is an odd characteristic. We exclude naturally this case, because then $\theta[g;h]((0))$ vanishes identically, as function of the branch points. Likewise if $[g_2, \cdots, g_p; h_2, \cdots, h_p]$ is an odd characteristic, we have

$$\exp[h_1\pi i] - \exp[-h_1\pi i] = 0;$$

hence $h_1 = 0$, and $[g_1, \cdots, g_p; h_1, \cdots, h_p] = [\frac{1}{2}, g_2, \cdots, g_p; 0, h_2, \cdots, h_p]$ is again an odd characteristic.

In conclusion, we have the following result: *The function* $\theta[g;h]((0))$ *possesses a zero of order* g_1^2 *at* $k = k_1$, *if* $g_1 \leqq \frac{1}{2}$, *and of order* $(1 - g_1)^2$, *if* $g_1 \geqq \frac{1}{2}$. *The only exception corresponds to the case in which* $g_1 = 0$ or $\frac{1}{2}$ *and* $[g_2, \cdots, g_p; h_2, \cdots, h_p]$ *is a characteristic* $[\epsilon_2]$ *such that* $\bar{\theta}[\epsilon]_2((m\bar{u}_\sigma(\delta)))$ *vanishes identically. Here* $m = 1$, *if* $g_1 = \frac{1}{2}$, *and* $m = 0$, *if* $g_1 = 0$.

We examine now this exceptional case.

6. *Examination of the exceptional case.* The values of the integrals $\bar{u}_\sigma$ $(\sigma = 2, 3, \cdots, p)$ at the branch point k_i $(i = 2, 3, \cdots, 2p+1)$ of the curve $\bar{C}$ constitute a system $\{\bar{a}_i\}$ of half periods. It is easily seen that the characteristic $(\bar{a}_i)$ is obtained from the characteristic (a_i) of the table (11) (§ 3) suppressing the two numbers of the first column. Every characteristic $[g_2, \cdots, g_p; h_2, \cdots, h_p] = [\epsilon]_2$ is of the form

$$[\bar{\kappa} + \sum^{p-1-s} \bar{a}_i],$$

where $[\bar{\kappa}] = [1^{p-1}; p-1, p-2, \cdots, 1]$, s is one of the members $0, 1, \cdots, p-1$, and $\sum^{p-1-s} \bar{a}_i$ indicates the sum of $p - 1 - s$ conveniently chosen characteristics $(\bar{a}_i)$.

We recall the following well-known theorems, with reference to the curve C of genus p:[*]

I. Indicating by $\sum^{p-s} a_i$ the sum of any $p - s$ characteristics of the table (11), the function $\theta[\kappa + \sum^{p-s} a_i]((mu_\sigma(\eta)))$ vanishes identically, if m is one of the integers $0, 1, \cdots, s-1$. However $\theta[\kappa + \sum^{p-s} a_i]((su_\sigma(\eta)))$ does not vanish identically.

---

[*] See (Kr.), p. 459.

II. Denoting by $[(s-1)/2]$ the integral part of $(s-1)/2$, the function $\theta[\kappa+\sum^{p-s} a_i]((v))$ and all its partial derivatives of order $1, 2, \cdots, [(s-1)/2]$, but not all of its partial derivatives of order $[(s+1)/2]$, vanish for the zero values of the arguments $v_1, v_2, \cdots, v_p$.

Let us consider now the function $\theta[g;h]((0))$, in which $g_1=0$ or $\frac{1}{2}$, h_1 is arbitrary, and $[g_2, \cdots, g_p; h_2, \cdots, h_p]$ is a characteristic $[\epsilon]_2$:

$$[g_2, \cdots, g_p; h_2, \cdots, h_p] = [\epsilon]_2 = [\bar{\kappa}+\sum^{p-1-s} \bar{a}_i].$$

Looking at (24) and (25) and applying the theorem I to the curve $\bar{C}$ of genus $p-1$, we find:

1) If $g_1=0$, then

$$\lim_{k\to k_1} A_{m_1}=0 \text{ for } m_1=0, \pm 1, \cdots, \pm [(s-1)/2],$$

and
$$\lim_{k\to k_1} A_{[(s+1)/2]} \neq 0.$$

2) If $g_1=\frac{1}{2}$, then

$$\lim_{k\to k_1} A_{m_1}=0 \text{ for } m_1=0, \pm 1, \cdots, \pm [(s-2)/2], -[s/2],$$

and
$$\lim_{k\to k_1} A_{[s/2]} \neq 0.$$

The characteristic $[\bar{\kappa}+\sum^{p-1-s} \bar{a}_i]$ coincides evidently with the last $p-1$ columns of the corresponding characteristic $[\kappa+\sum^{p-1-s} a_i]$. On the other hand, the addition of the characteristic (a), or of (a_1), or of their sum does not alter the $p-1$ last columns. We obtain thus the 4 characteristics

$$[\lambda, \epsilon_1, \cdots, \epsilon_p; \mu, \epsilon_2', \cdots, \epsilon_p']_2 \quad (\lambda, \mu = 0, 1),$$

and, it is seen at once, that

$$[\kappa+a+\sum^{p-1-s} a_i], \quad [\kappa+a_1+\sum^{p-1-s} a_i]$$

coincide, to within order, with the characteristics

$$[0, \epsilon_1, \cdots, \epsilon_p; 0, \epsilon_2', \cdots, \epsilon_p']_2, \quad [0, \epsilon_2, \cdots, \epsilon_p; 1, \epsilon_2', \cdots, \epsilon_p']_2,$$

while

$$[\kappa+\sum^{p-1-s} a_i], \quad [\kappa+a+a_1+\sum^{p-1-s} a_i]$$

coincide, to within order, with

$$[1, \epsilon_2, \cdots, \epsilon_p; 0, \epsilon_2', \cdots, \epsilon_p']_2, [1, \epsilon_2, \cdots, \epsilon_p; 1, \epsilon_2', \cdots, \epsilon_p']_2.$$

We consider separately the two cases: 1) $g_1 = 0$, 2) $g_1 = \frac{1}{2}$.

1) Let $g_1 = 0$. We have

$$\theta[g;h]((0)) = \sum_{-\infty}^{+\infty} \exp\,[2m_1h_1\pi i + a_{11}m_1^2] \cdot A_{m_1}. \tag{28}$$

We observe that A_{m_1} does not depend upon g_1, h_1. Applying the theorem II to the functions

$$\theta[\kappa + a + \sum^{p-1-s} a_i]((v)), \quad \theta[\kappa + a_1 + \sum^{p-1-s} a_i]((v))$$

and to their derivatives taken with respect to v_1, we find

$$\left.\begin{aligned} &\sum_{-\infty}^{+\infty}{}_{m_1} m_1^r \exp\,[a_{11}m_1^2] \cdot A_{m_1} = 0; \\ &\sum_{-\infty}^{+\infty}{}_{m_1} m_1^r \exp\,[m_1\pi i + a_{11}m_1^2] \cdot A_{m_1} = 0; \end{aligned}\right\} (r = 0, 1, \cdots, [(s-1)/2]. \tag{29}$$

We put for brevity $[(s-1)/2] = q$. The $2q+2$ equations (29) can be solved with respect

A_0, exp $[a_{11}] \cdot A_{\pm1}, \ldots,$ exp $[a_{11}q_1^2] \cdot A_{\pm q}$, exp $[a_{11}(q+1)^2] \cdot A_{-(q+1)}$,

the corresponding determinant being different from zero. Then (28) becomes

$$\begin{aligned} \theta[g;h]((0)) = c_1 \exp\,&[a_{11}(q+1)^2] \cdot A_{(q+1)} \\ &+ \exp\,[a_{11}(q+2)^2] \cdot \{c_2 A_{(q+2)} + c_{-2} A_{-(q+2)}\} + \cdots. \end{aligned}$$

We saw above that $\lim\limits_{k \to k_1} A_{(q+1)} \neq 0$. Hence, provided the coefficient c_1 is not 0, we have

$$\theta[g;h]((0)) \sim (k - k_1)^{(q+1)^2}. \tag{30}$$

To calculate c_1 in the simplest way, we consider the function $(e^z - e^{-z})^{q+1}$. Let

$$(e^z - e^{-z})^{q+1} = \sum_{-(q+1)}^{+(q+1)}{}_m \alpha_m e^{mz}. \tag{31}$$

Observing that $(e^z - e^{-z})^{q+1}$ and all its derivatives of order $1, 2, \cdots, q$ vanish for $z = 0$, or $z = \pi i$, we obtain the following equations:

$$\left.\begin{aligned} &\sum_{-(q+1)}^{+(q+1)}{}_m m^r \alpha_m = 0; \\ &\sum_{-(q+1)}^{+(q+1)}{}_m m^r \cdot e^{m\pi i} \alpha_m = 0; \end{aligned}\right\} \quad (r = 0, 1, \cdots, q). \tag{32}$$

The system of equations (32) does not differ from the system (29), if in the last one we neglect the terms of order $m_1 > q+1$, or $< -(q+1)$. Hence, comparing (31) with (28), we deduce,

$$c_1 = (e^{2h_1\pi i} - e^{-2h_1\pi i})^{q+1};$$

whence $c_1 = 0$, since evidently we must suppose $h_1 \neq 0, \frac{1}{2}$. We may write the result as follows:

$$(33)\quad \lim_{k\to k_1} \theta[g;h]((0)) \exp[-a_{11}(q+1)^2] = c_1\ddot{\theta}[\epsilon]_2(((2q+2)\bar{u}_\sigma(\delta))).$$

2) Let $g_1 = \frac{1}{2}$. We have now

$$(34)\quad \theta[g;h]((0)) = \sum_{-\infty}^{+\infty}{}_{m_1} \exp[(2m_1+1)h_1\pi i + a_{11}(m_1+\tfrac{1}{2})^2]\cdot A_{m_1}.$$

We apply the theorem II to the functions

$$\theta[\kappa + \sum^{p-1-s} a_i]((v)),\quad \theta[\kappa + a + a_1 + \sum^{p-1-s} a_i]((v)),$$

of which the first vanishes for $v = 0$ together with its partial derivatives of order $1, 2, \cdots, [s/2]$, and the second vanishes for $v = 0$ together with its partial derivatives of order $1, 2, \cdots, [(s-2)/2]$. It is easily seen that if s is odd, $[\kappa + \sum^{p-1-s} a_i]$ is the characteristic

$$[1, \epsilon_1, \cdots, \epsilon_p;\ 0, \epsilon_2', \cdots, \epsilon_p']_2,$$

and if s is even, it is the characteristic $[1, \epsilon_2, \cdots, \epsilon_p;\ 1, \epsilon_2', \cdots, \epsilon_p']_2$. Considering only the derivatives of both functions with respect to v_1, and putting, for brevity, $q = [(s-2)/2]$, we have in the first place the relations

$$(35)\quad \left.\begin{aligned} &\sum_{-\infty}^{+\infty}{}_{m_1} (2m_1+1)^r \exp[(m_1+\tfrac{1}{2})\pi i + a_{11}(m_1+\tfrac{1}{2})^2]\cdot A_{m_1} = 0;\\ &\sum_{-\infty}^{+\infty}{}_{m_1} (2m_1+1)^r \exp[a_{11}(m_1+\tfrac{1}{2})^2]\cdot A_{m_1} = 0;\end{aligned}\right\}$$

$$(r = 0, 1, \cdots, q).$$

One more equation arises from the vanishing of

$$\frac{\partial^{q+1}}{\partial v_1^{q+1}}\theta[\kappa + \sum^{p-1-s} a_i]((v))$$

for $v = 0$. Hence, if s is odd, we will have

$$(35')\quad \sum_{-\infty}^{+\infty}{}_{m_1} (2m_1+1)^{q+1} \exp[a_{11}(m_1+\tfrac{1}{2})^2]\cdot A_{m_1} = 0,$$

and if s is even, we will have

$$(35'')\quad \sum_{-\infty}^{+\infty}{}_{m_1} (2m_1+1)^{q+1} \exp[(m_1+\tfrac{1}{2})\pi i + a_{11}(m_1+\tfrac{1}{2})^2]\cdot A_{m_1} = 0.$$

In both cases we have a system of $2q+3$ equations, which we can solve with respect to the $2q+3$ functions $\exp\,[a_{11}(m_1+\frac{1}{2})^2]\cdot A_{m_1}$, $[m_1=0,\pm 1,\cdots,\pm q,-(q+1),-(q+2)]$, and which permit of expressing $\theta[g;h]((0))$ in (34) by means of the remaining functions. We have then

$$\theta[g;h]((0))=c_1\exp\,[a_{11}(q+3/2)^2]\cdot A_{q+1}+\cdots,$$

where the missing terms contain powers of $e^{a_{11}}$ higher than $(q+3/2)^2$. We saw above that $\lim\limits_{k\to k_1} A_{q+1}\neq 0$. Hence

$$\theta[g;h]((0))\sim(k-k_1)^{(q+3/2)^2}, \tag{30'}$$

provided that the coefficient c_1 is $\neq 0$. To calculate c_1 in a simple way, we consider, if, for instance, s is odd, the function

$$F(z)=(e^z-e^{-z})^{q+2}(e^z+e^{-z})^{q+1}.$$

We may write

$$F(z)=\sum_{-(q+2)}^{(q+1)}{}_m\,\alpha_m e^{(2m+1)z}. \tag{36}$$

We express the fact that $F(z)$ and all its derivatives of order $1, 2, \cdots, q$ vanish for $z=0$ or $\pi i/2$, and that moreover the derivative of order $q+1$ vanishes for $z=0$. We have thus the following systems of $2q+3$ equations:

$$\sum_{-(q+2)}^{q+1}{}_m\,(2m+1)^r\alpha_m=0 \qquad (r=0,1,\cdots,q+1)$$

$$\sum_{-(q+2)}^{q+1}{}_m\,(2m+1)^r\exp\,[(m+\tfrac{1}{2})\pi i]\cdot\alpha_m=0 \qquad (r=0,1,\cdots,q).$$

Comparing this system with the system of equations (35), (35′) in which we may neglect all terms of order $m_1>(q+1)$ or $<-(q+2)$, since we need only the coefficient of $\exp\,[a_{11}(q+3/2)^2]A_{q+1}$, we notice that these two systems of linear equations possess the same coefficients. Hence, comparing (34) and (36), we deduce

$$c_1=(e^{h_1\pi i}-e^{-h_1\pi i})^{q+2}\,(e^{h\pi_1 i}+e^{-h_1\pi i})^{q+1}.$$

If $h_1\neq 0,\ \frac{1}{2}$, then $c_1\neq 0$.

If s is even, we consider the function

$$(e^z-e^{-z})^{q+1}(e^z+e^{-z})^{q+2},$$

and we deduce in the same way

$$c_1=(e^{h_1\pi i}-e^{-h_1\pi i})^{q+1}\,(e^{h_1\pi i}+e^{-h_1\pi i})^{q+2}.$$

We may express the result as follows:

$$(33') \qquad \lim \theta[g;h]((0)) \exp\left[-a_{11}(q+3/2)^2\right] = c_1\bar{\theta}[\epsilon]_2(((2q+3)\bar{u}_\sigma(\delta_2))),$$

where

$$c_1 = (e^{h_1\pi i} + (-1)^s e^{-h_1\pi i})^{q+2} (e^{h_1\pi i} - (-1)^s e^{-h_1\pi i})^{q+1}$$

and $q = [(s-2)/2]$.

7. *The degree of the invariant I_r.* We return to the formula (17) of section 4, and put

$$(37) \qquad F_r(k, k_1, \cdots, k_{2p+1}) = D^h I_r^2,$$

where D is the discriminant of the polynomial $f(x)$ and I_r is the corresponding invariant of $f(x)$. We write I_r^2, because if $\theta[g;h]((0)) = 0$, then also $\theta[-g;-h]((0)) = 0$, and generally the vanishing of $\theta[g;h]((0))$ does not imply the vanishing of any other θ-function. From the algebraic point of view this means that if the polynomial satisfies the relation (10) or (10′), then generally the polynomials ϕ_{m-p-1} or ϕ_{m-p}, ψ_m and ψ_{p-1} are uniquely determined.*

Moreover we observe that F_r possesses a finite value for finite values of the k_i's. In fact, let us fix $k_1, \cdots, k_{2p+1}$. If k does not coincide with any k_i, then every θ-function of the product has a finite value, while the determinant $|A_{ij}|$ is $\neq 0$. Suppose now $k = k_i$; due to symmetry we may suppose $k = k_1$. Every function θ of the product possesses still a finite value. As far as the determinant $|A_{ij}|$ is concerned, its limit for $k = k_1$ can be calculated by using Thomae's formula (12), taking into consideration that $\lim\limits_{k\to k_1} \theta((0)) = \bar{\theta}((0))$ (see § 5). One finds then

$$\lim_{k\to k_1} |A_{ij}| = 2\pi i \; |\bar{A}_{\sigma\mu}| / \{\prod_{2}^{2p+1}{}_i (k_1 - k_i)\}^{1/2},$$

where the $\bar{A}_{\sigma\mu}$'s are the analogous periods relative to the limit curve $\bar{C}$. Hence this limit is still $\neq 0$, and therefore F_r is finite for $k = k_1$. This proves the correctness of (37).

---

* Take for instance the case $p = 5$, $r = 3$, i. e. a polynomial $f(x)$ of degree 12 and of the form $f(x) = j^2 + i^3$. Suppose we have another representation of the same polynomial: $f(x) = j_1^2 + i_1^3$. Then $i_1^3 - i^3 = (j - j_1)(j + j_1)$. If we fix i_1 and i (and this furnishes 8 arbitrary constants and one constant more as factor of proportionality in $i_1^3 - i^3$) we have ∞^1 solutions for j and j_1. It follows that the polynomials $f(x)$ which admit two different representations of the form $j^2 + i^3$ depend upon 10 arbitrary constants, while the general polynomial of this form depends upon 11 constants.

By means of the results of sections 5 and 6 we are able to evaluate the exponent h. In fact $|A_{ij}|$ does not vanish for $k \to k_1$, hence F_r vanishes for $k \to k_1$ to the same order, namely $2h$, as the product of the function θ. Let h be calculated. From the formula of Thomae we deduce that $|A_{ij}|^{1/2}$ vanishes at $k = \infty$ to order $p/4$, because, at $k = \infty$, $\theta((0))$ is $\neq 0$, while the radical which appears in the second member becomes ∞ to order $p/4$. Since every function $\theta[\kappa + \{\epsilon\}_r]((0))$ possesses generally for $k = \infty$ a value different from 0, it follows that F_r becomes ∞ at $k = \infty$ to order $N_r'p/4$, where N_r' is the number defined in section 4. Since the discriminant D is of degree $2(2p+1)$ in k, we deduce, denoting by g_r the degree of I_r:

$$2g_r = N_r'p/4 - 2h(2p+1). \tag{38}$$

Let us calculate h and then g_r in the case of p odd and r an odd prime number. There is no difficulty in calculating h in the most general case. Since r is a prime number and $r(p-1)$ is even, we have $N_r' = N_r$ and from (13)

$$N_r' = N_r = r^{2p} - 1. \tag{39}$$

The $r^{2p} - 1$ functions θ of the product are in this case all functions

$$\theta[\kappa + \{\epsilon\}_r]((0)), \tag{40}$$

in which, however, the characteristic has to be reduced modulo $2r$.

Let us denote the reduced characteristic by

$$[\eta]_{2r} = [\eta_1, \cdots, \eta_p; \eta_1', \cdots, \eta_p']_{2r}.$$

Here every η_i is one of the numbers $(r-2s)/2r$, $(r+2s)/2r$, where $s = 0, 1, \cdots, (r-1)/2$. All functions (40) for which $\eta_1 = (r-2s)/2r$ or $= (r+2s)/2r$ possess, for given $s \neq 0$, at $k = k_1$ a zero of order $(r-2s)^2/(2r)^2$ (see (26) and (26′), section 5). The number of such functions is equal to $2r^{2p-1}$, hence their product possesses at $k = k_1$ a zero of order

$$2r^{2p-1}(r-2s)^2/(2r)^2 = r^{2p-3}(r-2s)^2/2.$$

Giving to s the values $1, 2, \cdots, (r-1)/2$, we deduce that the product of all those functions (40), for which η_1 is $\neq \frac{1}{2}$, possess at $k = k_1$ a zero of order

$$(r^{2p-3}/2)\{1^2 + 3^2 + \cdots + (r-2)^2\} = r^{2p-2}(r-1)(r-2)/12. \tag{41}$$

We consider now those functions (40) in which $\eta_1 = \frac{1}{2}$. These possess at $k = k_1$ a zero of order $\frac{1}{4}$, unless the characteristic $[\eta_2, \cdots; \eta_2', \cdots]_{2r}$ is the characteristic $[\mathring{\kappa}]_2$, in which case we have special functions considered in section 6. The number of these special functions is $r-1$, and are obtained by giving to ϵ_1' the values $1, 2, \cdots, r-1$. Hence the number of the remaining non-special function is $r^{2p-1}-r$, and their product possesses at $k=k_1$ a zero of order

$$(r^{2p-1}-r)/4. \tag{42}$$

Finally every function

$$\theta[\kappa+\{0^p;\,\epsilon_1',\,0^{p-1}\}_r]((0))$$

possesses at $k=k_1$ a zero of order $p^2/4$, as a result of (30′), since in this case $s=p-1$, hence $q=(p-3)/2$. The product of these $r-1$ functions possesses then at $k=k_1$ a zero of order

$$(r-1)p^2/4. \tag{43}$$

Adding the orders written in (41), (42), (43) we deduce that the product of all $r^{2p}-1$ functions (40), and hence F_r, possesses at $k=k_1$ a zero of order

$$\begin{aligned} r^{2p-2}(r-1)(r-2)/12+(r^{2p-1}-r)/4+(r-1)p^2/4= \\ =(r^2+2)r^{2p-2}/12+(r-1)p^2/4-r/4. \end{aligned}$$

Hence this is the value of $2h$. Substituting this value of $2h$ and the value of N_r', given by (39), in (38) we find:

$$\begin{aligned} 2g_r=r^{2p-2}(pr^2-r^2-4p-2)/12 \\ -[(r-1)p^2-r](2p+1)/4-p/4. \end{aligned} \tag{44}$$

For $p=5$ and $r=3$ we find the degree of the invariant connected with the 12 tangents to a plane quartic curve. The degree of this invariant is equal

$$2\cdot 3^2\cdot 11\cdot 19.$$

THE JOHNS HOPKINS UNIVERSITY.

Certain Perfect Groups Generated by Two Operators of Orders Two and Three.

By H. R. Brahana.

Introduction

We have shown recently[*] that there is a one-to-one correspondence between the regular maps and groups generated by two operators of which one is of order two. Certain classes of groups came immediately to our attention, the symmetric and alternating groups, the meta-cyclic groups, and the linear fractional groups, $LF(2, p)$. Naturally one seeks other groups or classes of groups satisfying the given condition. Prof. Miller has shown that there exists an infinite number of finite groups generated by S and T satisfying the relation $T^2 = S^k = (ST)^l = 1$ for every pair of numbers k and l both greater than 3 or one equal to 3 and the other greater than 5. We undertook to examine some of the groups for $k = 3$ and $l = 7$. The immediate result was the series of theorems in § 1, concerned mostly with perfect groups. For want of a better method we decided to fix the order of the commutators of the generating elements and examine the resulting groups or classes of groups. The result (§ 2) was the discovery that if the order were fixed at 4 the simple group of order 168 is determined, and if fixed at 6 the simple group of order 1092 is determined, no other number less than 7 being possible. The work required by our method of procedure is tedious and promises to become more so as the order of the commutator is increased. Rather than continue in this manner we have taken the simple group of order 504 and found that it can be generated by two operators of orders two and three with product and commutator of orders seven and nine. We ask whether these conditions are sufficient to determine the group. We are unable to answer the question definitely, but we obtain sets of relations defining abstractly the generators of this group and also of the simple group of order 660. In each case we have been obliged to insert a condition in addition to fixing the orders of the generators, their product, and their commutator, without being able to prove that it is independent.

1. *The groups generated by S and T satisfying the relations* $S^3 = T^2 =$

---

[*] "Regular Maps and their Groups," *American Journal of Mathematics*, Vol. 49 (1927), pp. 266-284.

$(ST)^n = 1$. We consider two operators S and T satisfying the relations $S^3 = T^2 = (ST)^n = 1$, where each exponent is the smallest number for which the operator to which it is attached becomes identity. The group G, determined by these operators is completely determined if n is one of the numbers 3, 4, or 5.[*] On the other hand, if n is greater than 5 there exists an infinite number of finite groups each generated by an S and a T satisfying the given relations.[†] The groups for which $n = 6$ have been described in detail.[‡] It is our purpose to exhibit some of the properties of the group G for certain types of value of n greater than 6.

The commutators of T and powers of S are $\sigma_1 = TS^2TS$, $\sigma_2 = TSTS^2$ and their inverses. The commutator subgroup H of G is generated by σ_1 and σ_2.[§] The following relations exist among S, T and the σ's: $T\sigma_i = \sigma_i^{-1}T$, $S\sigma_1 = \sigma_2^{-1}S$, and $S\sigma_2 = S \cdot TS\,TS^2 = STS^2T \cdot TS^2TS \cdot S = \sigma_2^{-1}\sigma_1 S$.

Any element of G is of the form $S^\alpha TS^\beta T\, S^\gamma T\, S^\delta \cdots$ where $\alpha = 0$, 1, or 2, and each of the other exponents is 1 or 2. If we rewrite the above product in the form $S^\alpha\, (TS^\beta TS^{-\beta})\, S^{\beta+\gamma}\, (TS^\delta TS^{-\delta})\, S^{\delta+\epsilon} \cdots$ and make use of the relations among S, T, and the σ's, we see that every element of G is in one of the forms

$h, hS, hS^2, hT, hST, hS^2T, hSTS, hS^2TS, hSTS^2, hS^2TS^2$, where h is a product of the σ's and hence an element of H. Moreover, the last four types are included among the preceding six due to the relations: $STS = STS^2T \cdot TS^2 = STS^2T \cdot TS^2TSS^2T = \sigma_2^{-1}\sigma_1 \cdot S^2T$; $S^2TS = \sigma_1^{-1} \cdot T$; $STS^2 = \sigma_2^{-1}T$; and $S^2TS^2 = S^2TST \cdot TS = S^2TST \cdot TSTS^2 \cdot ST = \sigma_1^{-1}\sigma_2 \cdot ST$. These six types are distinct provided neither of the generators S and T is in the commutator subgroup of G.

We have the following theorem:

The order of G is at most six times the order of its commutator subgroup.

The successive powers of (ST) are:

$$\begin{aligned} ST &= ST \\ (ST)^2 &= \sigma_2^{-1}\sigma_1 \cdot S^2 \\ (ST)^3 &= \sigma_2^{-1}\sigma_1 \cdot T \\ (ST)^4 &= \sigma_2^{-1}\sigma_1\sigma_2 \cdot S \end{aligned}$$

---

[*] Cf. Burnside, *Theory of Groups* (1897), p. 291.

[†] Miller, *American Journal of Mathematics*, Vol. 24 (1902), pp. 96-100.

[‡] Miller, *Quarterly Journal of Mathematics*, Vol. 33 (1901), pp. 76-79.

[§] Miller, "On the Commutator Groups," *Bulletin of the American Mathematical Society*, Vol. 4 (1897), pp. 135-139.

$$
\begin{aligned}
(ST)^5 &= \sigma_2^{-1}\sigma_1\sigma_2 \cdot S^2T \\
(ST)^6 &= \sigma_2^{-1}\sigma_1\sigma_2\sigma_1^{-1} \\
(ST)^7 &= (\sigma_2^{-1}\sigma_1\sigma_2\sigma_1^{-1})(ST) \\
&\vdots \\
(ST)^{6k+l} &= (\sigma_2^{-1}\sigma_1\sigma_2\sigma_1)^k \cdot (ST)^l.
\end{aligned}
$$

From the above relations we see that if n, the order of (ST), is 6 the commutator sub-group of G is Abelian.* Moreover, we may state the following converse theorem:

If two non-commutative operators of orders two and three generate a group whose commutator sub-group is Abelian their product is of order six.

If (ST) is of order 7, then $(\sigma_2^{-1}\sigma_1\sigma_2\sigma_1^{-1})(ST) = 1$ and therefore (ST) is in H. If (ST) is in H we see from the values of $(ST)^3$ and $(ST)^4$ that both S and T are in H. Likewise S and T are in H if the order of (ST) is any number of the form $6k + 1$. Moreover, if the order of (ST) is of the form $6k - 1$ S^2T is in H, and since $S^2T \cdot ST = \sigma_1^{-1}$, (ST) and, consequently, S and T are in H. Hence, G and H coincide. Such a group is called perfect.† We have the following theorem:

Any group generated by two operators of orders two and three respectively the order of whose product is of the form $6k \pm 1$ *is perfect.*

We have, *a fortiori*, the theorem:

Two operators of orders two and three whose product is of prime order greater than three generate a perfect group.

Every perfect group is simple or else it is isomorphic $(\alpha - 1)$ to some simple group of composite order.‡ Hence,

No group generated by two operators of orders two and three the order of whose product is a prime $6k \pm 1$ *can have an order less than that of the smallest simple group whose order contains the factors* 2, 3, *and* $6k \pm 1$.

For, the simple group to which G is isomorphic will be generated by two operators which correspond in the isomorphism to the generators of G and therefore must be or orders two and three with product of order $6k \pm 1$.

---

* This fact was used by Prof. Miller in studying the groups in question. Cf. the reference above, σ_1 and σ_2 are the same as his s_3 and s_4.

† Miller, *American Journal of Mathematics*, Vol. 20 (1898), pp. 277. See footnote.

‡ Miller: Cf. the last reference.

We may also state the following theorem:

In a group whose order is not a multiple of the order of some simple group it is impossible to select an operator of order two and an operator of order three such that their product is of order $6k \pm 1$.

If the order of (ST) is of the form $6k \pm 2$, S is in H; and if of the form $6k + 3$, T is in H. Hence,

If a group generated by two operators of orders two and three whose product is of order $6k \pm 2$ *(or* $6k + 3$*) its order is at most* 2 *(or* 3*) times the order of its commutator subgroup.*

The transform of σ_2 by (ST) is $TS^2 \cdot TSTS^2 \cdot ST = TS^2TS = \sigma_1$ and therefore σ_1 and σ_2 are of the same order. Hence,

Any group generated by two operators of orders two and three whose product is of order $6k \pm 1$ *is also generated by two operators of equal orders.*

If the order of ST is of the form $6k$ the subgroup H may or may not contain either or both of the operators S and T. If it contains neither so that the sets H, HS, HS^2, HT, HST, and HS^2T are all distinct, it follows that the quotient group G/H is cyclic. As a result of this and the foregoing theorems we have the theorem:

The largest Abelian group to which a group generated by two operators of orders two and three is isomorphic is a cyclic group of order 1, 2, 3, *or* 6, *according as the order of the product of the generators is equivalent to* 1, 2, 3, *or* 0, *mod* 6.

From the value of $(ST)^7$ above we see that if $(ST)^7 = 1$ (ST) is the commutator of σ_1^{-1} and σ_2. Hence,

In any group generated by T *and* S *of orders two and three such that* ST *is of order* 7, ST *is the commutator of the commutators of* T *and* S *and of* S^2 *and* T.

2. *Relations defining generators of certain simple groups.* Any group G generated by two operators S and T of orders 3 and 2 whose product is of order 7 contains two operators σ_1 and σ_2 which also serve to generate the group. The elements σ_1 and σ_2 satisfy the following conditions:

(A) $(\sigma_2^{-1}\sigma_1\sigma_2\sigma_1^{-1})^7 = 1$;

(B) $(\sigma_2^{-1}\sigma_1\sigma_2\sigma_1^{-1})^3\sigma_2^{-1}\sigma_1 = \sigma_2\sigma_1^{-1}(\sigma_2^{-1}\sigma_1\sigma_2\sigma_1^{-1})^3$;

(C) $(\sigma_2^{-1}\sigma_1\sigma_2\sigma_1^{-1})^2\sigma_2^{-1}\sigma_1 \cdot (\sigma_2^{-1}\sigma_1\sigma_2\sigma_1^{-1})^2 = \sigma_1^{-1}\sigma_2(\sigma_2^{-1}\sigma_1\sigma_2\sigma_1^{-1})^5\sigma_1^{-1}\sigma_2$;

(D) $\sigma_1 = (\sigma_2^{-1}\sigma_1\sigma_2\sigma_1^{-1})^2\sigma_1^{-1}\sigma_2(\sigma_2^{-1}\sigma_1\sigma_2\sigma_1^{-1})^2$;

Conversely, *any group G generated by two operators σ_1 and σ_2 satisfying* (*A*), (*B*), (*C*), and (*D*) *is also generated by two operators of orders two and three whose product is of order* 7.

For, the element $T' = (\sigma_2^{-1}\sigma_1\sigma_2\sigma_1^{-1})^3\sigma_2^{-1}\sigma_1$, is of order two as a result of (B); the element $S' = (\sigma_2^{-1}\sigma_1\sigma_2\sigma_1^{-1})^2\sigma_2^{-1}\sigma_1$ is of order three as a result of (C); their product, $S'T' = (\sigma_2^{-1}\sigma_1\sigma_2\sigma_1^{-1})^6$, is of order 7 by (A); the group $[S', T']$ contains σ_1 as a result of (D); and $[S', T']$ contains σ_2 since it contains T', $S'T'$, and σ_1.

The above theorem states the sufficiency of the given conditions on σ_1 and σ_2 in order that the group be of the type in question. The conditions are of course necessary as may be seen by consideration of the group $[S, T]$ in the manner of § 1, but there arises the question of their independence, whether or not one or more of them may not be omitted. In attempting to answer the question two methods of procedure are possible. First, the independence would be shown by giving examples satisfying all but one of the conditions and contradicting the one; or, on the other hand, the redundancy of the conditions might be shown by deriving a certain number of them on the basis of the others. Both attempts have been made with no result other than a slight reflection on the ingenuity of the author. However, we may arrive at a plausible answer in the following manner: Conditions (B) and (C) insure the presence in $[\sigma_1, \sigma_2]$ of operators of orders two and three, condition (A) requires one pair of such operators to have a product of order 7, and condition (D) requires the group generated by the two operators just described to be the whole group.

In investigating the groups under consideration we make use of the co-sets with respect to a given subgroup, which should be as large as can be conveniently arranged so that the number of co-sets will be small, and which should be designated by simple expressions in the generators. For this reason we determine another set of generators of G.

Let $(\sigma_2^{-1}\sigma_1\sigma_2\sigma_1^{-1})$ be denoted by R and let $\sigma_2^{-1}\sigma_1$ be denoted by Q. Then $S' = R^2Q$ and $T' = Q^{-1}R^4$. Relations (*A*), (*B*), (*C*), and (*D*) become

(A′) $R^7 = 1$;

(B′) $Q^{-1}R^4 = R^3Q$;

(C′) $R^2QR^{-1} = Q^{-1}RQ$.

Condition (D) expressed in terms of Q and R is a consequence of (*A′*),

(B'), (C'), for if $S' = R^2Q$ and $T' = R^3Q$, $[S', T']$ contains both Q and R. This does not say, however, that condition (D) is a consequence of the three conditions which precede it, for presumably σ_1 and σ_2 might generate a group of which the group generated by Q and R or by S' and T' would be a subgroup.

Any group generated by two operators Q and R satisfying relations (A'), (B'), and (C'), is also generated by two operators of orders two and three, whose product is of order seven.

We shall now fix the order of Q and examine the resulting groups. First we observe that Q cannot be of order 2 for then (B′) would become $QR^4 = R^3Q$, whence $QR^4Q = R^3$ which requires the group to be the dihedral group of order 14. The dihedral group of order 14 contains no operator of order 3 and therefore condition (C′) cannot be satisfied. Hence,

There is no group generated by two operators of orders two and three whose product is of order 7 and whose commutator is of order 2.

Let us suppose that Q is of order 3. Then by (C) we have $QR^2 = RQRQ^2$. Applying this relation to $QRQR^2$ we get

$$\begin{aligned} QRQR^2 &= QR^2QRQ^2 \\ &= R(QR^2Q)Q \\ &= R(RQR)Q \\ &= R^2QRQ. \end{aligned}$$

This states that R^2 is permutable with QRQ as is also $(R^2)^4 = R$. As a result of this fact $QR^2 = RQRQ^2$ gives

$$QR^2 = R(QRQ)Q = QRQRQ$$

whence $R = QRQ$. This result applied to (B′) gives

$$\begin{aligned} R^4 &= QR^3Q \\ &= QRQ \cdot Q^2RQ^2 \cdot QRQ \\ &= R(QRQ)R = R. \end{aligned}$$

This contradicts (A'); therefore,

There is no group generated by two operators of orders two and three whose product is of order 7 and whose commutator is of order 3.

If Q is of order 4 the group is the simple group of order 168. To show this we shall show that the operators S and T satisfy the relations given by Burnside[*] to determine the simple group of order 168. We proceed to show that the condition $Q^4 = 1$ is equivalent under conditions (A′), (B′), and

---

[*] *Theory of Groups*, p. 422. (Second edition).

(C′) above to the condition $[(ST)^4T]^4 = 1$. The group $[Q, R]$ is generated by S and T whose commutator is of order 4. The commutator of T and S is TS^2TS, putting the fourth power of it equal to zero, we have $[TS^2TS]^4 = [TST \cdot TSTS]^4 = 1$, or

$TST \cdot TSTSTST \cdot TSTSTST \cdot TSTSTST \cdot TSTS = 1$. This implies that $(TSTSTST)^4 = 1$. Since $(ST)^7 = 1$, the last statement implies that $(STSTSTS)^4 = 1$, which is $[(ST)^4T]^4 = 1$, the condition given by Burnside. Therefore,

The group generated by two operators of orders two and three whose product is of order 7 *and whose commutator is of order* 4 *is the simple group of order* 168.

We shall proceed to show that $Q^5 = 1$ is impossible by showing that such an assumption leads, in view of relations (A′), (B′), and (C′), to the relation $QRQ = Q$.

$$(1) \qquad QRQ = QR^4 \cdot R^4Q = R^3 \cdot R^4QR^4 \cdot R^4QR^4 \cdot R^3.$$

The product $R^4QR^4 = R^3 \cdot RQR \cdot R^3$ by (C′) becomes $R^3 \cdot QR^2Q \cdot R^3$. This becomes by (B′) $Q^4R^4 \cdot R^2 \cdot R^4Q^4 = Q^4R^3Q^4$, which reduces the rightmost member of (1) to

$$(2) \qquad R^3 \cdot Q^4R^3Q^4 \cdot Q^4R^3Q^4 \cdot R^3 = R^3Q^4 \cdot R^3Q^3R^3 \cdot Q^4R^3.$$

The product $R^3Q^3R^3$ may be written as $R^3Q \cdot Q \cdot QR^3$ which by (B′) is $Q^4 \cdot R^4QR^4 \cdot Q^4$, which is $Q^3R^3Q^3$. Hence, the right member of (2) is

$$(3) \qquad R^3Q^4 \cdot Q^3R^3Q^3 \cdot Q^4R^3 = R^3 \cdot Q^2R^3Q^2 \cdot R^3.$$

The right member of (3) may be written as

$$(4) \qquad R^5 \cdot R^5Q^2R^5 \cdot R^5Q^2R^5 \cdot R^5.$$

The element $R^5Q^2R^5$ reduces to Q in the following manner.

$$(5) \qquad \begin{aligned} R^5Q^2R^5 &= R^3 \cdot R^2Q^2R^2 \cdot R^8. \\ R^2Q^2R^2 &= R^2Q \cdot QR^2Q \cdot Q^4 \\ &= R^2Q \cdot RQR \cdot Q^4 = R \cdot RQR \cdot QRQ^4 \\ &= R \cdot QR^2Q \cdot QRQ^4 = RQR \cdot RQ^2RQ^4 \\ &= QR^2Q \cdot RQ^2RQ^4 = QR \cdot RQR \cdot Q^2RQ^4 \\ &= QR \cdot QR^2Q \cdot Q^2RQ^4 = Q \cdot RQR \cdot RQ^3RQ^4 \\ &= Q^2R^2Q \cdot RQ^3R \cdot Q^4 = Q^2R \cdot RQR \cdot Q^3RQ^4 \\ &= Q^2R \cdot QR^2Q \cdot Q^3RQ^4 = Q^2RQR^2 \cdot Q^4RQ \cdot Q^3. \end{aligned}$$

The element Q^4RQ is by (C′) R^2QR^6 and hence $R^2Q^2R^2 = Q^2RQR^2 \cdot R^2QR^6 \cdot Q = Q^2RQR^4QR^6 \cdot Q^3 = Q^2RQR^3 \cdot RQR \cdot R^5Q^3$ which by (C′) again becomes $Q^2R \cdot QR^3Q \cdot R^2QR^5Q^3$. If in this we use (B′) to replace QR^3Q by R^4 we have

$$R^2Q^2R^2 = Q^2R \cdot R^4 \cdot R^2QR^5Q^3 = Q^3R^5Q^3.$$

Using this in (5) we get

(6) $R^5Q^2R^5 = R^3 \cdot Q^3R^5Q^3 \cdot R^3$ which may be written $R^3Q^3R^3 \cdot R^6 \cdot R^3Q^3R^3$. Replacing $R^3Q^3R^3$ by $Q^3R^3Q^3$ we get

(7) $R^5Q^2R^5 = Q^3R^3Q^3 \cdot R^6 \cdot Q^3R^3Q^3 = Q^3 \cdot R^3Q^3R^3 \cdot R^3Q^3R^3 \cdot Q^3$.

By means of the same substitution we get

(8) $R^5Q^2R^5 = Q^3 \cdot Q^3R^3Q^3 \cdot Q^3R^3Q^3 \cdot Q^3 = QR^3QR^3Q$, which by (B′) is Q.

Using this in (4) we get $QRQ = R^5 \cdot Q \cdot Q \cdot R^5 = Q$. Hence, $QR = 1$ which is impossible. Therefore,

There is no group generated by two operators of orders two and three whose product is of order 7 *and whose commutator is of order* 5.

If Q is of order 6 it turns out that the group is of order 1092. Our method was to determine the number of right co-sets with respect to the cyclic group generated by R. There are 156 such sets counting the group generated by R. Each set may be designated by a single product of Q's and R's where the first element in the product is a power of Q; for example, QR may be allowed to stand for the set $QR, RQR, \cdots, R^6QR$. To write the resulting 156 operators would require about two printed pages, so we shall adopt a notation to allow us to cut down the space required. We shall denote each co-set, whose symbol ends in R, by a letter, for example b is the co-set QR. For every set whose symbol ends in R there are five more sets obtained from it by multiplication on the right by powers of Q. For each such set we shall use the same letter as for the first one except that we shall affect it with a subscript equal to the index of the power of Q on the right. Thus, b_1 stands for QRQ, b_2 for QRQ^2. We shall not stop to prove that b_i and b_j are distinct, a fact which will be evident when we obtain the final result. Thus we shall be able to indicate the 156 co-sets by giving 26 of them.

The 156 co-sets are $a, a_1, a_2, a_3, a_4, a_5, b, b_1, b_2, b_3, b_4, b_5, \cdots, z, z_1, z_2, z_3, z_4, z_5,$ where

$a = 1$
$b = QR$
$c = QR^4$
$d = QR^5$
$e = Q^2R$
$f = Q^2R^2$
$g = Q^2R^4$
$h = Q^2R^5$
$i = Q^2R^6$

$j = Q^3R$
$k = Q^3R^2$
$l = Q^3R^5$
$m = Q^3R^6$
$n = Q^4R$
$o = Q^4R^2$
$p = QRQ^2R^4$
$q = QR^4Q^3R^2$
$r = QR^4Q^3R^5$

$s = QR^4Q^3R^6$
$t = QR^4Q^4R$
$u = QR^4Q^4R^2$
$v = QR^5Q^2R^5$
$w = QR^5Q^3R$
$x = QR^5Q^3R^2$
$y = Q^2RQ^2R^4$
$z = Q^2R^5Q^3R^6$

Multiplication on the right permutes the co-sets in the following manner:

$$Q \equiv (aa_1a_2a_3a_4a_5)(bb_1b_2b_3b_4b_5) \cdots (zz_1z_2z_3z_4z_5);$$

$$\begin{aligned} R \equiv\ & (a_1bb_5a_3cdc_1)(a_2efc_5ghi)(a_3jkg_5g_2lm) \\ & (a_4nog_1c_2f_2e_5)(b_1f_1b_4d_5i_5n_1d_1)(b_2l_5e_4i_4pj_1o_1)(b_3h_5m_5p_5n_2e_1k_1) \\ & (c_3m_1qn_5g_4rs)(c_4tug_3i_1q_2j_5)(d_2k_2l_4d_4p_4vp_1)(d_3wxp_3p_2x_4w_5) \\ & (e_2x_3l_3m_4yw_1v_1)(e_3v_5w_4y_5j_2k_3x_1)(f_3q_1f_5o_5s_5t_1h_1)(f_4r_5j_4s_4t_2m_2u_1) \\ & (h_2u_2r_4o_4s_3v_3t_3)(h_3w_3r_2s_2y_2k_4z)(h_4i_3y_5y_1n_3o_2x_2)(i_2s_1q_4t_5n_4z_1z_3) \\ & (j_3v_2z_5v_4m_3u_3r_3)(k_5u_5z_2r_1l_1q_3q_5)(l_2y_3t_4u_4w_2o_3z_4)(a)(x_5). \end{aligned}$$

The work required to show that the sets are permuted by R in the manner described above had to be done, but it is obviously too long to be given here. Each step was similar to, though fortunately not so long and involved as, the work done to show that Q could not be of order 5.

Assuming the correctness of the expression for R, all that is required to prove that the group $[Q, R]$ is of order 1092 is to make certain that the substitution group $[Q, R]$ is transitive. The transitivity follows from the facts that Q permutes the different subscripts on each letter cyclically and that in the first four cycles of R we have a with every letter up to o while every cycle of R begins with a letter which precedes o in the alphabet.

The group $[Q, R]$ is perfect and is of order 1092. If a perfect group contains an invariant subgroup the corresponding quotient group is simple and of composite order. Since there exists no simple group whose order is less than 1092 and is contained in 1092 it follows that the group $[Q, R]$ is simple. There being but one simple group of this order it is thereby completely identified. This group is described by Burnside.* We may describe the group as follows:

---

* *Proceedings of the London Mathematical Society,* Vol. 26 (1895), p. 338.

(1) *The group generated by two operators of orders two and three whose product is of order* 7 *and whose commutator is of order* 6 *is the simple group of order* 1092.

(2) *The group generated by two operators of order* 6 *satisfying conditions* (*A*), (*B*), (*C*), *and* (*D*) *is the simple group of order* 1092.

(3) *The group generated by two operators, Q of order* 6 *and R, satisfying conditions* (A'), (B'), *and* (C') *is the simple group of order* 1092.

Instead of continuing in this manner, assigning values to the order of Q and examining the resulting groups, we note that 504 contains the factors 2, 3, and 7 and inquire whether the simple group of that order may not be generated by two operators. The three operators $s = (1254673)$, $t = (12)(34)(56)(78)$, and $A = (23)(46)(57)(89)$ generate G_{504}.[*] We note that $s^3t = (1362587)$ and that $s^3tA = (127)(346)(598)$. If we change to our own notation by letting $s^3tA = S$ and $A = T$, we have S and T satisfying the relations $S^3 = T^2 = (ST)^7 = 1$. The commutator of S and T is $\sigma_1 = (187623459)$. The group generated by S and T is perfect, its order contains the factors 2, 7, and 9, as well as the order of some simple group, and its order is a factor of 504. The only possibility is that S and T generate the simple group of order 504. Therefore,

The simple group of order 504 *is generated by two operators of orders two and three whose product is of order* 7 *and whose commutator is of order* 9.

We inquire whether the converse of the above theorem is true, whether two abstract operators satisfying the above conditions necessarily generate G_{504}. We change to the generators Q and R and require Q to be of order 9.

Our answer to the question is unsatisfactory for the reason that we have found it necessary to impose a further condition on Q and R without being able to prove that the added condition is independent of the four given above. This condition is

(D′) $$R^2Q^6 = Q^3R^5.$$

This condition is satisfied by the generators of the substitution group given above, but seems not to be a consequence of relations (A′), (B′), and (C′). An attempt was made to establish its independence by replacing it by the condition that the order of Q^3R^5 be four instead of two; the attempt was abandoned because it seemed to be leading to a group whose order was almost unmanageably large, at least 15,000.

---

[*] Burnside, *Theory of Groups* (1897), p. 375.

As before, we determine the number of right co-sets with respect to the subgroup generated by R. We find that there are 72 sets which, by means of the same notation as was used before, may be designated by $a, a_1, a_2, \cdots, a_8, b, \cdots, h_8$, where

$$\begin{array}{ll} a = 1 & e = Q^7R \\ b = Q^8R & f = Q^7R^3 \\ c = Q^8R^2 & g = Q^7R^5 \\ d = Q^8R^5 & h = Q^7R^6. \end{array}$$

The sets are permuted by multiplication on the right according to the substitutions

$$\begin{aligned} Q^8 &\equiv (aa_1a_2a_3a_4a_5a_6a_7a_8) \cdots (hh_1h_2h_3h_4h_5h_6h_7h_8); \\ R &\equiv (a_1bcb_8a_8dd_1)(a_2ec_6fb_6gh)(a_3b_4a_6h_5f_8f_2h_3)(a_4a_5d_6g_6h_4g_5d_4) \\ &\quad (a_7h_8g_2b_1f_1c_3e_4)(b_2f_5b_6h_2e_3e_1h_6)(b_3c_5d_7f_3h_1e_6g_4)(b_5g_7e_7h_7f_7d_3c_4) \\ &\quad (c_1e_5e_8c_8d_8g_1d_2)(c_2g_8e_2g_3c_7f_6f_4)(a)(d_5). \end{aligned}$$

If we express (D′) in terms of S and T we may state the following theorem:

The group generated by two operators of orders two and three, whose product is of order 7 *and whose commutator is of order* 9 *and which satisfy the further condition*

$$(STSTS^2TS^2TSTS^2TS^2T)^2 = 1$$

generate the simple group of order 504.

We consider next the simple group of order 660. According to Cole[*] the substitution group on 11 letters is generated by $(abiegdhfcjk)$, $(aegci)(bfjhd)$, and $(ac)(ba)(eg)(fh)$. The product of the first by the last is $S = (adf)(big)(cjk)$. If we denote $(ac)(bd)(eg)(fh)$ by T, we have S and T of orders three and two with product of order 11. S and T must generate the whole group since there is no simple group of lower order whose order contains the factors 2, 3, and 11.

The commutator σ_1 is $(achif)(bjdeg)$. The simple group of order 60, the icosahedral group, is a subgroup of G_{660} and we note that $S' = (ST)^{-2} \cdot S \cdot (ST)^2 = (abk)(egh)(fji)$ and T have the product $(adbkc)(fjihg)$ and therefore must generate such a subgroup.

We inquire whether two abstract operators S and T subject to the conditions that they be of orders 3 and 2 with product of order 11 and commutator of order 5 necessarily generate the simple group of order 660. Again we change to operators Q and R, where R is defined as above but Q is $\sigma_1^{-1}\sigma_2$. Q and R satisfy the relations:

---

[*] *Quarterly Journal of Mathematics*, Vol. 27 (1894), p. 49.

$$
\begin{array}{ll}
(\mathrm{A}'') & R^{11} = 1; \\
(\mathrm{B}'') & QR^6 = R^5Q^4; \\
(\mathrm{C}'') & R^7Q^4R^7 = QR^4Q; \\
(\mathrm{D}'') & Q^5 = 1.
\end{array}
$$

Here again the conditions seem not to be sufficient and we have found it necessary to add the condition

$$(\mathrm{E}'') \qquad RQ^2 = Q^3R^{10}.$$

We use the same notation as before and denote the right co-sets with respect to the group generated by R as $a, a_1, a_2, a_3, a_4, \cdots, l, l_1, l_2, l_3, l_4$, where

$$
\begin{array}{lll}
a = 1 & e = QR^7 & i = Q^2R^4 \\
b = QR & f = QR^8 & j = Q^2R^5 \\
c = QR^2 & g = QR^{10} & k = QRQ^3R^2 \\
d = QR^3 & h = Q^2R^3 & l = QRQ^3R^4.
\end{array}
$$

The corresponding substitutions on the 60 letters are

$$
\begin{aligned}
Q &= (aa_1a_2a_3a_4) \cdots (ll_1l_2l_3l_4); \\
R &= (a_1bcdc_4b_2a_4eff_1g)(a_2a_3g_3hijb_1j_5i_2h_3e_2) \\
&\quad (b_3j_3ke_1ll_1g_4k_2j_1b_4c_2)(c_1d_3i_1d_2c_3l_2e_4h_1h_2g_1l_4) \\
&\quad (d_1f_2e_3k_3k_4g_2f_4d_4i_4j_2i_3)(a)(f_3)(h_4)(k_1)(l_3).
\end{aligned}
$$

The relation (E″) written in terms of S and T is $[(TS)^3(TS^2)^3]^2 = 1$.

The group generated by two operators of orders two and three with product of order 11 *and commutator of order* 5 *which satisfy the further relation* $[(TS)^3(TS^2)^3]^2 = 1$ *is the simple group of order* 660.

URBANA, ILLINOIS,
OCTOBER, 1927.

On Triadic Cremona Nets of Plane Curves.

By Fay Farnum.

I. Introduction.

1. A Cremona transformation is fixed when its scheme of f-points and f-curves is known.

Those which have in each plane only one group of fundamental points (f-points), composed of points of equal multiplicity, are called symmetric and are well known. A correspondence is called semi-symmetric if, in each of the two planes, there are only two groups of f-points; of these the Jonquières case is a familiar example; all others have been discussed by Montesano (D).*

A correspondence is called triadic if, in each of the two planes, there are three and only three groups of f-points; such correspondences have received little attention; one isolated case was discussed by Montesano, (C)* and a more general set by Ruffini (A).*

It is the purpose of this paper to determine additional triadic cases; in it are developed:

(a) some triadic correspondences as products of previously existing correspondences,

(b) all triadic correspondences having simple fundamental points in both planes,

(c) some triadic correspondences having simple fundamental points in one plane only.

2. *Characteristic Square.* The characteristic square of the transformation is:

$$T \equiv \begin{array}{ccccc} n & r_1 & r_2 & \cdots & r_p \\ s_1 & \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1p} \\ s_2 & \alpha_{21} & \alpha_{22} & \cdots & \alpha_{2p} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ s_p & \alpha_{p1} & \alpha_{p2} & \cdots & \alpha_{pp} \end{array}$$

in which n is the order of the transformation; $r_1, \cdots, r_p$ the orders of the f-points in one plane; $s_1, \cdots, s_p$ the orders of the f-points in the other plane;

---

* The reference is explained in section 3.

and α_{ij} is the number of times the fundamental curve (f-curve) of s_i goes through r_j, or also the number of times the f-curve of r_j goes through s_i.

The characteristic square may be simplified by indicating a group of α_i f-points of multiplicity r_i by the symbol $[\alpha_i, r_i]$ and a group of β_k f-curves of order s_k by $[\beta_k, s_k]$. In designating the number of passages of a group of curves through that group of points, such that each curve passes through all but one of the points L times, and through the remaining one $L \pm 1$ times, use is made of the symbol L followed by a plus sign or a minus sign, according as it passes through the remaining point $L+1$ or $L-1$ times. The simplified characteristic square for the isologous net is

$$\begin{array}{ccc} n & [1, n-1] & [2(n-1), 1] \\ [1, n-1] & n-2 & 1 \\ [2(n-1), 1] & 1 & 0+ \end{array} .$$

For a net of order n which has α_1 f-points of multiplicity $r_1, \cdots, \alpha_m$ of multiplicity r_m, the characteristic group of the transformation is given by

$$G_n \equiv [\alpha_1, r_1] \cdots [\alpha_m, r_m].$$

The values in T are subject to the following relations

(a) n, r_i, s_k are positive integers,

(b) α_{ij} are positive integers or 0,

(1) (c) $\sum \alpha_i r_i^2 = n^2 - 1$,

(2) $\sum \alpha_i r_i = 3(n-1)$.

These are called the equations of condition.

(d) Each f-curve is rational and is completely determined by its passages through the f-points.

(e) The f-curves intersect only in f-points.

(f) The total number of passages of f-curves through an i-fold f-point is $3i-1$.

3. *References.* The results necessary for this paper were obtained from the following articles, and will be referred to by the indicated letters:

(A) F. P. Ruffini: "Sullo risoluzione delle trasformazioni Cremoniane," p. 499, *Memoire Istituto Bologna,* Ser. 3, Vol. 8 (1877).

(B) D. Montesano: "Sur le retiomaloidiche di curve," *Rendiconti Accademia Napoli,* 1905.

(C) D. Montesano: "Sur i quadri caratteristici delle corrispondenze birationali piane," *Rendiconti Accademia Napoli,* Ser. 3, Vol. 21 (1915).

(D) D. Montesano: "Le corrispondenze birazionali piane emisimmetriche," *Rendiconti Accademia Napoli,* Ser. 3, Vol. 24 (1918).

II. CORRESPONDENCES DEVELOPED BY PRODUCTS.

4. Products of symmetric and Ruffini correspondences will be shown to give the following:

(a) $G_n \equiv [3, n/2], \quad [4, (n-2)/4], \quad [(n-2)/4, 2] \qquad n \equiv 2 \pmod{4},$

(b) $G_n \equiv [6, 2n/5], \quad [4, (n-5)/10], \quad [(n-5)/10, 2] \qquad n \equiv 5 \pmod{10},$

(c) $G_n \equiv [7, 3n/8], \quad [4, (n-8)/16], \quad [(n-8)/16, 2] \qquad n \equiv 8 \pmod{16},$

(d) $G_n \equiv [8, 3n/17], \quad [4, (n-17)/34], \quad [(n-17)/34, 2] \qquad n \equiv 17 \pmod{34}.$

(a) For any positive integral value of v there exists a Ruffini group $G_{2v+1} \equiv [4, v], [v, 2]$, which is geometric. Form the product of this group and the quadratic transformation, that is,

$$\begin{aligned} G_{2v+1} &\equiv [3, 0],\ [4, v],\ [v, 2], \\ G_2 &\equiv [3, 1]. \end{aligned}$$

The latter is so chosen that none of the f-points of the two fundamental systems are common as indicated by

$$\begin{array}{cccccc} 2v+1 & 0 & 0 & 0 & [4, v] & [v, 2] \\ 2 & 1 & 1 & 1 & & \\ 1 & 0 & 1 & 1 & & \\ 1 & 1 & 0 & 1 & & \\ 1 & 1 & 1 & 0 & & \end{array}$$

This product gives

$$G_n \equiv [3, r],\ [4, v],\ [v, 2],$$

where

$$n = 2(2v+1), \quad r = 2v+1;$$

hence we have

$$G_{4v+2} \equiv [3, 2v+1],\ [4, v],\ [v, 2], \qquad (v = 1, 2, 3, \cdots)$$

or, in terms of n,

$$G_n \equiv [3, n/2],\ [4, (n-2)/4],\ [(n-2)/4, 2] \qquad n \equiv 2 \pmod{4}.$$

For $n = 6, \qquad G_6 \equiv [3, 3],\ [4, 1],\ [1, 2],$

$n = 10, \qquad G_{10} \equiv [3, 5],\ [4, 2],\ [2, 2]$ or $G_{10} \equiv [3, 5],\ [6, 2].$

The latter degenerates into a semi-symmetric case, since $(n-2)/4=2$ for $n=10$, but this is true for no other value of n.

Since there are in this Cremona group $3P_{n/2}$, there will be 3 fundamental straight lines in π, hence in the fundamental system of the other plane, or in the conjugate group, there will be 3 simple f-points.

Let the multiplicities of the other two point groups of the conjugate group be x and y. They must satisfy (1) and (2) hence

$$x=n/2-1, \qquad y=4,$$

and the conjugate characteristic group is

$$G_n \equiv [4, n/2-1],\ [(n-2)/4, 4],\ [3, 1].$$

Since the fundamental straight lines are determined by two basis points of multiplicity $n/2$ and meet the fundamental system in no other points, the characteristic square may be indicated by

n	$[3, n/2]$,	$[4, (n-2)/4]$,	$[(n-2)/4, 2]$
$[4, n/2-1]$	a	$b\pm$	c
$[(n-2)/4, 4]$	d	e	$f\pm$
$[3, 1]$	$1-$	0	0

and, according to principles previously stated, the following equations must be satisfied:

$$\begin{aligned}
&n^2/2=4a(n/2-1)+4d(n-2)/4+2\\
&n(n-2)/4=3b(n/2-1)+(b\pm 1)(n/2-1)+4e(n-2)/4\\
&2n=4c(n/2-1)+4[(n-2)/4-1]f+4(f\pm 1)\\
&n(n-2)/8=3ab+a(b\pm 1)+(n-2)de/4\\
&n=4ac+[(n-2)/4-1]df+(f\pm 1)d\\
&(n-2)/2=3bc+(b\pm 1)c+[(n-2)/4-1]ef+e(f\pm 1).
\end{aligned}$$

The solutions of these equations give the following values for the characteristic squares:

If v is odd, let $v=2\alpha+1$ and $n=4v+2=8\alpha+6$ with $n\equiv 6 \pmod 8$, we have

n	$[3, n/2]$	$[4, (n-2)/4]$	$[(n-2)/4, 2]$
$[4, n/2-1]$	$(n-2)/4$	$(n-6)/8+$	1
$[(n-2)/4, 4]$	2	1	$0+$
$[3, 1]$	$1-$	0	0

or

$$\begin{array}{cccc}
8\alpha+6 & [3, 4\alpha+3] & [4, 2\alpha+1] & [2\alpha+1, 2] \\
[4, 4\alpha+2] & 2\alpha+1 & \alpha+ & 1 \\
[2\alpha+1, 4] & 2 & 1 & 0+ \\
[3, 1] & 1- & 0 & 0
\end{array}$$

If v is even, let $v = 2\alpha$. Then $n = 4v + 2 = 8\alpha + 2$, with $n \equiv 2 \pmod 8$ and we obtain

$$\begin{array}{cccc}
n & [3, n/2] & [4, (n-2)/4] & [(n-2)/4, 2] \\
[4, n/2-1] & (n-2)/4 & (n-2)/8- & 1 \\
[(n-2)/4, 4] & 2 & 1 & 0+ \\
[3, 1] & 1- & 0 & 0
\end{array}$$

or

$$\begin{array}{cccc}
8\alpha+2 & [3, 4\alpha+1] & [4, 2\alpha] & [2\alpha, 2] \\
[4, 4\alpha] & 2\alpha & \alpha- & 1 \\
[2\alpha, 4] & 2 & 1 & 0+ \\
[3, 1] & 1- & 0 & 0\ .
\end{array}$$

(b) Similarly form the product of $G_{2v+1} \equiv [6, 0]\ [4, v]\ [v, 2]$ and $G_5 \equiv [6, 2]$. We obtain the following correspondences:

For v odd, $n = 20\alpha + 15$ with $n \equiv 15 \pmod{20}$,

$$\begin{array}{cccc}
20\alpha+15 & [6, 8\alpha+6] & [4, 2\alpha+1] & [2\alpha+1, 2] \\
[4, 10\alpha+1] & 4\alpha+2 & \alpha+ & 1 \\
[2\alpha+1, 10] & 4 & 1 & 0+ \\
[6, 2] & 1- & 0 & 0\ .
\end{array}$$

For v even, $n = 20\alpha + 5$ with $n \equiv 5 \pmod{20}$,

$$\begin{array}{cccc}
20\alpha+5 & [6, 8\alpha+2] & [4, 2\alpha] & [2\alpha, 2] \\
[4, 10\alpha] & 4\alpha & \alpha- & 1 \\
[2\alpha, 10] & 4 & 1 & 0+ \\
[6, 2] & 1- & 0 & 0\ .
\end{array}$$

(c) The product of $G_{2v+1} \equiv [7, 0]\ [4, v]\ [v, 2]$ and $G_8 \equiv [7, 3]$ gives for v even, $n \equiv 8 \pmod{32}$, $n > 40$ with

$$\begin{array}{cccc}
32\alpha+8 & [7, 12\alpha+3] & [4, 2\alpha] & [2\alpha, 2] \\
[4, 16\alpha] & 6\alpha & \alpha- & 1 \\
[2\alpha, 16] & 6 & 1 & 0+ \\
[7, 3] & 1+ & 0 & 0\ ,
\end{array}$$

and for v odd, $n \equiv 24 \pmod{32}$,

$$\begin{array}{cccc}
32\alpha + 24 & [7, 12\alpha + 9] & [4, 2\alpha + 1] & [2\alpha + 1, 2] \\
[4, 16\alpha + 8] & 6\alpha + 3 & \alpha + & 1 \\
[2\alpha + 1, 16] & 6 & 1 & 0 + \\
[7, 3] & 1 + & 0 & 0 \ .
\end{array}$$

(d) The product of $G_{2v+1} \equiv [8, 0] \ \ [4, v] \ \ [v, 2]$ and $G_{17} \equiv [8, 6]$ gives, for v odd, $n \equiv 51 \pmod{68}$ with

$$\begin{array}{cccc}
68\alpha + 51 & [8, 24\alpha + 18] & [4, 2\alpha + 1] & [2\alpha + 1, 2] \\
[4, 34\alpha + 17] & 12\alpha + 6 & \alpha + & 1 \\
[2\alpha + 1, 34] & 12 & 1 & 0 + \\
[8, 6] & 2 + & 0 & 0 \ ,
\end{array}$$

and, for v even, $n \equiv 17 \pmod{68}$ with

$$\begin{array}{cccc}
68\alpha + 17 & [8, 24\alpha + 6] & [4, 2\alpha] & [2\alpha, 2] \\
[4, 34\alpha] & 12\alpha & \alpha - & 1 \\
[2\alpha, 34] & 12 & 1 & 0 + \\
[8, 6] & 2 + & 0 & 0 \ .
\end{array}$$

5. *Product of two Ruffini correspondences.* The product of two Ruffini ($G_{2v+1} \equiv [4, v] \ \ [v, 2]$) groups of the same order, made in such a way that in the intermediate plane, the two fundamental systems have the double points in common, gives

$$G_{4v^2+1} \equiv [4, 2v^2 - v] \ \ [v, 4v] \ \ [4, v].$$

Use is made of the theorem (D, p. 18): "Given two plane birational correspondences $K_{\pi\pi'}$, $K_{\pi'\pi''}$ represented respectively by the squares

$$Q \equiv \left|\begin{array}{ccc} n & [\alpha, r'] & \cdots \\ [\alpha, r] & w \pm & \cdots \\ \vdots & \vdots & \end{array}\right|^{\pi'}_{\pi} \qquad Q' \equiv \left|\begin{array}{ccc} n' & [\alpha, r_1'] & \cdots \\ [\alpha, r_1'] & w \pm & \cdots \\ \vdots & \vdots & \end{array}\right|^{\pi''}_{\pi'} .$$

If in the plane π' the α multiple points of order r' coincide with the α multiple points of order r_1', the resulting correspondence $K_{\pi\pi''}$ is given by

$$Q'' \equiv \underset{\pi}{\,} \left|\begin{array}{ccc} n'' & [\alpha, m_0''] & \cdots \\ [\alpha, m_0] & w'' \pm & \cdots \\ \vdots & \vdots & \end{array}\right|^{\pi''}$$

in which the term w'' has the sign $+$ or $-$ according as the signs of w, w'

are identical or opposite. Hence we obtain for v odd,

$$\begin{array}{cccc}
4v^2+1 & [4, 2v^2-v] & [v, 4v] & [4, v] \\
[4, 2v^2-v] & v^2-v & 2v-1 & (v-1)/2 \\
[v, 4v] & 2v-1 & 4\text{—} & 1 \\
[4. v] & (v-1)/2 & 1 & 0
\end{array},$$

for v even,

$$\begin{array}{cccc}
4v^2+1 & [4, 2v^2-v] & [v, 4v] & [4, v] \\
[4, 2v^2-v] & v^2-v & 2v-1 & v/2 \\
[v, 4v] & 2v-1 & 4\text{—} & 1 \\
[4, v] & v/2\text{—} & 1 & 0
\end{array}.$$

III. TRIADIC CORRESPONDENCES WITH SIMPLE POINTS IN BOTH PLANES.

6. *Classification.* Let there be given between two planes, π and π', a birational correspondence of order n with simple f-points in both planes. The number, β, of simple f-points in π equals the number of fundamental straight lines in π'. In general each fundamental straight line is determined by two f-points, of multiplicities whose sum is n, as p and $n-p$. There are two cases, according as

(a) $p < n/2$ (b) $p = n/2$ (n even).

(a) $p < n/2$. The value of p must be the same for all the fundamental straight lines in π', otherwise there would be more than one point $n-p$, where $n-p > n/2$, which is impossible. Hence there must be, in π, β points of multiplicity p and one of multiplicity $n-p$.

(b) $p = n/2$. In this case $\alpha_{n/2} = 1, 2, 3$, since a homaloidal net cannot have more than 3 basis points of multiplicity $n/2$ (B, p. 11).

(1) In π', $\alpha_{n/2} = 1$; then other f-points of π' must have multiplicity less than $n/2$, hence there is no pair of f-points whose sum is n, therefore no fundamental straight lines. Consequently in π there will be no simple f-points; so this case is excluded.

(2) In π', $\alpha_{n/2} = 2$; then in π' there will be one fundamental straight line, hence one simple f-point in π.

(3) In π', $\alpha_{n/2} = 3$; then in π' there will be three fundamental straight lines, consequently three simple f-points in π.

From these results we have the following theorem: *If between two planes, π and π', there exists a birational correspondence of order n, and if in π there are β simple f-points, then in π' there are*

(1) *β f-points of multiplicity p and one f-point of multiplicity $n-p$, or*

(2) *if $\beta = 1, 3$ there may be*
for $\beta = 1$, two f-points of multiplicity $n/2$,
for $\beta = 3$, three f-points of multiplicity $n/2$.

We will discuss the cases:

I. $\beta =$ any positive integer, $p < n/2$,
II. $\beta = 1, 3$, $p = n/2$ (n even).

7. Case I. $\beta =$ any positive integer, $p < n/2$. The correspondence may be designated by

$$\begin{array}{lll} \text{in } \pi & G_n \equiv [1, m] \ \ [\alpha, s] \ \ [\beta, 1], & \\ \text{in } \pi' & G_n \equiv [1, n-p] \ \ [\beta, p] \ \ [\alpha, 1], & m + s = n. \end{array}$$

The case of $p = 1$ will not be considered since there the correspondence becomes a Jonquières, semi-symmetric correspondence. If p and n are used as parameters, the only set of solutions satisfying the equations of condition is

$$s = (n-1)/p, \qquad m = n - (n-1)/p,$$
$$\beta = 2(n-1)/p - 1, \qquad \alpha = 2p - 1.$$

These values will be positive integers if and only if $n \equiv 1 \pmod{p}$ and $n > 1$. By methods previously used, we obtain the following as the characteristic square of the correspondence. $k = (n-1)/p$.

$$\begin{array}{cccc} kp+1 & [1, kp-p+1] & [2k-1, p] & [2p-1, 1] \\ [1, kp-k+1] & kp+2-p-k & p-1 & 1 \\ [2p-1, k] & k-1 & 1 & 0+ \\ [2k-1, 1] & 1 & 0+ & 0\ . \end{array}$$

The correspondence has so far only been shown to satisfy the equations of condition; it yet remains to prove it geometric. A correspondence is geometric if it is the product of geometric correspondences. The series of products which result in a correspondence is called its series of origin. The reverse of the series of origin of the correspondence here considered is a series of products of the original by the quadratic correspondence, formed so that the three simple f-points of G_2 coincide with the three f-points of highest multiplicity in the multiplicand, and is as follows:

$$\begin{aligned}
G_n &\equiv [1, n-p] \quad [2(n-1)/p-1, p] \quad [2p-1, 1],\\
G_{n-p} &\equiv [1, n-2p] \quad [2(n-1)/p-3, p] \quad [2p-1, 1],\\
G_{n-2p} &\equiv [1, n-3p] \quad [2(n-1)/p-5, p] \quad [2p-1, 1],\\
&\cdots\cdots\cdots\cdots\cdots\cdots\\
G_{n-ip} &\equiv [1, n-(i+1)p] \quad [2(n-1)/p-(2i+1), p] \quad [2p-1, 1];
\end{aligned}$$

and for $i = (n-1)/p - 1$, we have

$$G_{p+1} \equiv [1, n-(n-1)p] \quad [1, p] \quad [2p-1, 1].$$

The last correspondence reduces to

$$G_{p+1} \equiv [1, p] \quad [2p, 1],$$

which is the well-known geometric isologous correspondence. Hence G_n is a geometric correspondence.

If $p = 2$, the result is the general Ruffini correspondence (D, p. 499).

That the above correspondence be self-conjugate, the necessary and sufficient condition is that $k = p$.

8. Case II. $\beta = 1, 3 \quad p = n/2$.

There are the following possible cases:

(1) $G_n \equiv [1, n-q] \quad [\alpha, q] \quad [3, 1], \qquad q < n/2$
$G_n \equiv [3, n/2] \quad [1, s] \quad [\alpha, 1],$

(2) $G_n \equiv [3, n/2] \quad [\alpha, s] \quad [3, 1],$
$G_n \equiv [3, n/2] \quad [\alpha, p] \quad [3, 1],$

(3) $G_n \equiv [3, n/2] \quad [2, r] \quad [1, 1],$
$G_n \equiv [2, n/2] \quad [1, s] \quad [3, 1],$

(4) $G_n \equiv [2, n/2] \quad [1, m] \quad [1, 1],$
self-conjugate,

(5) $G_n \equiv [2, n/2] \quad [1, m] \quad [\alpha, 1],$
$G_n \equiv [1, n-p] \quad [\alpha, p] \quad [1, 1].$

The equations of condition determine the unknown values with the following results:

(1) No triadic correspondence except the Ruffini case,

(2) The correspondence is given by

$$\begin{array}{cccc} n & [3, n/2] & [9-72/(n+4), n+4/6] & [3,1] \\ [3, n/2] & n/4 & (n+4)/12 & 1 - \\ [9-72/(n+4), n+4/6] & (n+4)/12 & z \pm & 0 \\ [3,1] & 1 - & 0 & 0 \end{array}$$

where $z = (n+4)(n/4 \pm 1)/9(n-4)$, $n = 68, 32, 20, 8$ only,

(3) No triadic case,

(4) $\alpha = 8 + 32/(n+2)$, hence $n = 2, 6, 14, 30$, giving correspondences for $n = 6, 14, 30$ only,

(5) No solution.

IV. SOME TRIADIC BIRATIONAL CORRESPONDENCES WITH SIMPLE FUNDAMENTAL POINTS IN ONE PLANE ONLY.

9. *Classification.* Let $\beta =$ number of simple f-points in plane π, then, by the theorem previously discussed, there will be in π',

I. For β any integer, β f-points of multiplicity p $(p < n/2)$ and one f-point of multiplicity $n - p$, or

II. For $\beta = 1$ and n even, two f-points of multiplicity $n/2$, or

III. For $\beta = 3$ and n even, three f-points of multiplicity $n/2$.

10. Case I. The correspondence may be represented by

$$\begin{array}{ccccc} n & [1, v] & [\alpha, s] & [\beta, 1] & \\ [1, n-p] & a & b & 1 & \\ [\beta, p] & c & d & 0 + & \\ [\alpha, t] & e & f \pm & 0 & t \neq 1. \end{array}$$

The cases for which $p = 1$ and $p = 2$ give the semi-symmetric Jonquières and general Ruffini triadic correspondences, respectively, hence, will not be discussed.

From the characteristics stated in § 2 we obtain the following equations:

by (d) (3) $c^2 + \alpha d^2 = p^2$,
(4) $(n-p)^2 + 1 = a^2 + \alpha b^2 + \beta$,
(5) $t^2 = e^2 + \alpha f^2 \pm 2f$,

by (f) (6) $e + \alpha f \pm 1 = 3t - 1$,
(7) $c + \alpha d = 3p + 2$,
(8) $3(n-p) - 1 = a + \alpha b + \beta$,

by (e) (9) $ca + \alpha db + 1 = p(n-p)$.

Let $p = 3$, then $c < p$, hence $c = 0$, 1 or 2; similarly $d < p$, hence $d = 0$, 1, or 2. The equations (3) and (7) are satisfied by $c = 2$, $\alpha = 5$, and $d = 1$, giving $b = (n-2)/3$ or $(n-4)/3$. In equations (5) and (6) we must have

$$t < p, \ \therefore\ t = 2; \quad e < t, \ \therefore\ e = 0 \text{ or } 1; \quad f < t, \ \therefore\ f = 0 \text{ or } 1.$$

Under these conditions the only solution is with a minus sign for f, and is given by $t = 2$, $e = 1$, $f = 1$. By methods previously used we complete the following correspondence, where the parameter is $k = (n-2)/3$:

$$\begin{array}{ccccc}
3k+2 & [1,2k] & [5,k+1] & [2(k-1),1] & \\
[1,3k-2] & 2(k-1) & k & 1 & \\
[2(k-1),3] & 2 & 1 & 0+ & \\
[5,2] & 1 & 1- & 0 & (k = 2, 3, 4 \cdots).
\end{array}$$

The series of origin is given by

$$\begin{aligned}
G_{k+1} &\equiv & & & [1,k] & \ [2k,1], \\
G_{2k+2} &\equiv & & [1,k] & [3,k+1] & \ [2k,1], \\
G_{3k+2} &\equiv & [1,2k] & \ [2,k+1] & [3,k+1] & \ [2(k-1),1].
\end{aligned}$$

Similarly for $p = 4$, we obtain the following correspondences:

$$\begin{array}{cccccc}
4k+2 & [1,3k] & [7,k+1] & [2(k-2),1] & & k = (n-2)/4 \\
[1,4k-2] & 3(k-1) & k & 1 & & \\
[2(k-2),4] & 3 & 1 & 0+ & & \\
[7,3] & 2 & 1- & 0 & , &
\end{array}$$

and

$$\begin{array}{cccccc}
4k+1 & [1,3k+1] & [7,k] & [2k-1,1] & & k = (n-1)/4 \\
[1,4k-2] & 3(k-1) & k & 1 & & \\
[2(k-2),4] & 3 & 1 & 0+ & & \\
[7,3] & 2 & 1- & 0 & . &
\end{array}$$

11. *General Correspondence.* The correspondences for $p = 3$, 4 suggest the generalized case $p = p$ and the following analogous correspondence:

$$\begin{array}{ccccc}
kp+2 & [1,k(p-1)] & [2p-1,k+1] & [2(k+2-p),1] & \\
[1,kp+2-p] & (p-1)(k-1) & k & 1 & \\
[2(k-p+2),p] & p-1 & 1 & 0+ & \\
[2p-1,p-1] & p-2 & 1- & 0 & k = (n-2)/p.
\end{array}$$

It is necessary that $k + 2 - p > 0$, hence we may have $p = 3, 4, \cdots$, and

$k = 2, 3, \cdots$ This correspondence can be shown to check in the necessary equations and the following gives the series of reductions by quadratic inversions to the Jonquières correspondence:

$$G_{kp+2} \equiv [1, k(p-1)]\ \ [2p-1, k+1]\ \ [2(k-p+2), 1],$$
$$G_{k(p-1)+2} \equiv [1, k(p-2)]\ \ [2p-3, k+1]\ \ [2(k-p+3), 1],$$
$$\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot$$
$$G_{k(p-i)+2} \equiv [1, k\{p-(i+1)\}]\ \ [2p-(2i+1), k+1]\ \ [2\{k-p+(i+2)\}, 1].$$

The last gives for $i = p - 2$

$$G_{2k+2} \equiv [1, k]\ \ [3, k+1]\ \ [2k, 1],$$

from which a quadratic inversion gives

$$G_{k+1} \equiv [1, k]\ \ [2k, 1].$$

12. Case II. $\beta = 1$, n even, two f-points of multiplicity $n/2$.

The possible solutions are indicated by:

A. n	$[2, n/2]$	$[\alpha, s]$	$[1, t]$	*B.* n	$[2, n/2]$	$[2, s]$	$[1, t]$
$[2, p]$	$a +$	b	c	$[2, p]$	a	$b \pm$	c
$[\alpha, u]$	d	$e \pm$	f	$[2, u]$	$d +$	e	f
$[1, 1]$	1	0	0	$[1, 1]$	1	0	0 .

From the principles stated in § 2 we have for *A*,

$$t^2 + 1 = 2c^2 + \alpha f^2 \tag{10}$$

$$3t - 1 = 2c + \alpha f. \tag{11}$$

The solving of (10) and (11) for t in terms of α and c gives

$$t = \{-3(2c+1) \pm \alpha^{1/2}[2(\alpha-7)c^2+4c+10-\alpha]^{1/2}\}/(\alpha-9), \qquad \alpha \neq 9. \tag{12}$$

In order that t be rational it is necessary that

$$2(7-\alpha)c^2 - 4c + \alpha - 10 \leq 0$$

which gives for $c = 0$, $\alpha \leq 10$

$$c \geq 1, \ \ \alpha \geqq 7 - (4c+3)/(2c^2-1).$$

The latter relation shows that, for $\alpha < 7$, it is necessary to consider only those values of c which make $(4c+3)\ (2c^2-1) > 1$, that is, $c = 1, 2$. Hence the following cases will be discussed:

(a) $c = 0$, (b) $c = 1$, (c) $c = 2$, (d) $\alpha \geq 7$, $\alpha = 7, 8, 9, 10 \cdots$.

(a) $c=0$, $\alpha \leqq 10$, $\alpha \neq 9$. By the equations of condition, we find that the only new triadic correspondence is a G_{10} having in π the characteristic group $[2,5]$ $[5,3]$ $[1,2]$ and in π', $[2,3]$ $[5,4]$ $[1,1]$.

(b) $c=1$, $\alpha \neq 9$. Equation (12) gives $t=1$, or $-1+18/(9-\alpha)$; hence the only solutions are given by

$$\begin{array}{llll} \alpha = 3 & 6 & 7 & 8 \\ t = 2 & 5 & 8 & 17 \\ n = 4 & \left\{\begin{matrix} 10 \\ 22 \end{matrix}\right. & \left\{\begin{matrix} 16 \\ 72 \end{matrix}\right. & 34. \end{array}$$

All of these give geometric correspondences.

(c) $c=2$. For the given equations, the only rational values for t are 1 and 8. The former has been discussed and the latter gives $r=19/3$, which is excluded. Hence this case adds no triadic correspondence.

(d) $\alpha \geqq 7$. If $\alpha=7$, equations (10) and (11) give

$$t = \{3(2c+1) \pm [7(4c+3]^{1/2}\}/2.$$

In order that t be rational it is necessary that $c \equiv 1 \pmod 7$, that is, $c=7x+1$; it is also necessary that $7(4c+3)=7(28x+7)=49(4x+1)$ be a square. The resulting values are

$$\begin{array}{ll} t = 21k^2+28k+8 & \qquad \text{and} \qquad t = 21k^2+14k+1 \\ f = 7k^2+10k+3 & \qquad\qquad\qquad f = 7k^2+4k \\ c = 7k^2+7k+1 & \qquad\qquad\qquad c = 7k^2+7k+1 \\ r = 7 & \qquad\qquad\qquad r = 7. \end{array}$$

The first gives $t=1$, and the second a mixed number for s, hence we have no new correspondences.

$\alpha=8$. Similarly it can be shown that this case gives no new correspondences.

$\alpha=9$. From (12) we find that for c to be rational, it is necessary that

$$4(3t-1)^2+(8+6t)$$

be a square, which gives 1 as the only integral value of t.

$\alpha=10$. The only value possible for t is 1, hence no new triadic correspondence.

Further values of α $(\alpha > 10)$ could be similarly investigated.

B. The only possible value for t is 1.

13. Case III. $\beta = 3$, n even, 3 f-points of multiplicity $n/2$. If we let $s = 2$, $b = 0$, we obtain as one correspondence for this case,

$$G_n \equiv [3, n/2] \;\; [4, (n-2)/4] \;\; [(n-2)/4, 2],$$

which is the same correspondence that was previously developed as a product of the Ruffini and quadratic correspondences.

14. *Conclusion.* This completes the discussion of all triadic correspondences involving simple points in both planes, hence of all self-conjugate triadic correspondences with simple points. It also gives some solutions for each of the cases of triadic correspondences involving simple points in one plane only.

Number Relations between Types of Extremals Joining a Pair of Points.[*]

By D. E. Richmond.[†]

---

1. *Introduction.* Writers on the calculus of variations have heretofore been primarily concerned with minimizing or maximizing a given integral. Accordngly, attention has been centered on extremal arcs which have at most one conjugate point to a given point. The Euler equation, however, gives merely the condition that the integral be stationary. A new field has been opened up by H. M. Morse[‡] in a paper which considers extremal arcs bearing an arbitrary number of conjugate points. In this paper, Morse develops a parallelism between types of critical points and types of extremal arcs, in which the type of an extremal arc, whose end-points are not conjugate to each other, is determined by the number of points within the arc conjugate to the initial point. Hence, a given extremal arc has a type number equal to the number of such conjugate points.

The primary object of the present paper is to establish the completeness of a set of relations obtained by Morse, between the numbers of extremals of different type, joining two fixed points in an ordinary calculus of variations problem. Morse's relations are obtained for the parametric case under certain general boundary conditions and under the hypothesis of the existence of a field of extremals covering the region of operation.

By the aid of the field of extremals which are taken as the curves $x =$ constant, Morse first reduces the problem to one in non-parametric form. When Morse does not wish to exclude the case where the end-points of an extremal arc are mutually conjugate, he assumes that the integrand is analytic. We shall exclude this possibility and accordingly require less in the way of hypothesis. Reference to Morse's paper will show that the relations in question hold for the following non-parametric problem. Given the integral

$$\int_{x_0}^{x_1} f(x, y, y')\,dx, \tag{1}$$

---

[*] Presented to the Society, May 7, 1927.

[†] National Research Fellow in Mathematics.

[‡] To appear in the June issue of the *Transactions of the American Mathematical Society.*

where $f(x, y, y')$ is of class C'''' for any number y' and for (x, y) in an open region R of the (x, y) plane including the x-axis. In R let there be given two curves

$$\begin{array}{ll} \gamma_1 & y = y_1(x), \quad -\infty < x < \infty, \\ \gamma_2 & y = y_2(x), \quad y_1(x) < y_2(x), \end{array}$$

in which $y_1(x)$ and $y_2(x)$ are analytic in x.

I. We suppose that the curves γ_1 and γ_2 are extremal-convex relative to the region R'

$$y_1(x) \leqq y \leqq y_2(x), \quad -\infty < x < \infty,$$

in the sense that extremals tangent to γ_1 and γ_2 lie outside of R' in the neighborhood of the point of contact.

II. We suppose that there exists a positive constant e so small that two points in R' on any two lines $x = k$ and $x = k + e$, respectively, can be joined by a unique extremal arc of (1), upon which there exists no pair of conjugate points.

That (I) and (II) are in fact satisfied in the examples we consider, will appear in Lemmas 2 and 4.

Let A and B be any two points in R' not on the same line $x =$ constant. It follows from I and II that there exists at least one extremal E joining A to B in R'. We suppose that on no extremal E is B conjugate to A. Let n be the maximum type of any extremal joining A to B. Let M_i $(i = 0, 1, \cdots, n)$ be the number of extremals E of type i. Then the following theorem holds.

THEOREM. *Between the numbers M_i corresponding to the set of extremals joining any two fixed points in R' there exist the following relations:*

$$\begin{array}{ll} & M_0 \geqq 1, \\ & M_0 - M_1 \leqq 1, \\ (2) & \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ & \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ & (-1)^{n-1} [M_0 - M_1 + \cdots (-1)^{n-1} M_{n-1}] \geqq 1, \\ & M_0 - M_1 + \cdots (-1)^n M_n = 1. \end{array}$$

Morse has given a proof, not yet published, that if $n > 0$, $M_0 \geqq 2$. No use will be made of this fact, other than to restrict the relations whose completeness is to be proved. No further restrictive relations are known. Let (2′) denote the set of relations (2), in which the first inequality is replaced by $M_0 \geqq 2$.

An obvious method of proving the absence of new independent relations is to show that, given an arbitrary set of positive integers M_i $(i = 0, 1, \cdots, n)$ satisfying (2′), there exists a calculus of variations problem of the form (1) such that the number of extremal arcs of type i joining a given pair of points is equal to M_i. The present paper will give such an existence proof for $n = 2$. Work is in progress on the relations involving extremals of higher type.

It will be well to point out that among the relations (2) or (2′), the next to the last may be omitted. For example, if $n = 2$, $M_0 - M_1 \leqq 1$ follows from $M_0 - M_1 + M_2 = 1$.

We shall show that there exists an example corresponding to each set of positive integers M_0, M_1, and M_2 satisfying

$$\begin{gathered} M_0 \geqq 2, \\ M_0 - M_1 + M_2 = 1. \end{gathered} \tag{3}$$

2. *The integral.* All of our examples will be obtained by an appropriate choice of $\phi(y)$ and b in the integral

$$\int_0^b [y'^2 - \phi(y)]\, dx. \tag{4}$$

In every case, we require that:

(A) *The function* $\phi(y)$ *be of class* C'''' *in an interval*

$$(I)\; -h' - e' \leqq y \leqq h + e,$$

where h, h', e, *and* e' *are positive constants to be more explicitly chosen later;*

(B) $\phi(y)$ *have a minimum at* $y = 0$,

maxima at $y = h$ *and* $y = -h'$, *and* $\phi'(y) \neq 0$ *elsewhere in* (I);

(C) $\phi(0) = 0$,

$\phi(y) > 0, \quad y \neq 0;$

(D) $\phi(-y) = \phi(y)$, $\quad 0 \leqq y \leqq \gamma$, $\quad \gamma \leqq h$ *and* $\leqq h'$, *where* γ *is a positive constant.*

3. *The extremals.* We shall describe the extremals of (4), making use of the general properties of $\phi(y)$ given in § 2.

The Euler equation corresponding to (4) is

$$y'' = -\phi'(y)/2. \tag{5}$$

Let us first consider the extremals only insofar as they remain in the strip S

$$(S) \quad -h' \leqq y \leqq h, \quad -\infty < x < \infty.$$

In the first place, $y = h$ and $y = -h'$ are extremals. The only other extremal of the form $y =$ constant, is $y = 0$. By the existence theorem for differential equations, there exist no other solutions of (5) tangent to $y = h$, $y = -h'$, or $y = 0$.

Because of the property (B) of $\phi(y)$, *any extremal, other than* $y = 0$, *which has points within* S, *must be concave toward the x-axis and hence must intersect it at least once.*

Since x does not occur explicitly in (5), it is clear that if $y = g\ (x)$ is an extremal in S, so also is $y = g\ (x + \alpha)$, where α is an arbitrary constant. It will suffice for our purposes to consider extremals which pass through the origin.

Now (5) admits the first integral

$$(6) \qquad y'^2 + \phi(y) = c^2.$$

Consider the extremal E through the origin with the initial slope y_0'. Since $\phi(0) = 0$, it follows from (6) that $c^2 = y_0'^2$ so that c in (6) may be interpreted as the initial slope. Let E, the extremal through $(0, 0)$ with initial slope c, be represented by $y = g\ (x, c)$.

For $c = 0$, we obtain simply the x axis. For $c > 0$, the discussion will be divided into three cases, according as $c^2 < \phi(h)$, $c^2 = \phi(h)$, or $c^2 > \phi(h)$.

Case I. $c^2 < \phi(h)$. From (6) and (5) it follows that as the ordinate y on E increases, y' decreases until for some constant $\bar{y} < h$, $\phi(\bar{y}) = c^2$. At the point $P = (\bar{x}, \bar{y})$ on E,

$$y' = g_x(\bar{x}, c) = 0.$$

Since E is concave toward the x axis, it has a maximum at P. From (6), the slope of E is numerically the same at all points with the same ordinate, and E is symmetric with respect to P. Hence, for $c^2 < \phi(h)$, the extremal $y = g\ (x, c)$ intersects the x axis again at the point $(2\bar{x}, 0)$.

As c^2 increases from 0 to $\phi(h)$, the maximum ordinate $\bar{y}$ of E, given by $\phi(\bar{y}) = c^2$, increases from 0 to h, since

$$\phi'(\bar{y}) > 0, \quad 0 < \bar{y} < h.$$

Case II. $c^2 = \phi(h)$. Here (6) becomes

$$y'^2 = \phi(h) - \phi(y).$$

Then $y' \neq 0$ for $0 \leqq y < h$, so that y increases steadily on E. Clearly E cannot have a maximum for $y < h$. Now $y' = 0$ for $y = h$, but E is not tangent to $y = h$ because of the existence theorem. It follows that E approaches $y = h$ asymptotically. Obviously then, E does not intersect the x axis again.

Case III. $c^2 > \phi(h)$. The corresponding extremals through $(0, 0)$ pass out of S, intersecting $y = h$. After leaving the origin, they do not intersect each other or the extremal considered in Case II. Let us follow these extremals into the strip,

$$h < y \leqq h + e, \quad -\infty < x < \infty.$$

In this e strip, the extremals are concave away from the x axis, since $\phi'(y) < 0$. Hence they do not return to S before intersecting $y = h + e$. Therefore, in Case III, as in Case II, the extremals through the origin do not intersect the x axis again.

If we now consider negative values of c, we can make a similar classification of extremals. Extremals through the origin intersect the x axis again, if $c^2 < \phi(-h')$, otherwise not.

We may summarize our results in the following lemma.

LEMMA 1. *Given the integral*

$$\int_0^b [y'^2 - \phi(y)]\, dx,$$

where $\phi(y)$ has the properties prescribed in § 2. An extremal which passes through the origin with the slope $c \neq 0$ intersects the x axis again if

$$c > 0 \quad \textit{and} \quad c^2 < \phi(h),$$
$$\textit{or } c < 0 \quad \textit{and} \quad c^2 < \phi(-h'), \qquad \textit{otherwise not.}$$

Let a region S' be defined by

$$(S') \quad -d' - h' \leqq y \leqq h + d, \quad -\infty < x < \infty, \quad 0 < d < e,\ 0 < d' < e'.$$

Then S' is extremal-convex, for since $\phi'(h + d) < 0$, it follows from (5) that an extremal tangent to $y = h + d$ lies outside of S' in the neighborhood of the point of contact. A similar statement applies at the boundary, $y = -h' - d'$.

LEMMA 2. *The region S' is extremal-convex.*

4. *Intersection of extremals with x axis.* We shall now consider an extremal E through the origin, whose initial slope satisfies the inequality

$$c^2 < \phi(\gamma),$$

where γ is the constant of § 2 (D), such that

$$\phi(-y)=\phi(y), \quad 0 \leqq y \leqq \gamma.$$

Then on E, y is a periodic function of x. The first intersection of E with the positive x axis occurs at $x=2\bar{x}$ (cf. § 3), or more explicitly $x=2\bar{x}(c)$. For convenience in notation we shall hereafter let

$$\eta(c)=2\bar{x}(c).$$

For $c^2<\phi(\gamma)$, the intersections of E with the positive x axis occur at

$$x=m\eta(c), \tag{7}$$

m a positive integer. By integrating (6), we have for $c>0$,

$$\eta(c)=2\bar{x}(c)=2\int_0^{\bar{y}(c)} \frac{dy}{[c^2-\phi(y)]^{1/2}}, \quad 0<c^2<\phi(h). \tag{8}$$

Here $\bar{y}(c)$ is obtained by solving

$$\phi(\bar{y})=c^2, \quad \phi'(\bar{y}) \neq 0, \quad 0<\bar{y}<h.$$

Equation (8) is valid in the indicated c interval although for the application in (7) we require that $c^2<\phi(\gamma) \leqq \phi(h)$. For negative values of c, $\eta\ (c)$ is defined by a formula similar to (8).

Now the integral (8) is improper because the integrand becomes infinite for $y=\bar{y}$. But in the neighborhood of $y=\bar{y}$,

$$[c^2-\phi(y)]^{1/2}=(\bar{y}-y)^{1/2}\psi(y), \quad \psi(\bar{y}) \neq 0,$$

where $\psi(y)$ is continuous. The convergence of (7) follows according to well-known rules.

5. *Type of extremal arc.* Let $y=g\ (x,c)$ represent the one parameter family of extremals through $(0,0)$, and $y=g(x,c_0)$ $0<c_0{}^2<\phi(\gamma)$, a particular extremal E of this family. If for abbreviation we let

$$m\eta(c_0)=a,$$

then by (7) the extremal arc E_a

$$(E_a) \quad y=g(x,c_0), \quad 0 \leqq x \leqq a,$$

joins two points on the x axis. We wish to determine its type, i. e., the number of points on E_a conjugate to $(0,0)$. This will require a study of the

Jacobi equation.[*] Now all extremals of (4), for which $-h' < y < h$, are represented by

$$y = g(x + \alpha, c).$$

The Jacobi equation for (4), set up for the particular extremal E, $y = g(x + \alpha_0, c_0)$, is

$$w'' + (\phi''/2) g(x + \alpha_0, c_0) w = 0. \tag{9}$$

It is easily seen[†] that for $c_0 \neq 0$,

$$w_1(x) = g_a(x + \alpha_0, c_0) \text{ and } w_2(x) = g_c(x + \alpha_0, c_0) \tag{10}$$

are linearly independent solutions of (9). But $w_1(x) = g_a(x + \alpha_0, c_0) = g_x(x + \alpha_0, c_0)$ is the slope of E. For E_a, an extremal arc through the origin, $\alpha_0 = 0$, and $c_0^2 < \phi(\gamma)$. If $P, P', P'', \cdots$ are successive maxima and minima on E_a, P' is the first conjugate point to P, P'' is the second, and so on.

Since $w_2(0) = 0$ and $w_2'(0) \neq 0$, the conjugate points to $A = (0, 0)$ on E_a are given by the zeros of $w_2(x)$. By Sturm's separation theorem, the roots of linearly independent solutions of (9) separate each other. Hence the first conjugate point A' to the origin A lies between P and P' on E_a, the second between P' and P'' and so on. On E_a, therefore, there exist not less than $(m - 1)$ nor more than m points conjugate to A. Hence the *type of E_a is either $(m - 1)$ or m.* The decision rests with the sign of $\eta'(c)$ as we proceed to show.

6. *Type of extremal arc (cont.).* Now $\eta(c)$ satisfies the following identity in c.

$$g(m\eta(c), c) \equiv 0, \quad 0 < c^2 < \phi(\gamma).$$

Differentiating with respect to c,

$$g_c(m\eta(c), c) + m\eta'(c) g_x(m\eta(c), c) \equiv 0.$$

On setting $c = c_0$ and $m\eta(c_0) = a$, we have by (10),

$$w_2(a) = - m\eta'(c_0)\, w_1(a).$$

In the following discussion, we shall drop the subscript on c_0. There are three cases, depending on the sign of $\eta'(c)$. For simplicity, we restrict ourselves to positive values of c.

Case I. If $\eta'(c) > 0$, $w_2(a)$ and $w_1(a)$ have opposite signs.

---

[*] Cf. Bolza, *Vorlesungen über Variationsrechnung* (1909), p. 60. This book will hereafter be referred to as B.

[†] Cf. B., p. 74.

(a) If m is even, $w_1(a)$ is positive. Then $w_2(a)$ is negative, which means that $w_1(x)$ vanishes an odd number of times between $x = 0$ and $x = a$, hence $(m - 1)$ times.

(b) If m is odd, $w_1(a)$ is negative. Then $w_2(a)$ is positive which means that $w_2(x)$ vanishes an even number of times between $x = 0$ and $x = a$, hence $(m - 1)$ times.

For Case I, therefore, the type is $(m - 1)$.

Case II. If $\eta'(c) < 0$, it follows similarly that the type is m, whether m is odd or even.

Case III. If $\eta'(c) = 0$, $w_2(a) = 0$ and the point $(a, 0)$ is conjugate to $A = (0, 0)$ on E_a. In the examples which we shall give extremal arcs whose end-points are conjugate will not occur.

LEMMA 3. *Given the extremal arc E_a through the origin represented by*

$$(E_a) \quad y = g(x, c_0), \quad 0 \leqq x \leqq a,$$

where $$a = m\eta(c_0).$$

Then $$g(a, c_0) = 0,$$

and E_a is of type $$\begin{array}{ll} m - 1, & \text{if } \eta'(c_0) > 0; \\ m, & \text{if } \eta'(c_0) < 0. \end{array}$$

7. *Extremals joining points on neighboring parallels.*

LEMMA 4. *There exists a positive constant e so small that two points in S' on any two lines $x = k$ and $x = k + e$ respectively can be joined by a unique extremal arc upon which there exist no pairs of mutually conjugate points.*

Since S' is an extremal-convex region (Lemma 2), any two points P and Q in S' may be joined by at least one extremal arc (§ 1).

Now since $\phi''(y)/2$ is continuous in $-h' - d' \leqq y \leqq h + d$, it is bounded in the same interval. Say $\phi''(y)/2 \leqq M^2$, $-h' - d' \leqq y \leqq h + d$. Comparing the Jacobi equation (9) with

$$z'' + M^2 z = 0,$$

it follows by Sturm's comparison theorem, that no two conjugate points are separated by an x interval less than $\pi/M = e$, say. Hence, if P and Q are on the lines $x = k$ and $x = k + e$, respectively, there exists no pair of conjugate points on any arc PQ.

It now follows that the extremal arcs passing through P and intersecting $x = k + e$, form a field,[*] except for the point P itself. Hence through P and Q there can exist not more than one extremal arc. The lemma is therefore proved.

8. *Required properties of* $\eta(c)$. Let us turn from general considerations to the specific task of constructing examples in which M_0, M_1 and M_2 are arbitrary positive integers satisfying the relations.

$$(3) \qquad \begin{gathered} M_0 \geqq 2, \\ M_0 - M_1 + M_2 = 1. \end{gathered}$$

We first note that all such integers are expressible in the form

$$(11) \qquad \begin{aligned} M_0 &= 2 + p, & p &\geqq 0, \\ M_1 &= 2 + p + q, & q &\geqq 0, \\ M_2 &= 1 + q. \end{aligned}$$

We have four cases to consider:

Case I. q even, p even;
Case II. q odd, p even;
Case III. q even, p odd;
Case IV. q odd, p odd.

We shall choose for the points A and B to be joined by extremal arcs, the origin $(0, 0)$ and the point $(b, 0)$, respectively, in conformity with the limits of the integral (4).

We shall now describe the character of the function $\eta(c)$ to be used for a given p and q in (11).

Case I. q even, p even. Let $q = 2q_1$ and $p = 2p_1$.
We shall so construct an example that:

(1) $b/3 < \eta(0) < b/2$, where $\eta(0) = \lim_{c \to 0} \eta(c)$;
(2) $\eta(c) = b/2$, $\eta'(c) > 0$, for $q_1 + 1$ values of $c > 0$;
(3) $\eta(c) = b/2$, $\eta'(c) < 0$, for q_1 values of $c > 0$;
(4) $\eta(c) = b$, $\eta'(c) > 0$, for $p_1 + 1$ values of $c > 0$;
(5) $\eta(c) = b$, $\eta'(c) < 0$, for p_1 values of $c > 0$;
(6) $\eta(c) > b/3$;
(7) $\eta'(c) \neq 0$ when $\eta(c) = b$ or $b/2$;
(8) $\eta(-c) = \eta(c)$.

---

[*] Cf. B., p. 307.

In (1), $\eta(0)$ gives the first conjugate point* to $(0, 0)$. According to (1) therefore, the extremal $y = 0$ is of type 2. Using this fact and Lemma 3, (where a is to be replaced by b) it follows that for an example for which $\eta(c)$ satisfies the conditions above

$$\begin{aligned} M_0 &= 2(p_1 + 1) = 2 + p, \\ M_1 &= 2p_1 + 2(q_1 + 1) = 2 + p + q, \\ M_2 &= 1 + 2q_1 = 1 + q, \end{aligned}$$

as required.

Case II. q odd, p even. Let $q = 2q_1 + 1$ and $p = 2p_1$. An example will be constructed so that:

(1) $b/2 < \eta(0) < b$;
(2) $\eta(c) = b/2$, $\eta'(c) > 0$, for $q_1 + 1$ values of $c > 0$;
(3) $\eta(c) = b/2$, $\eta'(c) < 0$, for $q_1 + 1$ values of $c > 0$;
(4) $\eta(c) = b$, $\eta'(c) > 0$, for $p_1 + 1$ values of $c > 0$;
(5) $\eta(c) = b$, $\eta'(c) < 0$, for p_1 values of $c > 0$;
(6) $\eta(c) > b/3$;
(7) $\eta'(c) \neq 0$ when $\eta(c) = b$ or $b/2$;
(8) $\eta(-c) = \eta(c)$.

It follows from (1) that the extremal $y = 0$ is of type 1. Using Lemma 3, it follows that for an example for which $\eta(c)$ satisfies the requirements above

$$\begin{aligned} M_0 &= 2(p_1 + 1) = 2 + p, \\ M_1 &= 1 + 2p_1 + 2(q_1 + 1) = 2 + p + q, \\ M_2 &= 2(q_1 + 1) = 1 + q. \end{aligned}$$

Case III. q even, p odd. Let $q = 2q_1$ and $p = 2p_1 + 1$. We seek an example for which

$$\begin{aligned} M_0 &= 2 + 2p_1 + 1, \\ M_1 &= 2 + 2p_1 + 2q_1 + 1, \\ M_2 &= 1 + 2q_1. \end{aligned}$$

Suppose all of the values of c at which the function $\eta(c)$ described in Case I takes on the values b and $b/2$ lie in the interval D,

$$(D) \quad -c_1 \leqq c \leqq c_1.$$

In the present case we start with a function $\eta(c)$ of the same character on D as the function $\eta(c)$ of Case I. Because of the nature of $\eta(c)$ on D

---

* Cf. B., p. 237.

we will obtain extremals whose type numbers sum up as in Case I. In addition we need in this case one extremal of type zero and one of type one. These we shall obtain by requiring that for two values of c we shall have respectively:

$$\eta(c) = b, \quad \eta'(c) > 0, \quad c > c_1;$$
$$\eta(c) = b, \quad \eta'(c) < 0, \quad c > c_1.$$

Case IV may now be treated in a manner similar to Case III, making use this time, however, of the results under Case II.

We have now described an $\eta(c)$ which is suitable for any example involving extremals of type 2 joining A to B, with no extremals of higher type. It is now necessary to so choose $\phi(y)$ that $\eta(c)$ has the character thus described.

9. *Outline of procedure.* We shall define a function $\psi(y)$ by means of different analytic functions over successive y intervals. The function $\psi(y)$ will later be modified to give our final function $\phi(y)$. Let the curve $z = \psi(y)$ be drawn in the zy plane. For the $\psi(y)$ to be described (§ 10), the segments of the curve through $y = 0$, $y = h$ and $y = -h'$ will be parabolic, while the intermediate segments will be linear. The resulting $\psi(y)$ will be continuous but in general there will be a discontinuity of $\psi'(y)$ at each junction. We first obtain a $\psi(y)$ of this description such that the integral

$$\zeta(c) = 2 \int_0^{\bar{z}} \frac{dy}{(c^2 - \psi(y))^{1/2}}, \quad c^2 = \psi(\bar{z}), \tag{12}$$

defines a function $\zeta(c)$, which satisfies the requirements made on $\eta(c)$ in § 8. We shall then modify $\psi(y)$ so as to obtain a function $\phi(y)$ of class C'''' such that the corresponding function $\eta(c)$ has the same properties as does $\zeta(c)$.

10. The *function* $\psi(y)$. The $\psi(y)$ used will be such that $\zeta(0) = \pi$. Hence in choosing b, the upper limit of (4), in order to satisfy (1) of Case I or III (§ 8), we require that

$$b/3 < \pi < b/2,$$

while to satisfy (1) of Case II or IV, we require that

$$b/2 < \pi < b.$$

A value of b, once appropriately chosen for the case under consideration, is to remain fixed during the subsequent discussion.

In any case, in defining $\psi(y)$ for positive y, we begin by assuming that

(A) $\psi(y) = y^2, \quad 0 \leqq y \leqq a_1,$

where a_1 is an arbitrary positive constant. We have

$$\zeta(c) = 2\int_0^c \frac{dy}{(c^2-y^2)^{1/2}} = \pi, \quad 0 < c^2 \leqq \psi(a_1).$$

Now,

$$\lim_{c\to 0} \zeta(c) = \pi, \text{ and accordingly,}$$

we set

$$\zeta(0) = \pi.$$

Let us suppose for the present that we are dealing with Case I. To obtain one of the points satisfying (2) Case I, we next define

$$\text{(B)} \quad \psi(y) = 2a_1 y - a_1^2, \quad a_1 < y \leqq a_2,$$

where a_2 will be chosen in a manner to be explained in the subsequent discussion. It will be noted that $\psi'(y)$ is continuous at $y = a_1$. Using (A) and (B), we have

$$\zeta(c) = 2\int_0^{a_1} \frac{dy}{(c^2-y^2)^{1/2}} + 2\int_{a_1}^{\bar{z}} \frac{dy}{(c^2+a_1^2-2a_1y)^{1/2}}, \quad \psi(a_1) < c^2 \leqq \psi(a_2),$$

where $\psi(\bar{z}) = c^2$. Integrating and differentiating,

$$\text{(13)} \quad \begin{aligned} \zeta(c) &= 2\sin^{-1}(a_1/c) + (2/a_1)(c^2-a_1^2)^{1/2}, \\ \zeta'(c) &= 2/(c^2-a_1^2)^{1/2}\,[c/a_1 - a_1/c], \quad \psi(a_1) < c^2 \leqq \psi(a_2). \end{aligned}$$

Let $F(c)$ denote the right hand member of (13) for any $c > a_1$. The function $F(c)$ becomes positively infinite with c. Furthermore, $F'(c) > 0$. Now let c_2 and the preceding constant a_2 be successively chosen in accordance with the conditions

$$\begin{aligned} b/2 < F(c_2) &= \zeta(c_2) < b, \\ \psi(a_2) &= c_2^2. \end{aligned}$$

The linear segment (B) of $\psi(y)$ is thus terminated so that $\zeta(c)$ does not exceed $\zeta(c_2)$ and hence never equals b. The next linear segment of $\psi(y)$ will be so chosen that $\zeta(c)$ immediately decreases from $\zeta(c_2)$.

We next define

$$\text{(C)} \quad \psi(y) = \beta_2 + 2\mu_2 y, \quad \mu_2 > a_1, \quad a_2 < y \leqq a_3,$$

where β_2, μ_2, and a_3 are constants and in terms of $\psi(y)$ as defined by (B),

$$\psi(a_2) = \beta_2 + 2\mu_2 a_2.$$

This insures the continuity of $\psi(y)$ at $y=a_2$. Now

$$\zeta(c)=2\int_0^{a_1}\frac{dy}{(c^2-y^2)^{1/2}}+2\int_{a_1}^{a_2}\frac{dy}{(c^2+a_1^2-2a_1y)^{1/2}}$$
$$+2\int_{a_2}^{\bar{z}}\frac{dy}{(c^2-\beta_2-2\mu_2y)^{1/2}},$$
$$\psi(a_2)<c^2\leqq\psi(a_3),\quad \psi(\bar{z})=c^2.$$

Integrating and differentiating

$$(14)\qquad \zeta(c)/2=\sin^{-1}(a_1/c_1)+(1/a_1)(c^2-\psi(a_1))^{1/2}$$
$$+(c^2-\psi(a_2))^{1/2}(1/\mu_2-1/a_2),$$

$$\frac{\zeta'(c)}{2}=\frac{(c/a_1-a_1/c)}{(c^2-\psi(a_1))^{1/2}}+\frac{c(1/\mu_2-1/a_1)}{(c^2-\psi(a_2))^{1/2}}.$$

Set the right hand member of (14) equal to $G(c)$ for $c^2>\psi(a_2)$. Computation reveals that $G''(c)>0$. As c^2 increases from $\psi(a_2)$, $G'(c)$ increases from $-\infty$ through 0 approaching $1/\mu_2$ as c^2 becomes infinite. Hence $G(c)$ has one and only one minimum at $c=c_m$, say. The constant a_3 which terminates the interval of definition of (C) is to be chosen sufficiently large so that $\psi(a_3)>c_m^2$. Hence the domain of definition of $\zeta(c)$ in (14) will include $c=c_m$. Another requirement on a_3 will follow.

Now c_m and $\zeta(c_m)$ will of course depend upon the value chosen for μ_2. The required continuity of $\psi(y)$ determines the value of β_2 which is to correspond to a given μ_2. Now

$$\frac{d\zeta(c_m)}{d\mu_2}=\frac{\partial\zeta(c_m)}{\partial c_m}\cdot\frac{dc_m}{d\mu_2}+\frac{\partial\zeta(c_m)}{\partial\mu_2}=\frac{\partial\zeta(c_m)}{\partial\mu_2},$$

since

$$\frac{\partial\zeta(c_m)}{\partial c_m}=\zeta'(c_m)=0.$$

But by (14)

$$\frac{\partial\zeta(c_m)}{\partial\mu_2}=\frac{-2}{\mu_2^2}(c_m^2-\psi(a_2))^{1/2}<0.$$

Hence $\zeta(c_m)$ is a decreasing function of μ_2. It can be shown that as μ_2 is increased from a_1 to ∞, $\zeta(c_m)$ decreases from $\zeta(c_2)$ to 0. We shall now choose μ_2 so that

$$b/3<\zeta(c_m)<b/2.$$

For $c>c_m$, $\zeta(c)$ increases steadily. Let c_3 and the previous constant a_3 be determined by the relations

$$\psi(a_3)=c_3^2,$$

where

$$G(c_3)=\zeta(c_3)=\zeta(c_2).$$

Then by using the definition (C) of $\psi(y)$ over the range $a_2 < y \leqq a_3$, $\zeta(c)$ rises to $\zeta(c_2)$. As before, the next linear segment will cause $\zeta(c)$ to decrease immediately. For $\psi(a_2) < c^2 < \psi(a_3)$, we have $\zeta(c) = b/2$, twice, once with $\zeta'(c) > 0$ and once with $\zeta'(c) < 0$.

The character of $\zeta(c)$ for the range $\psi(a_2) < c^2 \leqq \psi(a_3)$ is typical for the remaining linear segments of $\psi(y)$ to be chosen. To proceed we define

$$\text{(D)} \quad \begin{array}{l} \psi(y) = \beta_3 + 2\mu_3 y, \quad \mu_3 > \mu_2, \quad a_3 < y \leqq a_4, \\ \psi(a_3) = \beta_3 + 2\mu_3 a_3 \end{array}$$

where β_3, μ_3 and a_4 are constants. As before, $\zeta(c)$ has one and only one minimum at $c^2 = c_m{}^2 > \psi(a_3)$. By exactly similar steps, μ_3 may be so chosen that

$$b/3 < \zeta(c_m) < b/2.$$

To terminate (D), choose c_4 and the previous constant a_4 in accordance with the relations

$$\begin{array}{l} \zeta(c_4) = \zeta(c_3), \quad c_4 > c_3, \\ \psi(a_4) = c_4{}^2. \end{array}$$

For $\psi(a_3) < c^2 < \psi(a_4)$, we have $\zeta(c) = b/2$ twice, once with $\zeta'(c) > 0$, once with $\zeta'(c) < 0$.

We continue in this manner, using q_1, segments such as (C) until $\zeta(c) = b/2$ a total of $(q_1 + 1)$ times with $\zeta'(c) > 0$ and q_1 times with $\zeta'(c) < 0$, thus satisfying (2) and (3) of Case I. (§ 8).

For Case III, we can use so far exactly the same $\psi(y)$. For Cases II or IV, if we use $q_1 + 1$ segments like (C), our choice of b relative to $\zeta(0)$ makes $\zeta(c) = b/2$, a total of $q_1 + 1$ times with $\zeta'(c) > 0$ and $q_1 + 1$ times with $\zeta'(c) < 0$. In all cases, for the range of c for which $\zeta(c)$ is now defined, $\zeta(c)$ oscillates between $\zeta(c_2)$ and $\zeta(c_m)$ and hence never equal b or $b/3$.

11. *The function $\psi(y)$ (continued).* Returning to Case I, we now wish to obtain the required intersections with $\zeta = b$. After the last intersection of $\zeta(c)$ with $\zeta = b/2$, instead of allowing $\zeta(c)$ to stop at $\zeta = \zeta(c_2) = \zeta(c_3)$, etc., let us continue to use the last linear segment of $\psi(y)$ until $\zeta(c) > b$. Then by using a proper succession of p_1 linear segments defining $\psi(y)$, $\zeta(c)$ will intersect $\zeta = b$ a total of $p_1 + 1$ times with $\zeta'(c) > 0$ and p_1 times with $\zeta'(c) < 0$, in such a manner that $\zeta(c)$ oscillates between K_1 and K_2 where

$$\begin{array}{c} b/2 < K_1 < b, \\ K_2 > b. \end{array}$$

For a $\psi(y)$ so defined, let $\psi(-y) = \psi(y)$.

If we are dealing with Case II, this part of the procedure is exactly the same. For Cases III or IV, it is merely necessary to add to a $\psi(y)$ thus defined, an extra linear segment like (C) for $y > 0$.

In Cases I and II, we now finish off $\psi(y)$ for $y > 0$ with a parabolic segment having a maximum at $y = h$, where h is an arbitrary constant greater than any value of y prior to this segment. We then choose $h' = h$ and $\psi(-y) = \psi(y)$.

More explicitly, let k be the largest value of y for which $\psi(y)$ has been previously defined. We set

$$\text{(15)} \qquad \psi(y) = \alpha - \beta(h - y)^2, \quad k < y \leqq h + e$$

where e is a positive constant and α and β are so chosen that the curve $z = \psi(y)$ is continuous at $y = k$.

The contribution to $\zeta(c)$ arising from (15) for $c^2 < \psi(h)$, is

$$I(c) = \int_k^{\bar{z}} \frac{dy}{[c^2 - \alpha + \beta(h - y)^2]^{1/2}}, \quad \psi(k) < c^2 < \psi(h), \quad \psi(\bar{z}) = c^2.$$

Integrating,

$$I(c) = (1/\beta^{1/2}) \log \{(h - k)/m + [((h - k)/m)^2 - 1]^{1/2}\},$$
$$m^2 = (\alpha - c^2)/\beta.$$

It is readily proved that $I'(c) > 0$, so that for $\psi(k) < c^2 < \psi(h)$, $\zeta(c)$ continues to increase, becoming infinite as c^2 approaches $\psi(h) = \alpha$.

In cases III and IV, we take the curve $z = \psi(y)$ as defined prior to this paragraph and add a parabolic segment such as the preceding to the part of the curve $y > 0$ and a similar segment to the part $y < 0$.

Except for continuity in its first four derivatives, $\psi(y)$ satisfies the requirements of § 2. The constant γ there introduced is to be chosen in the following manner. For Cases I and II, let $\gamma = h = h'$. For Cases III and IV, we let γ be the largest value of y for which $\psi(-y) = \psi(y)$ in the function $\psi(y)$ which we have described.

THEOREM I. *A continuous function $\psi(y)$ defined by different analytic functions over successive y intervals, and satisfying (B) (C) (D) § 2, may be so chosen that*

$$\zeta(c) = 2 \int_0^{\bar{z}} \frac{dy}{(c^2 - \psi(y))^{1/2}}, \quad \psi(\bar{z}) = c^2,$$

has the properties assigned to $\eta(c)$ in § 8.

12. *A function $\phi(y)$ approximating $\psi(y)$.* Having specified a function

$\psi(y)$ such that $\zeta(c)$ has the properties of the function $\eta(c)$ described in § 8, it remains to define a function $\phi(y)$ of class C'''' such that the corresponding

$$\eta(c) = 2\int_0^{\bar{y}} \frac{dy}{(c^2 - \phi(y))^{1/2}}, \quad \phi(\bar{y}) = c^2,$$

has the same properties. This we proceed to do by setting $\phi(y) = \psi(y)$ outside of an arbitrarily small interval including each junction, $y = a_i$. In the neighborhood of $y = a_i$, $\phi(y)$ will be so chosen that $\phi'(y)$ increases steadily from the slope of the preceding linear segment to the slope of the succeeding linear segment. Then in this neighborhood $\phi(y) > \psi(y)$.

In order to compare $\eta(c)$ and $\zeta(c)$ we require the following lemma.

LEMMA 5. *Given the integral*

$$I = \int_{\bar{y}-e}^{\bar{y}} \frac{dy}{(\phi(\bar{y}) - \phi(y))^{1/2}},$$

in which

$$\phi'(y) > m, \quad \bar{y} - e < y \leqq \bar{y}.$$

Then,

$$I < 2(e/m)^{1/2}.$$

Proof. Using the law of the mean,

$$\phi(\bar{y}) - \phi(y) = \phi'(y^*)(\bar{y} - y) > m(\bar{y} - y)$$

where

$$y < y^* < \bar{y}.$$

Hence,

$$I < (1/m^{1/2}) \int_{\bar{y}-e}^{\bar{y}} \frac{dy}{(\bar{y} - y)^{1/2}} = 2(e/m)^{1/2}.$$

In comparing $\eta(c)$ and $\zeta(c)$, it will suffice to consider the difference produced by the modification in the neighborhood of a single junction, $y = a_i$. For brevity let

$$a_i - (e/2) = r, \quad a_i + (e/2) = s, \quad \psi'(r) = m.$$

We are choosing $\phi(y)$ so that

$$\begin{aligned} &\phi(y) = \psi(y), \quad y \leqq r, \quad s \leqq y; \\ (16) \qquad &\phi(y) > \psi(y), \quad \phi'(y) > m, \quad r < y < s. \end{aligned}$$

Now for $y \leqq r$, since $\phi(y) = \psi(y)$, we have $\bar{y}(c) = \bar{z}(c)$ and $\eta(c) = \zeta(c)$. For $y \geqq s$,

$$\eta(c) - \zeta(c) = 2\int_r^s \frac{dy}{(c^2 - \phi(y))^{1/2}} - 2\int_r^s \frac{dy}{(c^2 - \psi(y))^{1/2}}.$$

Then

$$0 < \eta(c) - \zeta(c) < 2 \int_r^s \frac{dy}{(c^2 - \phi(y))^{1/2}}$$
$$\leqq 2 \int_r^s \frac{dy}{(\phi(s) - \phi(y))^{1/2}} < 4(e/m)^{1/2},$$

using (16) and Lemma 5.

For $r < y < s$,

$$\eta(c) - \zeta(c) = 2 \int_r^{\bar{y}} \frac{dy}{(c^2 - \phi(y))^{1/2}} - 2 \int_r^{\bar{z}} \frac{dy}{(c^2 - \psi(y))^{1/2}} .$$

Then

$$\eta(c) - \zeta(c) < 2 \int_r^{\bar{y}} \frac{dy}{(c^2 - \phi(y))^{1/2}} < 4[(\bar{y} - r)/m]^{1/2} < 4(e/m)^{1/2}.$$

Also

$$\zeta(c) - \eta(c) < 2 \int_r^{\bar{z}} \frac{dy}{(\psi(\bar{z}) - \psi(y))^{1/2}} .$$

While $\psi(y)$ has a discontinuity in its derivative at $y = a_i$, we find by a slight modification of Lemma 5 that

$$\zeta(c) - \eta(c) < 4(e/m)^{1/2}.$$

Hence, *by a proper choice of* e, $|\eta(c) - \zeta(c)|$ *may be made arbitrarily small, uniformly for all values of* c *in the interval*

$$-[\psi(-h')]^{1/2} + e \leqq c \leqq [\psi(h)]^{1/2} - e, \quad e > 0.$$

We now compare $\eta'(c)$ and $\zeta'(c)$ for a value of $c^2 > \psi(s) = \phi(s)$ for which $\zeta(c) = b$ or $b/2$. Let $c^2 - \phi(s) = k$. Now

$$\zeta'(c) - \eta'(c) = 2 \int_r^s \frac{c\,dy}{(c^2 - \phi(y))^{1/2}} - 2 \int_r^s \frac{c\,dy}{(c^2 - \psi(y))^{1/2}} .$$

Then by (16)

$$0 < \zeta'(c) - \eta'(c) < 2 \int_r^s \frac{c\,dy}{(c^2 - \phi(y))^{3/2}}$$
$$< 2c \int_r^s \frac{c\,dy}{(c^2 - \phi(y))^{3/2}} = \frac{2ce}{K^{3/2}} .$$

Hence *by a proper choice of* e, $\zeta'(c) - \eta'(c)$ *may be made arbitrarily small in the neighborhood of an intersection of* $\zeta(c)$ *with* $\zeta = b$ *or* $\zeta = b/2$.

As a result of the foregoing, e may be chosen sufficiently small so that the curve $z = \eta(c)$ intersects $z = b$ and $z = b/2$ the same number of times as does $z = \zeta(c)$, and $\eta'(c)$ and $\zeta'(c)$ have the same sign at corresponding intersections.

THEOREM 2. *A function $\phi(y)$, satisfying the requirements of § 2, may be so chosen that the function*

$$\eta(c) = 2\int_0^{\bar{y}} \frac{dy}{(c^2 - \phi(y))^{1/2}}, \quad \phi(\bar{y}) = c^2,$$

has the properties assigned to $\eta(c)$ in § 8.

13. *Conclusion.* As a result of Theorem 2 we have the important theorem

THEOREM 3. *A function $\phi(y)$ of class C'''' and a constant b may be so chosen that the integral*

$$\int_0^b [y'^2 - \phi(y)]\, dx$$

possesses extremals joining $(0, 0)$ and $(b, 0)$ of which M_i $(i = 0, 1, 2)$ are of type i, where M_0, M_1, and M_2 are arbitrary positive integers satisfying

$$(3) \qquad \begin{aligned} &M_0 \geqq 2, \\ &M_0 - M_1 + M_2 = 1. \end{aligned}$$

Finally,

THEOREM 4. *Between the numbers M_i corresponding to a set of extremals joining any two fixed points in an extremal-convex region S', there exist no relations independent of* (3).

By a similar procedure it is easily shown that the relations,

$$\begin{aligned} &M_0 \geqq 2, \\ &M_0 - M_1 = 1 \end{aligned}$$

are complete for the case $n = 1$.

HARVARD UNIVERSITY.

An Intrinsic Treatment of Poisson's Integral.[*]

BY FRED W. PERKINS.

Introduction. In many phases of mathematical work it is desirable to have the statements of fundamental propositions available in a form which is independent of any particular coördinate system. Such a property not only broadens the range of applicability of a formula, but often, by suggesting a simple intuitive interpretation, throws light on the meaning of a relation. A number of such interpretations of Poisson's integral have been given. The present paper generalizes these, and gives special attention to the case in which the circle is replaced by the half plane, where the integral can be given a particularly simple form.[†] It will appear that in general the representations given for Poisson's integral are valid without change of form for circle and half plane.

Section 1 is devoted to the study of the case of the half plane and to the consideration of a question of uniqueness. Section 2 gives several intrinsic formulations of Poisson's integral, embracing as special cases those given by Schwarz, Neumann, and Bôcher. Section 3 develops an intrinsic formula for the directional derivative of a harmonic function in terms of its boundary values. In Section 4 these results are used to generalize certain theorems regarding the oscillation of harmonic functions, while Section 5 is devoted to a discussion of the spatial analogues of some of these theorems of oscillation.

1. *A Formula Related to Poisson's Integral.* The solution of the first boundary problem of potential theory for a half plane is obtainable at once from results due to Evans and Bray [‡] and to Evans [§] by a conformal mapping

---

[*] Parts of this paper were presented to the American Mathematical Society, January 2, 1926, in two communications entitled "A Function Related to Poisson's Integral" and "On the Oscillation of Harmonic Functions (Second Paper)."

[†] The transformation of the circle into the half-plane has been used by Osgood (*Funktionentheorie,* 4th edition, Leipzig, 1923, p. 638) as a basis for the discussion of Poisson's integral. But full use has not been made of the geometry of the situation.

[‡] "Sur l'intégrale de Poisson généralisée," *Comptes Rendus,* t. 176 (1923), p. 1042; "La Formule de Poisson et le Problème de Dirichlet," *ibid.,* p. 1368.

[§] "Sur l'Intégrale de Poisson," *ibid.,* t. 177, p. 241.

of the half plane on the interior of the unit circle, and the use of Bôcher's interpretation of Poisson's integral.*

Let the half plane be taken as either the upper half or the lower half of the plane of x and y, and P be one of its points ($y \neq 0$). Let p be a variable point of the x-axis and ω' the angle from the perpendicular to the x-axis from P to the tangent at P of the circle through P and p orthogonal to the x-axis. Half this angle is the angle ω from the perpendicular to the x-axis at P to the line Pp. The results cited then enable us to state the following theorem:

THEOREM. *If $f(p)$ is any function of p, measurable (L) as a function of x, and bounded, the Lebesgue integrals*

$$u(P) = (1/2\pi) \int_{-\pi}^{\pi} f(p)\, d\omega' = (1/\pi) \int_{-\pi/2}^{\pi/2} f(p)\, d\omega,$$

where $d\omega$ and $d\omega'$ are counted positive in the direction of increasing x, define a function of P which is bounded and harmonic throughout the interior of either half plane, and assumes the boundary values $f(p)$ for approach along the circles ω' at almost all p. It is the only such function.

From Fatou's theorem † we have also the following result:

The function $u(P)$ approaches the boundary value $f(p)$ as P approaches p in any direction which makes a positive angle with the boundary, for almost all p; if $f(p)$ is continuous at p_0, $u(P)$ approaches the boundary value $f(p_0)$ at p_0 for the general two dimensional sense of approach.

Suppose that we restrict ourselves to the two dimensional sense of approach, using the expressions "assumes" or "approaches the boundary values" only in this sense. The theorem then yields the following statement:

If $f(p)$ is continuous almost everywhere, and bounded, the above formulas give us a function which is bounded and harmonic throughout the interiors of the half planes, and assumes the boundary values at all points of continuity. It is the only such function.

It should be noted that the uniqueness property does not hold in the form stated above [*] if the requirement that $f(p)$ be continuous almost everywhere on the boundary be dropped, as is shown by the following example. Moreover it is impossible, in this example, to remove the difficulty merely by redefining $f(p)$ on a point set of measure zero. It may also be noted that the function $f(p)$ here constructed is of class 1, in the sense of Baire. We shall find it convenient to study the problem with reference to the interior of a circle; the result can be extended to a half plane by inversion.

---

* *Bulletin of the American Mathematical Society*, Ser. 2, Vol. 4 (1897-98), p. 424; *Annals of Mathematics*, Ser. 2, Vol. 7 (1906), p. 92. See also Osgood, *loc. cit.*, p. 635.

† Fatou, "Séries Trigonométriques et Séries de Taylor," *Acta Mathematica*, Vol. 30 (1906), pp. 335-400, see p. 345 and p. 357.

We construct a point set e on an arc of the circumference C which is perfect and nowhere dense, yet such that the measure of the points common to e and any interval with mid-point at a point of e shall be positive. This can be done, for instance, by starting with an open arc of C, removing the open quarter with center at the mid-point of the arc, removing in the same way the open sixteenths of the remaining arcs, removing in the same way the open sixty-fourths of the four then remaining arcs, and so on indefinitely. The set e shall consist of the points of the initial arc never so removed. There is no difficulty in verifying the fact that it has the stated properties.

We now define $f(p)$ as 1 on e and 0 on the rest of C. Let $v(P)$ be the function defined by Poisson's integral for the values $f(p)$ on C, the integral being taken in the sense of Lebesgue. This function is bounded and harmonic throughout the interior of C; it is positive throughout this region, but it assumes the boundary value 0 continuously at each point p of the boundary at which the function $f(p)$ is continuous, namely all the points of C except those in e. Thus $v(P)$ and 0 are two distinct functions harmonic within C and assuming the boundary value $f(p)$ at every point of continuity of that function.

Moreover, the function $f(p)$ cannot be so altered at the points of any set of measure 0 that it uniquely determines a bounded harmonic function approaching it at every point of continuity. For suppose that $f^*(p)$ is any second function differing from $f(p)$ only on a set of measure 0. Poisson's integral, formed for $f^*(p)$ approaches $f^*(p)$ at each point of continuity. But the same is true of the function 0, since the only points at which $f^*(p)$ is continuous are among those for which it vanishes.

---

[*] For a discussion of a modified form of the uniqueness theorem for Poisson's Integral, see Fatou, *loc. cit.*, see also Plancherel, *Bulletin des Sciences Mathématiques*, Ser. 2, Vol. 34 (1910), p. 111. Necessary and sufficient conditions on a harmonic function for representation by Poisson's integral have been obtained by Evans and Bray and by Evans, *loc. cit.* See also the book of the latter on *Logarithmic Potential—Discontinuous Dirichlet and Neumann Problems*, in the Colloquium series of the American Mathematical Society.

To see this we need to verify that $f^*(p)$ is discontinuous at all the points of e. Let p_1 be any point of e. Every interval with p_1 as center contains infinitely many points at which $f(p)$ and $f^*(p)$ each have the value 1. For every such interval contains a point set of positive measure at which $f(p) = 1$, and $f^*(p) \neq f(p)$ only on a point set of measure zero. On the other hand, every such interval contains intervals exterior to the set e, and hence infinitely many points at which $f(p)$ and $f^*(p)$ have the common value zero. Hence, f^* is discontinuous at p_1. Thus $f^*(p)$ can be continuous only at points outside e, where it differs from 0 only at the points of a set of measure 0, and so cannot be continuous at any point where it differs from 0.

2. *Interpretations of Poisson's Integral.* In this section we shall derive a number of extensions and reformulations of known interpretations of Poisson's integral, obtaining certain results independent of coördinate systems. We shall consider functions which are bounded and harmonic in either the interior or exterior of a proper circle, or in a half-plane. From the point of view of the geometry of inversion a straight line is simply a circle through the point at infinity. Since many of the theorems under discussion can be formulated in such a way as to be applicable to a region bounded by this special type of circle as well as to the ordinary region, we shall use the terms "circle" and "point" as in the geometry of inversion, unless the contrary is explicitly indicated.

We shall confine our attention to the case that the boundary values are piece-wise continuous. In this case Poisson's integral,[*] or the formula of the first theorem in the preceding section, gives the unique bounded function which is harmonic throughout the plane except on the circle under consideration and assumes continuously the given boundary values at all points where the latter are continuous. For brevity, we shall refer to this harmonic function as the one "determined" by the given boundary values.

Our first theorem is an extended form of Schwartz's interpretation of Poisson's integral.[†]

THEOREM I. *Let p, p' be the intersections of a fixed circle C and a variable circle c through two distinct fixed points P, Q, not on C. Let $f(p)$ be a piece-wise continuous function on C, and let $g(p) = f(p')$. Let u, v be the*

---

[*] Here absolute value signs should be used for the coefficient of the boundary values $f(p)$ in the integrand, in order that the integral may give the solution of the exterior as well as the interior problem.

[†] Schwartz, *Gesammelte Werke,* Vol. 2, p. 360.

harmonic functions determined by the boundary values $f(p)$, $g(p)$ *respectively. Then*

$$\begin{cases} u(P) = v(Q)\,, \\ u(Q) = v(P)\,. \end{cases}$$

Schwarz's Theorem is equivalent to the special case in which C is a proper circle and Q the point at infinity; for when it is recalled that the value of v at infinity is the same as its value at the center of C, namely the average of the function $f(p')$, the theorem tells us that the value of u at P is the average of the values obtained by transferring the boundary value at p to the point p' on C collinear with P and p. Schwarz stated this result merely for the case in which P is inside C, but his demonstration can readily be extended to cover the case in which P lies outside C. His proof is based upon the formula for Poisson's integral. Another proof, free from computation, is given below.

In the case that C is a proper circle, and Q the point at infinity, we infer from a well-known theorem that the product $\overline{Pp} \cdot \overline{Pp'}$ has a constant value, k^2, as p traces C. If P lies outside C, this means that p, p' are conjugate points with respect to the circle Γ with center at P and radius k. If we invert in Γ the function u, finite and harmonic in C with boundary values $f(p)$ *on* C, is carried into a function v, bounded and harmonic within C, with boundary values $g(p) = f(p')$. Furthermore P is carried into the point at infinity, whence $u(P) = v(\infty)$, i. e. the average of the function $g(p)$. If P is an interior point of C, a rotation about P through an angle π followed by an inversion in Γ carries p into p', P into the point at infinity, and u into a function v bounded and harmonic exterior to C with the boundary values $g(p)$, whence the desired result follows.

The generalization to the theorem as stated above is obtained immediately by making an inversion with center at Q. Since Q does not lie on C, C will always be carried into a proper circle by this transformation; u, v are carried into two functions, u', v' such that

$$\begin{cases} u'(P') = v'(\infty), \\ u'(\infty) = v'(P'), \end{cases}$$

where P' is the point corresponding to P. Inverting again so as to return to the original figure, and using the fact that piece-wise continuous boundary values determine a bounded harmonic function uniquely, we obtain the desired result.

We note incidentally, as a limiting form of the above theorem, that if the boundary values on C are interchanged at each pair of points in which it is cut by a variable circle tangent to a fixed line at a fixed point P, not on C, then the corresponding harmonic functions have the same value at P. The truth of this is immediately apparent on making an inversion with center at P.

THEOREM II. *Let P, Q be any two distinct fixed points not on the fixed circle C. Let ψ be the directed angle* from some fixed direction at P to that directed arc $\overset{\smile}{Pp}$ of the circle c, determined by P, Q and the variable point p on C, which does not pass through Q. Let u be the harmonic function determined by the piece-wise continuous boundary values $f(p)$ on C. Then*

$$u(P) = (1/\pi) \int_C f(P)\, d\psi \pm u(Q),$$

the upper or lower sign being chosen according as P, Q lie in the same region or different regions bounded by C. The integration in each case is taken in such a sense that the region containing P lies on the left.

We discuss first the case in which Q is the point at infinity, and C, therefore, a proper circle.

Here the theorem is obviously true if P is chosen as the center of the circle, so we may exclude this case from further consideration. We take as the fixed direction from which to measure ψ the direction from P to O, the center of C. Let p_0 be the first point in which the ray $\overline{PO}$, when traced in the positive sense from P, meets C, and let p_0' be the other point of intersection. Let p' be the second intersection of C and the line Pp. Denote the directed angle from Op_0 to Op by ϕ, that from Op'_0 to Op' by ϕ'. In all cases we have, for suitable choices of the determinations of ϕ, ϕ', the relation

$$2\psi = \phi + \phi'.$$

Hence we have, according as P lies outside or inside C, either

$$(1/\pi) \int_C f(p)\, d\psi = (1/2\pi) \int_0^{-2\pi} f(p)\, d\phi + (1/2\pi) \int_0^{2\pi} f(p)\, d\phi',$$

or

$$(1/\pi) \int_C f(p)\, d\psi = (1/2\pi) \int_0^{2\pi} f(p)\, d\phi + (1/2\pi) \int_0^{2\pi} f(p)\, d\phi'.$$

---

* We may choose the determination of ψ arbitrarily, except for the condition that $\psi(p) \to \psi(q)$ as $p \to q$ for all points q on C, excepting one point (in the case that ψ is increased by 2π when p traces C in a suitable sense). A similar remark applies to θ in Theorem II-A below.

If P and Q are finite points and their roles are interchange the new angle ψ' replacing ψ may be so chosen that $\psi' = -\psi$. This fact suggests as a definition of ψ, in the case in which P is at infinity, the angle obtained by interchanging P and Q reversed in sign. With this convention the theorem is true in this special case.

Now

$$(1/2\pi)\int_0^{-2\pi} f(p)\,d\phi = -(1/2\pi)\int_0^{2\pi} f(p)\,d\phi = -u(\infty),$$

and

$$(1/2\pi)\int_0^{2\pi} f(p)\,d\phi' = u(P).$$

Hence,

$$u(P) = (1/\pi)\int_C f(p)\,d\psi \pm u(\infty)$$

according as P lies outside or inside[*] C.

Now the expression $(1/\pi)\int_C f(p)\,d\psi$ (as defined in the general case) has a meaning which is independent of an inversion. That is, if we carry the function $f(p)$ on C into a function $f'(p')$ on C' by an inversion, carrying also P, Q into P', Q' respectively, then it follows that this integral formed for one figure has exactly the same value as it has when formed with reference to the other. (The reversal of the sign of angles is compensated by the convention in regard to the sense of integration.) Hence, the general theorem follows by inversion from the special case treated above.

It will be noted that this result includes Bôcher's interpretation of Poisson's integral as the special case that P, Q are conjugate points with respect to C.

The following is another formulation of the same result, which is more convenient for some purposes:

THEOREM II-A.[†] *Let P, Q be any two distinct points not on the circle C,*

---

[*] This result, in the case that P lies inside C is essentially Neumann's interpretation of Poisson's integral, *Abelsche Integrale*, 2d edition, (1884) p. 410 ff. Neumann's proof is based on the method of the arithmetic mean. It will be noted that when P lies inside C the integral on the right represents simply the average of the boundary values on C as they appear to an observer at P. In the most general case in which P, Q lie in different regions bounded by C a similar meaning may be attached to the integral if we think of light as being refracted so that it comes to P along arcs of circles passing through Q. This device is similar to one used by Bôcher in the *Annals* paper cited above.

[†] This theorem has been given, for the special case in which P and Q are interior points of a proper circle, by Darboux, "Un peu de Géométrie à propos de l'Intégrale de Poisson," *Bulletin des Sciences Mathématiques*, 2d series, Vol. 34, (1910), p. 287 ff. (see particularly formula 12, on page 294). It should be noted that the angle AMB appearing in the form of the theorem there given must be interpreted as a *directed* angle. Failure to take account of this leads to inaccuracies in the application of the theorem on pages 294 and 295, although the final result, which we shall discuss under Theorem I of Section 4, is correct (the angles AEB and AHB being essentially positive).

and denote by θ *the angle from* $\overline{pQ}$ *to* $\overline{pP}$. *Let* u *be the harmonic function determined by the piece-wise continuous boundary values* $f(p)$ *on* C. *Then*

$$u(P) = (1/\pi) \int_C f(u)\, d\theta \pm u(Q)$$

according as P, Q *lie in the same region or different regions bounded by* C. *The integration is to be taken in such a sense that the region containing* P *lies on the left.*

The proof of the theorem for the case in which P and Q are finite points consists merely in noting that if the angle ψ of the preceding theorem is measured from the directed line $\overline{QP}$, then ψ is equal to θ, inasmuch as each is equal to one-half of that directed arc $\overset{\smile}{QP}$ of c which does not go through p, measured in terms of the radius.* In each case, the angle is positive or negative according as the sense from Q to P on this arc is counter-clockwise or clockwise.

If one of the points P or Q is chosen as the point at infinity the corresponding ray $\overline{pP}$ or $\overline{pQ}$ is indeterminate; it is readily verified, however, that the theorem still holds if the ray in question is understood to have a fixed direction as p traces C.

3. *A Representation of the Derivative of Certain Harmonic Functions.*

THEOREM. *Let* p *be a variable point on a fixed circle* † C, *and* c *the circle through* p *tangent at the fixed point* P, *not on* C, *to the fixed directed line* L. *Let the curvative of* c *be represented by* $\pm K(p)$, *the sign being chosen so as to agree with the sign of that determination of the angle from* L *to the ray* $\overline{Pp}$ *which is numerically not greater than* π. *Let* u *be the harmonic function determined by the piece-wise continuous boundary values* $f(p)$ *on* C. *Then the derivative of* u *in the direction of* L *is given by*

$$[du/d\xi]_P = (1/2\pi) \int_C f(p)\, dK(p),$$

the sense of integration being so chosen that the region containing P *lies on the left.*

---

* The relation is true when c goes through the point at infinity inasmuch as θ and ψ are continuous on C.

† In particular, C may pass through the point at infinity. The conventions adopted at the beginning of Section 2 regarding the use of this and certain other terms are retained in this section.

We note that this representation of the derivative is independent of coördinate systems.

Let Σ be an arc on C lying entirely in the finite part of the plane.[*] Divide Σ into n arcs Δs_i, $\overset{\frown}{P_{i-1}P_i}$, $i = 1, 2, \cdots, n$, the points of division being numbered in their order on Σ, traced in the positive sense.[†] We choose the points of division so as to include such points of discontinuity of $f(p)$ as lie on Σ.

We shall denote by f_i the value of $f(p)$ at the mid-point of Δs_i and by U_i the harmonic function determined by the boundary values f_i on Δs_i and 0 elsewhere on C.

Let P' be a point at a small distance $\Delta\xi$ from P on the line L, the direction $\overline{PP'}$ being the positive direction, ξ, on L. Denote by θ the angle [‡] from $\overline{pP}$ to $\overline{pP'}$.

By Theorem II-A of Section II, we know that if $\Delta\xi$ is sufficiently small to insure that P, P' lie in the same region bounded by C, then

$$U_i(P') - U_i(P) = (1/\pi) \int_C f_i \, d\theta.$$

Hence, denoting the value of θ at $P = P_i$ by θ_i we have

$$\frac{U_i(P') - U_i(P)}{\Delta\xi} = \frac{f_i(\theta_i - \theta_{i-1})}{\pi\Delta\xi}.$$

Excluding for the moment the special case that one of the ends of Δs_i lies on L, denote the radius of the circle through P, P' and p_i by r_i. Then

$$\Delta\xi = 2r_i \sin|\theta_i|,$$

and

$$\frac{U_i(P') - U_i(P)}{\Delta\xi} = \frac{f_i}{2\pi}\left[\frac{1}{r_i}\,\frac{\theta_i}{\sin|\theta_i|} - \frac{1}{r_{i-1}}\,\frac{\theta_{i-1}}{r_{i-1}\sin|\theta_{i-1}|}\right].$$

We note that this expression for the difference quotient is valid even in the previously excluded case that p_i or p_{i-1} lies on L, provided we agree to replace the corresponding fraction $1/r_i$ or $1/r_{i-1}$ by 0.

When $P' \rightarrow P$, the fraction $1/r_i$ approaches as a limit the curvature of the circle through p_i tangent to L at P. In all cases we obtain, on noting that θ and $K(p)$ have the same sign,

---

[*] If C is in particular a proper circle, Σ may be the whole of C.

[†] That is so that the region (bounded by C) containing P lies on the left.

[‡] The determination should be chosen at some arbitrarily chosen point p, on Σ but not on L, so that if p is held fast and P' allowed to approach P, then $\lim \theta = 0$. This determination is then extended continuously along C.

$$[dU_i/d\xi]_P = (f_i/2\pi)\ [K(p_i) - K(p_{i-1})] = (f_i/2\pi)\Delta K_i.$$

We introduce now the harmonic function $U^*(P)$ determined by the boundary values $f(p)$ on Σ, 0 on the remainder of C.

Allow n, the number of the arcs Δs_i on Σ to become infinite in such a manner that the length of the largest arc Δs_i approaches zero as a limit. On C, and consequently throughout each of the regions bounded by C, the difference $U^*(P) - \sum_{i=1}^{n} U_i$ ultimately becomes uniformly less in absolute value than any preassigned positive ϵ. From this, we infer that $(d/d\xi)\,[\sum_{i=1}^{n} U_i]_P$ converges to $[dU^*/d\xi]_P$. On the other hand, we know that

$$[(d/d\xi)\ (\sum_{i=1}^{n} U_i)]_P = (1/2\pi)\ \sum_{i=1}^{n} f_i\,\Delta K_i,$$

whence

$$[dU^*/d\xi]_P = (1/2\pi)\int_{\Sigma} f(P)\,d\,K(P),$$

the sense of integration being the positive sense on Σ.

This virtually completes the proof for the case in which C is a proper circle, inasmuch as Σ may here be chosen as the whole of C. If, on the other hand, C extends to infinity, we must examine the behavior of U^* when the ends of Σ are allowed to recede indefinitely in opposite directions on C. It is sufficient to show that in some neighborhood of the point P the function $U^*(Q)$ converges uniformly to $\bar{U}(Q)$. Throughout any finite connected region enclosing P but containing no points of C, the angle subtended at Q by the ends of Σ can be made to differ from π by less than any preassigned positive quantity, by requiring merely that the ends of Σ lie beyond certain fixed points the positions of which depend on the region S, but not on the choice of Q within S. The desired result now follows immediately from the first theorem of the paper.

4. *On the Oscillation of Harmonic Functions in the Plane.* This section contains applications of the above results in obtaining extensions of theorems on the oscillation of harmonic functions.[*] The extensions consist largely in

---

[*] See Schottky, "Ueber die Wertschwankung der harmonischen Funktionem zweier reellen Veränderlichen und der Funktionen eines Complexen Arguments," *Journal für die Mathematik*, Vol. 117 (1896-1897), p. 225 ff.; Koebe, "Ueber das Schwarzsche Lemma und einige damit zusammenhängende Ungleichheitsbeziehungen der Potentialtheorie und Funktionentheorie," *Mathematische Zeitschrift*, Vol. 6 (1920), p. 52 ff.;

a reformulation of the theorems which renders possible their utilization under less simple conditions in the plane of the geometry of inversion.[*]

THEOREM I.[†] *Let P, Q be any two distinct fixed finite points in the same region S bounded by a circle C. Denote by p_1, p_2, the points of contact with C of the two circles through P, Q tangent to C, and by θ_1, θ_2, the directed angles $\sphericalangle Qp_1P$ and $\sphericalangle Qp_2P$, respectively. Let u be any function bounded and harmonic in S; then*

$$| u(P) - u(Q) | \leqq (| \theta_2 - \theta_1 |/\pi)\, D(u, C)$$

where $D(u, C)$ means the oscillation[‡] *of u on C.*

If, further, u is continuous on C, but not a constant, then the upper sign always holds, but the inequality is the best which is applicable to all such functions.

This theorem is a generalization of a theorem given by Darboux, also discussed in the author's paper in the *Annals of Mathematics* cited above. In the case there treated—that in which P and Q are interior points of a proper circle C—one of the angles θ_1 and θ_2 is positive and the other negative, and so $| \theta_2 - \theta_1 |$ is the sum of the numerical values of the angles θ_1 and θ_2, which we represented by $A_1 + A_2$ in our earlier notation.

Jensen, "Investigation of a class of fundamental inequalities in the theory of analytic functions," *Annals of Mathematics*, Vol. 21 (1919), pp. 1-29; Darboux (*loc. cit.*), and Schwarz, *Gesammelte Abhandlungen*, Vol. II, p. 361.

Other properties of the oscillation of harmonic functions, of a type somewhat different than those studied here are given by Bouligand in a paper entitled "Sur la continuité et les approximations en dynamique des liquides," which is to appear in a forthcoming number of the *Journal de l'Ecole Polytéchnique.*

Further references to the literature will be found in the author's paper "On the Oscillation of Harmonic Function," *Annals of Mathematics*, 2d. Series, Vol. 27 (1925), pp. 159-170.

[*] We therefore use the terminology of the geometry of inversion, as in Sections 2 and 3.

[†] Since the publication of my paper in the *Annals of Mathematics*, mentioned above, I have learned that this result was given by Darboux (*loc. cit.*) for the case in which P and Q are interior points of a proper circle C, and the boundary values are piecewise continuous. In regard to the discussion given by Darboux see the footnote to Theorem II-A of Section 2 of the present paper.

[‡] If u is discontinuous on C the oscillation of u on C is defined as the difference between the upper and lower bounds of u in S.

In the first part of the proof of this case previously treated, a relation is obtained which can be written in the form

$$| u(P) - u(Q) | \leqq | U(P) - U(Q) | D(u, C)$$

where U is the function bounded and harmonic within the proper circle C, assuming the boundary values 1 and 0 respectively on the two arcs of C determined by the points we now call p_1 and p_2. Furthermore, if u is continuous on C, but not a constant, the upper sign holds, but the inequality is the best which is applicable to all such functions. That these relations hold in the cases in which the region of definition of u is the exterior of a proper circle, or a half-plane, can be verified immediately by inversion. The general theorem under consideration can now be obtained by the use of Theorem II-A of Section 2 of this paper.

We shall now give a second proof of this theorem which is independent of the earlier demonstration.

We note that θ, which is continuous on C, cannot have a maximum or minimum except at p_1 and p_2, for at any other point p_0 the circle c through P, Q, and p_0 cuts C at an angle not zero, which clearly implies that θ either increases or decreases when p passes through the point p_0. Moreover, since C does not go through P and Q, θ cannot be constant throughout any arc of C. From these facts it follows that θ varies monotonically when p traces either of the arcs* $\overset{\smile}{p_1 p_2}$ of C. For, assuming that this is not true, we infer that there exist on γ, one of these arcs, two points r_1 and r_2 such that p_1, r_1, r_2, p_2 occur in the order named and the quantities† $\theta(p_1) - \theta(p_2)$ and $\theta(r_1) - \theta(r_2)$ do not have the same sign. From the continuity of θ it follows that there are two distinct points r_1' and r_2' on γ such that $\theta(r_1') = \theta(r_2')$, whence we deduce that θ is either constant on at least a part of the arc $\overset{\smile}{r_1' r_2'}$ or else attains a maxium or minimum value, in the strict sense, at some interior point of the arc $\overset{\smile}{r_1', r_2'}$. But we know that neither of these situations is possible, so we conclude that θ is monotonic on each arc $\overset{\smile}{p_1 p_2}$ of C.

---

* This is true even if C passes through the point at infinity. In such a case it is convenient to consider the behavior of the function θ on the corresponding arc of a proper circle obtained from C by inversion. In the new figure the function θ does not in general have the same geometrical meaning as before, but this is of no importance in the part of the proof now under consideration.

† We denote by $\theta(p)$ the value of θ at the point p of C.

Let u be any function harmonic in S and continuous on the boundary C.[*] Using the fact that θ is monotonic on each of the arcs $\overset{\frown}{p_1 p_2}$, we have

$$u(P) - u(Q) = (1/\pi) \int_C f(p)\,d\theta = (1/\pi)\{f(p')[\theta_2 - \theta_1] + f(p'')[\theta_1 - \theta_2]\},$$

or
$$u(P) - u(Q) = (1/\pi)[\theta_2 - \theta_1]\,[f(p') - f(p'')],$$

where p' is a point of one of the arcs $\overset{\frown}{p_1 p_2}$ and p'' a point of the other. From this it follows that

$$| u(P) - u(Q) | \leqq (| \theta_2 - \theta_1 | / \pi) D(u, C).$$

Now $f(p')$ can be the maximum or minimum value of $f(p)$ on the corresponding arc $\overset{\frown}{p_1 p_2}$ only if $f(p)$ is constant on that arc; a similar remark applies to $f(p'')$. Since $f(p)$ is continuous on C it is not possible that $f(p')$ and $f(p'')$ should each be an extremum of $f(p)$ on C unless $f(p)$ is constant on C. Consequently in the relation given above the lower sign can hold (for functions u continuous on C) only in the trivial case in which u is a constant. That the inequality is the best applicable to all functions u harmonic in S and continuous on the boundary is readily verified by considering the harmonic function determined by the boundary values given below:

$$\left\{\begin{array}{l} f(p) = 0 \text{ on one arc } \overset{\frown}{p_1 p_2}, \\ f(p) = 1 \text{ on a part, } \overset{\frown}{r_1 r_2}, \text{ of the remaining arc } \overset{\frown}{p_1 p_2}, \\ f(p) \text{ is continuous and monotonic on the remainder of } C \text{ joining con-} \\ \quad \text{tinuously to the boundary values specified for the other portions of } C. \end{array}\right.$$

If, now, in the inequality above, the factor $| \theta_2 - \theta_1 | / \pi$ is replaced by a smaller quantity, then we can choose r_1 and r_2 so that $\overset{\frown}{r_1 r_2}$ comprises such a large part of $\overset{\frown}{p_1 p_2}$ that the new inequality fails to hold for the harmonic function determined by the continuous boundary values given above.

The case in which u is bounded and harmonic in S, but not necessarily continuous on C, can be treated by introducing a region S' which includes the points P and Q, and which, together with the proper circle C' forming its boundary, lies entirely in the open region S. Since u is continuous on C', we may write

$$| u(P) - u(Q) | \leqq (| \theta_2' - \theta_1' | / \pi)\, D(u, C'),$$

using primes throughout for quantities associated with S'.

---

[*] If the point at infinity lies on C, we shall agree that the restriction of continuity on C requires that the relation $\lim f(p) = f(q)$ holds even when q is the point at infinity, as $p \rightarrow q$.

Let R be any point in the region Σ' bounded by C' complementary to S'. Consider the behavior of the function $\theta(R) = \sphericalangle QRP$ in Σ'. We note that $\theta(R)$ is harmonic at each finite point of Σ'; moreover, it is continuous at infinity, (if the point at infinity is a point of Σ'), and it is continuous on C'. It follows that, even if Σ' extends to infinity,[*] the function $\theta(R)$ attains its extreme values only on C'. These extrema are θ_1' and θ_2'.

If now the center and the radius of C' are allowed to vary continuously in such a way that S' expands so that any given point of S ultimately becomes and remains a point of S', then the extreme values of $\theta(R)$ on S approach those on S, and consequently $|\theta_2' - \theta_1'| \to |\theta_2 - \theta_1|$. Moreover $D(u, C') \to D(u, C)$; hence we have as the limiting form of our inequality

$$|u(P) - u(Q)| \leqq (|\theta_2 - \theta_1|/\pi)\, D(u, C).$$

It should be noted that when discontinuities on C are allowed, it is not necessary that u be a constant in order that the lower sign hold. Such is obviously the case for the harmonic function determined by the boundary values unity on one arc $\overset{\smile}{p_1 p_2}$, zero on the other.

THEOREM II.[†] *Let C_1 and C_2 be any two fixed circles having no point in common; denote by S_1 the region bounded by C_1 containing C_2, by S_2 the region bounded by C_2 containing C_1. Let u_1, u_2 be any functions other than constants which are bounded and harmonic in S_1 and S_2 respectively. The ratios $q_1 = D(u_1, C_2) / D(u_1, C_1)$ and $q_2 = D(u_2, C_1) / D(u_2, C_2)$ depending*

---

[*] This case can be treated by an inversion.

[†] This theorem is a generalization of a theorem on the oscillation of harmonic functions on concentric circles due to C. Neumann. See the author's paper in the *Annals of Mathematics* cited above, and the papers by Schottky and Koebe, also cited above.

The constant q may be identified with the Hilbert configuration constant for the multiply connected region bounded by C_1 and C_2, in the case in which these are proper circles. See the discussion of the Hilbert constant (generalized for multiply connected regions with finite boundary) by E. R. Neumann, "Zur Integration der Potentialgleichung vermittelst C. Neumanns Methode des arithmetischen Mittels," *Mathematische Annalen*, Vol. 56 (1903) p. 49 ff. See particularly pp. 107-112 inc., where the theory is applied to the region bounded by two proper circles. The value obtained in the case in which the circles are concentric is identical with the constant q of the theorem now under consideration. Since each constant is independent of an inversion, they are identical in the general case. Neumann gives a formula for the Hilbert constant for the region bounded by two non-concentric circles, depending on the difference in the dipolar coördinates of the circles.

on the arbitrary functions u_1 *and* u_2 *respectively, have a common least upper bound* q*, determined by the choice of the fixed circles* C_1 *and* C_2*. There exist functions* u_1, u_2 *such that* $q_1 = q_2 = q$*. If, however, we make the further hypothesis that* u_1 *and* u_2 *are continuous on* C_1 *and* C_2 *respectively, then, although* q_1 *and* q_2 *have the same least upper bound* q *as before, this bound is not attained.*

If C_1 *and* C_2 *are proper circles of radii* a_1 *and* a_2 *respectively, and with their centers at a distance* k *from each other, then*

$$q = (2/\pi) \sin^{-1} (2a_1a_2/| a_1^2 + a_2^2 - k^2 |).$$

If C_1 *or* C_2 *is a straight line and the other a proper circle of radius* a *with center at a distance* h *from this line, then*

$$q = (2/\pi) \sin^{-1}(a/h).$$

The quantity q *may be given the following geometric interpretation: let* C *be any circle cutting* C_1 *and* C_2 *orthogonally.*[*] *Draw the two circular arcs* c_1 *and* c_2 *tangent to one of the given circles* C_1 *and* C_2 *and intersecting the other in the same points as does* C*. The number* q *is the ratio to* π *of the angle between* c_1 *and* c_2*.*

The numbers q_1 and q_2 associated with a pair of circles C_1 and C_2 and a pair of functions u_1 and u_2 are unchanged by an inversion, and so their least upper bounds for all admissible functions u_1 and u_2 have the same property. To prove, therefore, that q_1 and q_2 have the same least upper bounds it is sufficient to show that C_1 and C_2 can be interchanged by an inversion T. This is always the case, for by an inversion T_1 we can carry C_1 into a straight line C_1' and C_2 into a proper circle C_2'. Now C_1' and C_2' are conjugate with respect to the circle C' with center at that point of C_2' at the greatest distance from C_1' and radius equal to the geometric mean of this distance and the diameter of C_2'. It follows that C_1 and C_2 are conjugate with respect to the circle C corresponding to C' by the transformation T_1^{-1}.

In determining the value of q we shall consider first the case in which C_1 is a straight line. Let P and Q be any pair of points on C_2. Draw the circles through P and Q tangent to C_1; denote by p_1 and p_2 their points of contact with C_1. Make an inversion † in the circle orthogonal to C_2 with center at p_1. Now C_1 and C_2 are unaltered by this transformation; the points P and Q are carried into P^* and Q^*, equidistant from C_1, and so p_2^*, the inverse of p_2, is the foot of the perpendicular to C_1, drawn from the center of C_2. The function u_1 is transformed into a function u_1^*, bounded and harmonic in the same region. By the first theorem of this section we have

---

[*] If each of C_1 and C_2 is a proper circle, C may conveniently be chosen as a straight line through their centers; if one of C_1, C_2 is a straight line, C may be chosen as the straight line perpendicular to this through the center of the other. C_1, C_2 cannot both be straight lines since by hypothesis they have no point in common.

† If p_1 is the point at infinity, this inversion is unnecessary.

$$| u_1^*(P^*) - u_1^*(Q^*) | \leqq (| \sphericalangle Q^*p_2^*P^* |/\pi)\, D(u_1^*, C_1).$$

But $| \sphericalangle Q^*P_2^*P^* |$ is at most as great as ω, the angle between the tangents to C_2 drawn from the point p_2^*. It follows that the least upper bound of $D(u_1^*, C_2)/D(u_1^*, C_1)$, and so also of $D(u_1, C_2)/D(u_1, C_1)$ is $q = \omega/\pi$; this bound is attained if, and only if, u is allowed to be discontinuous on C. We thus obtain the evaluation of q for the case that one of the circles C_1 or C_2 passes through the point at infinity; we also deduce, by inversion, the geometric interpretation of q given in the last paragraph of the statement of the theorem.†

It remains to compute the value of q for two proper circles. We shall first consider the case in which one of these circles, say C_2, lies inside the other. We invert in the circle of radius $(2)^{1/2}a_1$ with center at that intersection of C_1 and the line of centers of C_1 and C_2 which lies nearer C_2.‡ By this transformation C_1 is carried into a straight line C_1' through the center of C_1, while C_2 is carried into a proper circle with center at a distance

$$h = a_1^2/(a_1 - a_2 - k) + a_1^2/(a_1 + a_2 - k) - a_1$$

from C_1', and of radius

$$a = a_1^2/(a_1 - a_2 - k) - a_1^2/(a_1 + a_2 - k).$$

Since the value of q is independent of an inversion we can compute q for the circles C_1 and C_2 by substituting these values of h and a for the formula

$$q = (2/\pi) \sin^{-1} (a/h),$$

which gives the value of q for C_1' and C_2'. We thus find that for two proper circles, of which one lies inside the other,

$$q = (2/\pi) \sin^{-1} [2a_1a_2/(a_1^2 + a_2^2 - k^2)].$$

---

† This interpretation is readily established for the case considered in Neumann's theorem (see the paper by Koebe cited above). It can then be extended to the general case by inversion, using the fact that the circles C_1 and C_2 can always be transformed into concentric circles by an inversion.

‡ If C_1 and C_2 are concentric we may use as center any point of C_1.

In case C_1 and C_2 are mutually exterior, we invert in a circle of radius $(2)^{1/2}a_1$ with center at the intersection of C_1 and the line segment joining the centers of C_1 and C_2.

In this case we have

$$\begin{cases} h = a_1^2/(k - a_1 - a_2) + a_1^2/\,(k - a_1 + a_2) + a_1, \\ a = a_1^2/(k - a_1 - a_2) - a_1^2/(k - a_1 + a_2)\,; \end{cases}$$

whence,

$$q = (2/\pi)\sin^{-1}\,[2a_1a_2/(k^2 - a_1^2 - a_2^2)].$$

In the last formula we have $a_1^2 + a_2^2 - k^2 < 0$, whereas, in the corresponding equation for the case that one of the circles, C_1 and C_2, lies inside the other, we have $a_1^2 + a_2^2 - k^2 > 0$. Hence, for any two proper circles we may write

$$q = (2/\pi)\sin^{-1}\,(2a_1a_2\,/\,|\,a_1^2 + a_2^2 - k^2\,|).$$

We note that, if $k = 0$ and $a_2 < a_1$, we have

$$q = (2/\pi)\sin^{-1}\,[2a_1a_2/(a_1^2 + a_2^2)] = (4/\pi)\tan^{-1}\,(a_2/a_1)$$

in accordance with the extended form of Neumann's theorem due to Koebe.*

THEOREM III.† *Let P be any fixed finite interior point of a region S bounded by a circle C. Let L be a fixed directed line through P. Let $\pm K_1$, $\pm K_2$ denote the curvatures of the two circles tangent to L at P and to the circle C, the sign in each case agreeing with that of the determination numerically not greater than π of the angle from L to the ray joining P to the point of contact with C. Let u be any function bounded and harmonic in S, and denote by $[du/d\xi]_P$ the derivative of u in the direction of L at P. Then*

$$|\,du/d\xi\,|_P \leqq (1/2\pi)\,|\,K_2 - K_1\,|\,D(u, C).$$

The lower sign may hold for certain functions discontinuous on C. If, however, u is continuous on C, but not a constant, then the upper sign always holds. Moreover this is the best inequality for the directional derivative applicable to all such functions.

The expression on the right is, in reality, independent of the direction ξ.

---

* See the paper in the *Mathematische Zeitschrift,* cited above. Another proof is given in the author's paper in the *Annals of Mathematics,* also cited above.

† This theorem is an extension of that given in § 3 of the author's paper in the *Annals,* cited above.

If C is a proper circle of radius a, and P is at a distance ρ from O, the center of C, then the above relation may be written in the form [*]

$$|\, du/d\xi\, |_P \leqq (2/\pi)\,(a/|\,a^2-\rho^2\,|)\; D(u,C).$$

If C is a straight line and P is at a distance δ from C, then the relation takes the form

$$|\, du/d\xi\, |_P \leqq D(u,C)/\pi\delta.$$

The function $K(p)$, introduced in the third section of this paper, is continuous and monotonic on each of the arcs of C bounded by p_1 and p_2, the points of contact with C of the two circles through P tangent to L and to C. It follows, then, from the theorem of that section that if u has the continuous boundary values $f(p)$ on C, we deduce from the relation

$$[du/d\xi]_P = (1/2\pi)\int_C f(p)\,dK(p)$$

the following equation:

$$[du/d\xi]_P = (1/2\pi)\,\{[K_2-K_1]f(p') + [K_1-K_2]f(p'')\},$$

or
$$[du/d\xi]_P = (1/2\pi)\,[K_2-K_1]\,[f(p')-f(p'')],$$

where p' is a point of one arc $\overset{\smile}{p_1p_2}$, and p'' a point of the other. We have then

$$|\, du/d\xi\, |_P \leqq (1/2\pi)\;|\,K_2-K_1\,|\; D(u,C).$$

The other facts given in the first paragraph of the statement of the result are also obvious from the same type of reasoning as that used in the proof of the first theorem of this section.

We shall now discuss the geometrical problems of showing that the value of $|\,K_2-K_1\,|$ is independent of the direction ξ, and of expressing this quantity in terms of a and ρ, or of h, according to the nature of C. For any choice of P and ξ, the center of any circle tangent at P to a line L with the direction ξ

---

[*] An analogous relation for a regular function of a complex variable is virtually contained in a relation given by Schottky (*loc. cit.*). It is shown that if $\phi(x)$ is such a function in the circle with center at $x=o$ and radius R, and if $D(\phi)$ is the maximum value of $|\,\phi(y)-\phi(z)\,|$ for two arbitrary points y, z on the boundary, and r any positive number less than R, then

$$|\,\phi'(r)\,| \leqq D(\phi)\;.\;(2/\pi)\;.\;R/(R^2-r^2).$$

This result can obviously be generalized. It will be noted that $D(\phi)$ is ordinarily greater than the oscillation on C of the real or pure imaginary part of $\phi(x)$.

lies on the line L' through P perpendicular to L. The center of any circle through P tangent to C lies on a certain conic. If C is a proper circle with center at O and radius a, then, according as P lies inside or outside C, this conic is an ellipse or a hyperbola with foci at O and P and major axis a; if C is a straight line, this conic is a parabola with P as focus and C as directrix. These facts follow immediately from elementary geometrical considerations, through the properties of the sum or difference of the focal radii in the case of the central conics.

In all cases except that in which P lies outside a proper circle C and L' has an intersection with the corresponding hyperbola lying on that branch farther from P, we know that K_1 and K_2, if different from zero, have opposite signs. For in these cases the centers of the circles through P tangent to L and to C, and therefore the points p_1 and p_2, lie on opposite sides of L. Here, then, $| K_2 - K_1 |$ represents the sum of the reciprocals of the segments into which P divides the chord of the conic lying on L'. This is true when only one of the intersections is finite, provided the reciprocal of the half line now replacing a finite segment is interpreted as zero. On setting up a polar coördinate system with pole at P and prime vector along the line normal to the corresponding directrix but directed away from this line, we have as the equation of the conic,[*] or the part thereof involved,

$$r = em/(1 - e \cos \phi).$$

Hence, we see that the reciprocals under consideration may be written in the form

$$1/r_1 = (1 - e \cos \phi)/em,$$
$$1/r_2 = [1 - e \cos(\phi_1 + \pi)]/em,$$

or

$$1/r_2 = (1 + e \cos \phi)/em,$$

and so we have

$$| K_1 - K_2 | = 2/em, \text{ or } | K_1 - K_2 | = 4/l,$$

where l is the length of the latus rectum of the conic.

In the case that the conic is a hyperbola, the branch farther from P has as its equation with reference to the coördinate system used above,

$$r = - em/(1 + e \cos \phi).$$

---

[*] We use a familiar notation, regarding r as essentially positive. This equation cannot be applied in the special case that P coincides with O as the ellipse then becomes a circle. Here, however, the validity of the final result is immediately obvious.

In order that L' meet this branch,[*] it is necessary that L', (directed from $\dot{P}$ to this point of intersection) make an angle ϕ_1 with the prime vector such that $e \cos \phi_1 < -1$, which means that the same ray has a second finite intersection with the branch $r = em/(1 - e \cos \phi)$. Hence, K_1 and K_2 have the same sign in this case and

$$| K_1 - K_2 | = (1 - e \cos \phi_1)/em + (1 + e \cos \phi_1)/em,$$

or

$$| K_1 - K_2 | = 4/l.$$

Hence, we see that in all cases $| K_1 - K_2 |$ has the value $4/l$, independent of the direction ξ.

If C is a proper circle of radius a, and P is at a distance ρ from its center, the square of the semi-minor axis of the corresponding ellipse or hyperbola is $| a^2 - \rho^2 |/4$, and so

$$l = | a^2 - \rho^2 |/a, \text{ and } | K_2 - K_1 | = 4a/| a^2 - \rho^2 |.$$

Hence in this case the relation

$$| du/d\xi |_P \leqq (1/2\pi) \, | K_2 - K_1 | \, D(u, C)$$

can be written in the form

$$| du/d\xi |_P \leqq (2/\pi)(a/| a^2 - \rho^2 |) D(u, C).$$

In the case in which C is a straight line, and the point P is at a distance δ from C, the corresponding conic is a parabola and we have $l = 2\delta$. Hence in this case $| K_2 - K_1 | = 2/\delta$, and the relation

$$| du/d\xi |_P \leqq (1/2\pi) \, | K_2 - K_1 | \, D(u, C)$$

takes the form

$$| du/d\xi |_P \leqq D(u, C)/\pi\delta.$$

5. *On the Oscillation of Harmonic Functions in Space.* This section contains relations in space analogous to some of the results obtained for the plane in Section 4, though now given in a less general form. Given a function, bounded and harmonic in the interior[†] of a proper sphere S, we define $D(u, S)$

---

[*] By making suitable conventions as to signs, this case could be united with that previously discussed. It seems clearer, however, to give separate discussions.

[†] It will be noted that if $D(u, S)$ were similarly defined for a function bounded and harmonic *exterior* to S, then it would not necessarily agree with the usual definition of oscillation in the case that u is continuous on S. For example, if $u(P) = 1/r$ is the distance from the center of a unit sphere S to the point P exterior to S, we would have $D(u, S) = 1$ whereas the difference between the maximum and minimum

as the difference between the upper and lower bounds of u within S. The following result corresponds to Theorem I of the preceding section.

THEOREM. *Let S be a sphere of radius A, and P_1, P_2 any two distinct points within this sphere, at distances l_1, l_2, respectively, from the center O. Let σ be the sphere or plane which is the locus of all points Q such that*

$$\frac{\overline{P_1Q}}{\overline{P_2Q}} = \left(\frac{A^2 - l_1^2}{A^2 - l_2^2}\right)^{1/3}.$$

Denote by s the part of the surface S which lies within σ, (or, if σ is a plane,[*] *on either side of σ). Let P be any arbitrary point within S; denoting the distances from P to O and to an arbitrary point p of s by l and ρ respectively, set*

$$U(P) = (1/4\pi A) \int\int_s \frac{A^2 - l^2}{\rho^3} \, dS.$$

If, now, $u(P)$ is any function bounded and harmonic within S, then

$$| u(P_1) - u(P_2) | \leqq | U(P_1) - U(P_2) | \, D(u, S).$$

If, further, $u(P)$ is continuous on S, but not a constant, then the upper sign holds, but this is the best appraisal of $| u(P_1) - u(P_2) |$ applicable to all such functions.

Let us first consider the case that u is continuous on S. If u is a constant, the validity of the first paragraph is obvious, so we may exclude this case from further consideration. We may, without loss of generality, suppose that u has the maximum and minimum values 1 and 0, respectively, on S. For if u' is any function harmonic within and continuous on S, having as extreme values on S the numbers M, m (where $m < M$), then

$$u = \frac{u' - m}{M - m}$$

is another function of the same sort with the maximum 1 and the minimum 0. If the theorem is true for the function u, it is also true for the function u', as can readily be seen from the relation

---

values of u on S is 0. In the treatment of these problems in space, it is not desirable to use the terminology of the geometry of inversion; by a sphere we mean always a proper sphere; by a point, a finite point, etc.

[*] It is readily shown by considerations of symmetry that in this case the value of $| U(P_1) - U(P_2) |$ is the same no matter which of the hemispheres of S determined by the plane σ is chosen as s.

$$u'(P_1) - u'(P_2) = [u(P_1) - u(P_2)]\, D(u', S).$$

Representing the boundary value of u at an arbitrary point p of S by $f(p)$, we have, from Poisson's integral

$$u(P_1) - u(P_2) = (1/4\pi A) \int\!\!\int_S f(p)\, F(P_1, P_2)\, d\, S,$$

where $F(P_1, P_2) = \dfrac{A^2 - l_1^2}{\rho_1^3} - \dfrac{A^2 - l_2^2}{\rho_2^3} = \dfrac{A^2 - l_2^2}{\rho_1^3}\left[\dfrac{A^2 - l_1^2}{A^2 - l_2^2} - \dfrac{\rho_1^3}{\rho_2^3}\right]$,

ρ_1 and ρ_2 designating the values of ρ at P_1 and P_2 respectively.

A similar relation would hold in the case of a harmonic function constant within S, whence we infer that $F(P_1, P_2)$ is positive on a part of S and negative on a part of S. This means that σ always cuts S, and so s always exists. Furthermore P_1, P_2, lie on opposite sides of σ. If, in the inequality to be established, the points P_1, P_2 are interchanged, then neither side is affected in value, s being unaltered. Hence we may assume that P_1 lies on the same side of σ as does s, which implies that $F(P_1, P_2)$ is positive on s and negative on S-s (the remainder of s), points of the common boundary of these regions being disregarded. We have

$$\begin{aligned} &u(P_1) - u(P_2) \\ &= (1/4\pi a) \int\!\!\int_s f(p)\, F(P_1, P_2)\, dS + (1/4\pi A) \int\!\!\int_{S-s} f(p)\, F(p_1, p_2)\, dS. \end{aligned}$$

Recalling that $0 \leqq f(p) \leqq 1$, we see that

$$\begin{cases} (1/4\pi A) \displaystyle\int\!\!\int_s f(p)\, F(P_1, P_2)\, dS \leqq (1/4\pi A) \int\!\!\int_s F(P_1, P_2)\, dS, \\ (1/4\pi A) \displaystyle\int\!\!\int_{S-s} f(p)\, F(P_1, P_2)\, dS \leqq 0. \end{cases}$$

Moreover, the continuity of $f(p)$ on S implies that the lower sign cannot hold in both these relations. Consequently,

$$u(P_1) - u(P_2) < (1/4\pi A) \int\!\!\int_s F(P_1, P_2)\, dS = U(P_1) - U(P_2).$$

A similar relation holds for the function $v(P) = 1 - u(P)$, which is finite and harmonic within S with the extreme values 0 and 1. This gives

$$-[u(P_1) - u(P_2)] < U(P_1) - U(P_2).$$

We therefore have

$$| u(P_1) - u(P_2) | < U(P_1) - U(P_2),$$

which implies the validity of the desired inequality for any function harmonic within and continuous on S.

To prove that this is the best possible inequality it is sufficient to show that, given any $\epsilon > 0$, there exists a $u(P)$ such that the relation

$$| U(P_1) - U(P_2) | - | u(P_1) - u(P_2) | < \epsilon$$

holds.

Denote by δ the distance from an arbitrary point p of s to the plane determined by the circle forming the boundary of s. Given any positive δ_0, consider the function $u_1(P)$, bounded and harmonic within S, determined by the continuous boundary values given below:

$$\begin{cases} f_1(p) = 0, \text{ on } S - s, \\ f_1(p) = \delta/\delta_0 \text{ when } p \text{ is a point of } s \text{ such that } \delta \leqq \delta_0, \\ f_1(p) = 1 \text{ on the remainder } s' \text{ of } s. \end{cases}$$

Assuming, as before, that s and P_1 lie on the same side of σ, we have

$$| U(P_1) - U(P_2) | - | u_1(P_1) - u_1(P_2) | \leqq (1/4\pi A) \iint_{s-s'} F(P_1, P_2)\, dS$$
$$< \frac{1}{4\pi A} \cdot \frac{A^2 - l_1^2}{(A - l_1)^3} \cdot 2\pi A \delta_0 = \frac{(A + l_1)\delta_0}{2(A - l_1)^2}.$$

Hence, if we choose $\delta_0 = \dfrac{2(A - l_1)^2}{(A + l_1)}\,\epsilon$ we have the desired result.

We are now ready to discuss the case in which the restriction that u is continuous on S is replaced by the weaker condition that u is bounded within S. Make a transformation of space consisting of a shrinking toward the center of the sphere S, so that S is carried into sphere S' of radius $A' = \lambda A$, where $0 < \lambda < 1$. The function u, (which is not subjected to the transformation) is continuous on S', and so

$$| u(P_1') - u(P_2') | < | U'(P_1') - U'P_2') | D(u, S)$$

P_1' and P_2' being the points corresponding to P_1 and P_2, and U' the function associated with P_1', P_2', S' in the same way as is U with P_1, P_2, S. By considering the relations between A, ρ, l, ΔS and the corresponding quantities with reference to the transformed figure, it is seen that at any pair of corresponding points P, P' we have

$$U'(P') = U(P).$$

Hence,

$$| u(P_1') - u(P_2') | < | U(P_1) - U(P_2) | \, D(u, S').$$

On letting $\lambda \to 1$, we have,

$$| u(P_1) - u(P_2) | \leqq | U(P_1) - U(P_2) | \, D(u, S),$$

the lower sign holding if, for instance, u is chosen as the function U.

THEOREM II. *Let S, S' be any two fixed concentric spheres of radii A, A' respectively (where $A' < A$). Let u be an arbitrary function bounded and harmonic within S, but not a constant. The ratios $D(u, S')/D(u, S)$ for all such functions u have a least upper bound q. The bound q is attained; if, however, the restriction is added that u is continuous on S, then the bound q is unaltered, but is not attained.*

The value of q in terms of the ratio $\mu = A'/A$ is

$$q = \frac{1}{\mu} \left[1 - \frac{1 - \mu^2}{(1 + \mu^2)^{1/2}} \right],$$

or in terms of the angle $\alpha = \tan^{-1} \mu$,

$$q = \sin (3\alpha/2)/\cos (\alpha/2)$$

or

$$q = [3 - 4 \sin^2(\alpha/2)] \tan (\alpha/2).$$

The number q may be given the following geometric interpretation: draw the mutually perpendicular lines L, L' through O, the center of S and S', Let D be a point of intersection of L and S and D' a point of intersection of L' and *S'. Denote by D'' the second point in which the line $\overline{DD'}$ cuts S'. With center at D and radius $\overline{DD''}$ draw a circular arc in the plane of L, L' cutting L in the point Δ. Then q is the ratio of the length $\overline{O\Delta}$ to A'.*

Denote by P_1 and P_2 the points where an arbitrary one of the functions u assumes its extreme values of S'.

Let σ be the plane in which P_1, P_2 are symmetric, s the hemisphere of S bounded by σ and lying on the same side of σ as does P_1. Let

$$U(P) = \frac{1}{4\pi A} \iint_s \frac{A^2 - l^2}{\rho^3} \, dS,$$

(using the notation of the preceding theorem).

Represent by Q_1 and Q_2 the points in which the normal to σ at O meets S', Q_1, lying on the same side of σ as does P_1; then $U(P_1) \leqq U(Q_1)$, $U(P_2) \geqq U(Q_2)$ the lower signs holding in fact only when P_1 and Q_1 are identical. For, excluding this trivial case, let π be the plane of symmetry of P_1 and Q_1, σ' the reflection of σ in π, s_1 the larger part of s bounded by the intersections of σ and σ' with S, s_2 the remainder of s. Denote by U_1 the function given by Poisson's integral for the boundary value 1 on s_1, 0 elsewhere, and U_2 that for the boundary value 1 on s_2, 0 elsewhere. Then

$$\begin{cases} U_1(P_1) = U_1(Q_1), \text{ by symmetry,} \\ U_2(P_1) < U_2(Q_2), \text{ since each point of } s_2 \text{ is nearer } Q_1 \text{ than } P_1. \end{cases}$$

Hence $U(P_1) = U_1(P_1) + U_2(P_1) < U_1(Q_1) + U_2(Q_1) = U(Q_1)$.

Let
$$V(P) = \frac{1}{4\pi A} \iint_{S-s} \frac{A^2 - l^2}{\rho^3} \, dS,$$

then
$$\begin{cases} U(Q_2) = V(Q_1) = 1 - U(Q_1), \\ U(P_2) = V(P_1) = 1 - U(P_1), \end{cases}$$

and so $U(Q_2) - U(P_2) = U(P_1) - U(Q_1) < 0$, or $U(P_2) > U(Q_2)$.

The explicit formula for q in terms of A, A' can be obtained by integration from the relation

$$q = \frac{A^2 - A'^2}{4\pi A} \int_0^{\pi/2} \int_0^{2\pi} \left(\frac{1}{\rho_1^3} - \frac{1}{\rho_2^3} \right) A^2 \sin\theta \, d\phi \, d\theta$$

where $\rho_1 = (A^2 - 2AA' \cos\theta + A'^2)^{1/2}$, $\rho_2 = (A^2 + 2AA' \cos\theta + A'^2)^{1/2}$.

This yields

$$q = \frac{A}{A'} - \frac{A^2 - A'^2}{A'(A^2 + A'^2)^{1/2}} \quad \text{or } q = \frac{1}{\mu} - \frac{1 - \mu^2}{\mu(1 + \mu^2)^{1/2}} .$$

We note that we may write

$$q = \frac{A - \left(\frac{A^2 - A'^2}{(A^2 + A'^2)^{1/2}} \right)}{A'} .$$

Now $\overline{DD'} \cdot \overline{DD''} = A^2 - A'^2$; moreover, $\overline{DD'} = (A^2 + A'^2)^{1/2}$. Hence we see that the parenthesis in the numerator represents DD''. The interpretation for q given in the last paragraph of the theorem follows.

From this we may obtain readily the trigonometric formula for q. Drawing the line $D''\Delta$, it is readily shown that

$$\sphericalangle\, OD''\Delta = 3\alpha/2, \quad \text{and} \quad \sphericalangle\, O\Delta D'' = \pi/2 + \alpha/2;$$

hence on applying the Law of Sines to the triangle $OD''\Delta$ we have

$$q = \overline{O\Delta}/\overline{OD''} = \sin(3\alpha/2)/\cos(\alpha/2) = [3 - 4\sin^2(\alpha/2)]\tan(\alpha/2).$$

This completes the proof of the theorem.

HARVARD UNIVERSITY.

On the Invariant Combinants of Two Binary Quintics.[*]

BY THOMAS W. MOORE.[†]

The question of the invariant combinants of two binary quintics considered in this paper presented itself in a study of the rational space quintic curve. Such a curve in a space of three dimensions is defined by means of four binary quintic forms giving the parameters of the sets of five points in which each of the four reference planes cuts the curve. Then if some projective specialization occurs in the curve it will be disclosed by some invariant or covariant condition imposed on these four quintics. In this way a study of the geometric properties of the curve can be converted into an investigation of the covariant theory of these four forms in question.

Two binary forms of the nth order are apolar if their bilinear invariant (nth ordered transvectant) vanishes. The apolarity relation is a linear condition on the coefficients, and it is known that there are $(n-r+1)$ linearly independent forms of order n apolar to each of r such given forms. Hence there are two binary quintics apolar to the four given quintics defining the $R_3{}^5$. The curve may equally well be defined by these two quintics. The linear combination of these two forms in terms of a parameter determines what is called the fundamental involution on the curve.

This is interpreted geometrically by noticing that any plane (ξx) cuts the curve $R_3{}^5$ in five points whose parameters are given as roots of a form which is a linear combination of the four quintics defining the curve. Hence any plane section of the curve will be apolar to the two fundamental quintics, and a necessary and sufficient condition that five points of the curve be on a plane is that the binary quintic defining them be apolar to each of the two fundamental quintics. If $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5$ are the five points and $f=a_\lambda{}^5$, $f'=a'_\lambda{}^5$ the two fundamental quintics, then the conditions are

$$a_{\lambda_1}a_{\lambda_2}a_{\lambda_3}a_{\lambda_4}a_{\lambda_5}=0, \qquad a'_{\lambda_1}a'_{\lambda_2}a'_{\lambda_3}a'_{\lambda_4}a'_{\lambda_5}=0,$$

or if s_1, s_2, s_3, s_4, s_5 are the elementary symmetric functions of the five λ's:

---

[*] Presented to the American Mathematical Society, Feb. 26, 1927.

[†] National Research Fellow in Mathematics.

$$\left\{\begin{array}{l} a_0s_5 + a_1s_4 + a_2s_3 + a_3s_2 + a_4s_1 + a_5 = 0 \\ a'_0s_5 + a'_1s_4 + a'_2s_3 + a'_3s_2 + a'_4s_1 + a'_5 = 0 \end{array}\right\}$$

From this point of view the study of the rational quintic becomes a study of the system of invariant and covariant forms of the two binary quintics. Of especial interest are the covariant forms known as combinants. An invariant or covariant of any number of binary forms of the same order $f_1, f_2, \cdots, f_r$ is called a combinant if it be unaltered except as regards a factor independent of the forms when each form is replaced by a linear combination of the type $l_1f_1 + l_2f_2 + \cdots + l_rf_r$ in which the l's are constants.* The combinants are important geometrically because they are unaltered not only by a linear transformation applied to the variables, but also by any linear transformation effected on the coefficients. The natural geometric representation of the combinants is in the theory of rational curves, and the projective properties of the curve are defined by them.

The simultaneous system of covariant forms of two binary quintics has never been established. The complete systems of single binary forms up to the tenth order have been worked out by Gordan, Clebsch, von Gall, Sylvester and others. Gordan proved in general that the number of forms in either the complete system of a single form or of a simultaneous system of two forms is finite. The simultaneous systems of two forms as high as the fourth order were set up by Gordan first in his article "Die simultanen Systeme binärer Formen" in Vol. 2 of the *Mathematische Annalen.* For instance the system of two quartics is given on p. 275 of that volume.

We propose to concern ourselves here with the combinant forms of the two quintics, inasmuch as this phase arises from the geometrical question. One might be led to suspect that the systems of combinants of either the set of four quintics which are the coordinate forms of the quintic curve or the set of two apolar quintics are equivalent, since the curve may be given either way. This is indeed the case, as Meyer has shown.† In fact, by virtue of the relations existing between the fourth order determinants in one set and the second order determinants in the other we can secure one set of forms from the other, so far as coefficients are concerned.

The theory of the combinant forms has been the subject of extensive investigation by Gordan in a memorable article "Ueber Combinanten" published in Vol. 3 of the *Mathematische Annalen.* Here he proved the theorem that the complete system of combinants of two binary forms of the same order is identical with the complete system of covariant forms of the joint set of the first, third, fifth, etc., transvectants of the two forms. These odd-ordered transvectants are combinant forms, and we remark besides that every invariant or covariant of a combinant is also a combinant of the two quantics.

---

* Grace and Young, *Algebra of Invariants*, p. 314.

† W. F. Meyer, *Apolarität und Rationale Kurven*, § 11.

E. Stroh called attention to the fact that such a system of covariant forms is not the most general of its type, due to the syzygies which exist between these odd-ordered transvectants. Let us suppose that the two given quantics are ϕ and ψ of the mth order. These odd-ordered transvectants are then $(\phi\psi)$, $(\phi\psi)^3$, $(\phi\psi)^5$, etc., of orders $2m-2$, $2m-6$, $2m-10$, etc., respectively. Now Stroh has shown[*] that such a complete system is the same as the complete system of a single binary form of order $2m-2$, the unique form apolar to the two given quantics of order m. This apolar form of order $2m-2$ is the form discussed by Gordan in his paper on resultants[†] and it vanishes identically if ϕ and ψ have two or more roots in common.

Consequently we know exactly the number of forms in such a system provided we know the number in the system of the form of order $2m-2$. For example consider the case of two quartics. The first and third transvectants are a sextic and a quadratic respectively. The apolar form is a sextic and hence there are just 26 covariant forms in the simultaneous system, and thus 26 combinants of the two quartics. This system was given by Stéphanos in the *Comptes Rendus*.[‡]

We are interested here in the complete system of combinants of two quintics, and to the author's knowledge little has been done on this so far. The first, third and fifth transvectants are an octavic, a quartic, and an invariant respectively. The single apolar form is an octavic, and so we have sixty-nine combinant forms (von Gall). Apparently the only work published on the combinants of two quintics is a short paper by Berzolari,[§] in which he gives three syzygies connecting these odd-ordered transvectants above.

Let us denote the two quintics defining the fundamental involution by $Q = q_x^5$, $Q' = q'^5_x$. Then the first, third and fifth transvectants are respectively

---

[*] E. Stroh, *Mathematische Annalen*, Vol. 34, pp. 321 ff.

[†] P. Gordan, "Ueber die Bildung der Resultante zweier Gleichungen," *Mathematische Annalen*, Vol. 3 (1871), pp. 355-414.

[‡] C. Stéphanos, "Sur le système complet des combinants de deux formes binaires biquadratiques," *Comptes Rendus*, Vol. 97 (1883), p. 27.

[§] L. Berzolari, "Sui combinanti dei sistemi lineari di quintiche binarie," *Rendiconti di Palermo*, Vol. 7 (1893), p. 5.

$$f = a_x^8 = (qq')q_x^4 q'^4_x, \qquad \phi = \alpha_x^4 = (qq')^3 q_x^2 q'^2_x, \qquad \gamma = (qq')^5.$$

If we denote the apolar octavic by $F = F_x^8$, we have the following relations due to Stroh:[*]

$$(qq')q_x^4 q'^4_x = (1/1200R^3)\ [8i_2 + (9/7)k^2 - (1/3)(Ai + 2BF)]$$
$$(qq')^3 q_x^2 q'^2_x = -(1/200R^3)\ [(2/7)k_2 - i_k + (2/15)Ak]$$
$$(qq')^5 = (1/800R^3)\ [(1/3)A^2 - 2C]$$

where R is the resultant of the two quintics, and the forms on the right, all pertaining to F, will be explained later.

Berzolari's three syzygies are:

(I) $(ff)^4 - (4/7)(f\phi)^2 + (1/15)\gamma f - (15/98)\phi^2 = 0$

(II) $2(ff)^6 + (15/7)(f\phi)^4 - (50/49)H_\phi - (1/3)\gamma\cdot\phi = 0$

(III) $(ff)^8 + (5/7)i_\phi - (2/15)\gamma^2 = 0.$

The explicit forms of part of the forms above may be written down easily. We have

$$Q = q_x^5 = q_0x_1^5 + 5q_1x_1^4x_2 + 10q_2x_1^3x_2^2 + 10q_3x_1^2x_2^3 + 5q_4x_1x_2^4 + q_5x_2^5$$
$$Q' = q'^5_x = q'_0x_1^5 + 5q'_1x_1^4x_2 + 10q'_2x_1^3x_2^2 + 10q'_3x_1^2x_2^3 + 5q'_4x_1x_2^4 + q'_5x_2^5.$$

Let us set $p_{ij} = q_iq_j' - q_jq_i'$ $(i, j = 0, 1, \cdots, 5)$ where the q_i are the actual coefficients. Then we have:

$$\begin{aligned} f = (qq')q_x^4q'^4_x = p_{01}x_1^8 &+ 4p_{02}x_1^7x_2 + 2(3p_{03} + 5p_{12})x_1^6x_2^2 \\ &+ 4(p_{04} + 5p_{13})x_1^5x_2^3 + (p_{05} + 15p_{14} + 20p_{23})x_1^4x_2^4 \\ &+ 4(p_{15} + 5p_{24})x_1^3x_2^5 + 2(3p_{25} + 5p_{34})x_1^2x_2^6 \\ &+ 4p_{35}x_1x_2^7 + p_{45}x_2^8. \end{aligned}$$

$$\begin{aligned} \phi = (qq')^3q_x^2q'^2_x = (p_{03} &- 3p_{12})x_1^4 + 2(p_{04} - 2p_{13})x_1^3x_2 \\ &+ (p_{05} + p_{14} - 8p_{23})x_1^2x_2^2 + 2(p_{15} - 2p_{24})x_1x_2^3 \\ &+ (p_{25} - 3p_{34})x_2^4 \end{aligned}$$

$$\gamma = (qq')^5 = p_{05} - 5p_{14} + 10p_{23}.$$

The apolar octavic is best expressed as the bordered resultant, using Bézout's method of elimination.

---

[*] E. Stroh, *loc. cit.*, p. 326.

$$\left|\begin{array}{llllll}
5p_{01}, & 10p_{02}, & 10p_{03}, & 5p_{04}, & p_{05}, & x_2^4 \\
10p_{02}, & 10p_{03}+50p_{12}, & 5p_{04}+50p_{13}, & p_{05}+25p_{14}, & 5p_{15}, & -4x_1x_2 \\
10p_{03}, & 5p_{04}+50p_{13}, & p_{05}+25p_{14}+100p_{23}, & 5p_{15}+50p_{24}, & 10p_{25}, & 6x_1^2x_2 \\
5p_{04}, & p_{05}+25p_{14}, & 5p_{15}+50p_{24}, & 10p_{25}+50p_{34}, & 10p_{35}, & -4x_1^3x_2 \\
p_{05}, & 5p_{15}, & 10p_{25}, & 10p_{35}, & 5p_{45}, & x_1^4 \\
x_2^4, & -4x_1x_2^3, & 6x_1^2x_2^2, & -4x_1^3x_2, & x_1^4, & 0
\end{array}\right.$$

So according to the theory of Stroh we shall get the complete system of combinants of the quintics Q and Q' by considering the simultaneous system of covariants of f, ϕ, and γ. Since γ is invariant it may be neglected and we need consider only the joint system of the Jacobian octavic f and the quartic ϕ. This system is in turn equivalent to the single complete system of the apolar octavic. Now starting with the two forms f and ϕ it is our task to set up the joint system of covariants and reduce them to the least possible number by use of the three syzygies above due to Berzolari. We shall make use of a formula known as Gordan's series to effect this. Namely, if f, ϕ, ψ are three binary form of orders m, n, p respectively, we shall have the following:

$$\sum_i \frac{\binom{n-\alpha_1-\alpha_3}{i}\binom{\alpha_2}{i}}{\binom{m+n-2\alpha_3-i+1}{i}}\{(f,\phi)^{\alpha_3+i},\psi\}^{\alpha_1+\alpha_2-i}$$

$$=(-1)^{\alpha_i}\sum_i \frac{\binom{p-\alpha_1-\alpha_2}{i}\binom{\alpha_3}{i}}{\binom{m+p-2\alpha_2-i+1}{i}}\{(f,\psi)^{\alpha_2+i},\phi\}^{\alpha_1+\alpha_3-i}$$

represented schematically by

$$(f,\phi,\psi)\ (m,n,p;\ \alpha_1,\alpha_2,\alpha_3)$$

where $\alpha_2+\alpha_3\leq m,\quad \alpha_3+\alpha_1\leq n,\quad \alpha_1+\alpha_2\leq p,$

and either $\alpha_1=0 \quad \text{or} \quad \alpha_2+\alpha_3=m.$[*]

The simultaneous system of two forms should contain ordinarily the complete system of each form alone together with the minimum number of such forms as are secured by transvection of a form in one system with a form in the other. But here the complete system of combinants can contain only sixty-nine independent forms, and there are that many in the system of the Jacobian octavic alone. This Jacobian is a general octavic form and none of its forms will be reducible in the domain of its own coefficients. However certain covariants of this octavic are connected with the forms of the quartic ϕ and invariant γ by the three syzygies of Berzolari. Since the forms of the quartic are much simpler than those of the octavic, it appears expedient to attempt to express as many of the forms of the Jacobian octavic in terms of the covariants of the quartic as is possible. Normally one would expect to consider the invariants of the octavic first, for they are simpler to handle and of more immediate geometric interest.

---

[*] Grace and Young, p. 56.

Fundamentally then our object is to find a set of covariants involving forms of the quartic ϕ and the invariant γ and simpler cross-transvectants of f and ϕ and their lower degree forms with which to replace as many as possible of the forms in the system of the octavic f. Certain of the cross-transvectants appear in this procedure, and since we know the maximum number of possible forms, and as it will appear later in the case of the invariants that we get at least this number, we need not concern ourselves for the present with the problem of direct cross-transvection from the system of the octavic to that of the quartic, which would be somewhat of an endless task. Briefly, if we can show that we get all necessary forms from the system of the octavic alone, we need not go further. This then we shall consider first.

The complete system of the quartic ϕ contains five forms:

$$\phi = \alpha_x{}^4;\quad H_\phi = H_{\phi_x}{}^4 = (\phi\phi)^2;\quad t_\phi = (\phi H_\phi);\quad i_\phi = (\phi\phi)^4;\quad j_\phi = (\phi H_\phi)^4.$$

The complete system of the Jacobian octavic contains sixty-nine forms. It is given by von Gall in three articles in Vol. 17 of the *Mathematische Annalen.* The following table is taken from one of these articles (p. 149).

$$\begin{array}{lllll}
k = (ff)^6, & i = (ff)^4, & H = (ff)^2, & f_k = (fk)^4, & f_3 = (fk)^3,\ \ldots, \\
p = (fi)^2, & \theta = (fi), & T = (Hf), & \Delta = (kk)^2, & S = (Hi), \\
f_{kk} = (f_k k)^4, & \ldots, & f_\Delta = (f\Delta)^4, & f''' = (f\Delta)^3, & \ldots, \\
\mathrm{T} = (k\Delta), & \ldots, & f_{\Delta\Delta} = (f_\Delta \Delta)^4,\ \ldots, & & \ldots,
\end{array}$$

ORDER.

Degree	0	2	4	6	8	10	12	14	18	Totals
1					f					1
2	$A = (ff)^8$		k		i		H			4
3	$B = (fi)^8$		f_k	f_3	f_2	f_1	p	θ	T	8
4	$C = (kk)^4$		Δ, i_k	i_3	i_2	H_3, i_1	H_2	H_1	S	10
5	f_{kk}	f_{k3}	f_{k2}, f_Δ	f_{k1}, f'''	f''	θ_k, p_3, f'		p_1		11
6	i_{kk}	i_{k3}	i_{k2}, i_Δ	i_{k1}, i''', T	i''	i'				9
7	$f_{k\Delta}$	$f_k''', (f\mathrm{T})^6$	$f_k'', (f\mathrm{T})^5$	$f_k', (f\mathrm{T})^4, \theta_{kk}$						8
8	$i_{k\Delta}$	$i_k''', (i\mathrm{T})^6$	$i_k'', (i\mathrm{T})^5$	$i_k', (i\mathrm{T})^4$						7
9	$f_{\Delta\Delta}$	$(f_k\mathrm{T})^4, f_\Delta''', \theta_{kkk}$	f_Δ''							5
10	$i_{\Delta\Delta}$	$(i_k\mathrm{T})^4, i_\Delta'''$								3
11		$(f_\Delta\mathrm{T})^4, \theta_{kkk2}$								2
12		$(i_\Delta\mathrm{T})^4$								1
Totals	9	14	13	12	6	7	3	3	2	69

In undertaking the reduction we shall confine ourselves to the invariants, but it will be necessary to reduce some intermediate covariant forms as well. The reduction process for the invariants of higher degree becomes very involved and for these the method of attack and the results will be sketched very briefly. An attempt will be made to outline the methods as clearly as possible for the lower degree forms.

We begin with the invariant $A = (ff)^8$. From Syzygy (III) we notice at once that it can be written in terms of γ^2 and i_ϕ. Then we may say that A is reducible in our sense; that γ and i_ϕ are fundamental.

The next invariant is $B = (fi)^8$, where $i = (ff)^4 = i_x^8$. From Syzygy (I) i is expressed in terms of $(f\phi)^2$, γf, ϕ^2. Then in the transvectant form $(fi)^8$ we replace i by each one of these and attempt a further reduction by Gordan's series. We shall frequently use the following notation for this:

$$i \equiv \gamma f,\ \phi^2,\ (f\phi)^2 \quad \text{or also} \quad i \equiv (1),\ (2),\ (3);$$

that is, each of the constituent parts will be numbered serially and if there are several component parts of either member of the original transvectant, the transvectant of any two members will be indicated by compounding the numbers. The system of indexing will become fairly self-explanatory as we proceed.

Then we turn to the reduction of the invariant B.

B1: $\gamma(ff)^8 = \gamma A \equiv \gamma i_\phi,\ \gamma^3$; B2: $(f\phi^2)^8 = (f\phi^2)^8$;
B3: $\{(f\phi)^2 f\}^8 (8, 4, 8; 2, 6, 2) = \{(ff)^6 \phi\}^4 = (k\phi)^4$;
$k \equiv H_\phi, \gamma \cdot \phi, (f\phi)^4 \equiv (1),\ (2),\ (3)$;
B31: $(\phi H_\phi)^4 = j_\phi$; B32: $\gamma(\phi\phi)^4 = \gamma i_\phi$;
B33: $\{\phi f)^4 \phi\}^4 (4, 8, 4; 4, 0, 4) = (\phi^2 f)^8$.

So that if we take the forms γ, i_ϕ, j_ϕ, $(\phi^2 f)^8$ as fundamental, we have $B = (fi)^8$ reduced in terms of $\gamma \cdot i_\phi$, γ^3, $(\phi^2 f)^8$, j_ϕ.

The next invariant is of fourth degree, $C = (kk)^4$. From Syzygy (II), k is given in terms of $\gamma \cdot \phi$, H_ϕ, $(f\phi)^4$ [(1), (2), (3)]. So we take the fourth transvectant of k with each of its three members.

C1: $\gamma(\phi k)^4 \equiv \gamma j_\phi,\ \gamma^2 i_\phi,\ (\phi^2 f)^8$.
C2: $(kH_\phi)^4$; $k \equiv H_\phi,\ \gamma \cdot \phi,\ (f\phi)^4 \equiv (1),\ (2),\ (3)$.
C21: $(H_\phi H_\phi)^4 = \frac{1}{6} i_\phi^2$; C22: $\gamma(\phi H_\phi)^4 = \gamma j_\phi$;
C23: $\{(\phi f)^4 H_\phi\}^4 (4, 8, 4; 4, 0, 4) = (\phi H_\phi, f)^8$.
C3: $\{(f\phi)^4 k\}^4$; $k \equiv H_\phi,\ \gamma \cdot \phi,\ (f\phi)^4 \equiv (1),\ (2),\ (3)$.

C31: $\{(\phi f)^4 H_\phi\}^4 = (\phi H_\phi, f)^8$; C32: $\gamma\{\phi(\phi f)^4\}^4 = \gamma(\phi^2 f)^8$;

C33: $\{(f\phi)^4(f\phi)^4\}^4(8, 4, 4; 0, 4, 4) = \{(f(f\phi)^4)^4\phi\}^4$;

$$\{(f\phi)^4 f\}^4(8, 4, 8; 0, 4, 4) = (i\phi)^4 + (12/7)(k\phi)^2 + (1/5)A\cdot\phi.$$
$$\equiv (1), \qquad (2), \qquad (3).$$

C331: $\{(\phi i)^4\phi\}^4 = (\phi^2 i)^8$; $i \equiv \phi^2, \gamma f, (f\phi)^2 \equiv (a), (b), (c)$.

C331a: $(\phi^2\phi^2)^8 = (\phi\cdot\phi, \phi^2)^8 = \{(\phi\phi^2)^4\phi\}^4 = (18/35)i_\phi{}^2$; [*]

C331b: $(\phi^2, \gamma f)^8 = \gamma(\phi^2, f)^8$;

C331c: $\{(f\phi)^2\phi^2\}^8 = \{(\phi\phi^2)^2 f\}^8 = (5/7)(\phi H_\phi, f)^8$.

C332: $\{(\phi k)^2\phi\}^4 = (H_\phi k)^4 \equiv i_\phi{}^2, \gamma j_\phi, (\phi H_\phi, f)^8$; [Cf. C2].

C333: $A(\phi\phi)^4 = Ai_\phi \equiv \gamma^2 i_\phi, i_\phi{}^2$.

So the invariant $C = (kk)^4$ is reducible as $\gamma^2\cdot i_\phi$, γj_ϕ, $i_\phi{}^2$, $\gamma(\phi^2 f)^8$, $(\phi H_\phi, f)^8$.

The next following invariant is the one of fifth degree

$$f_{kk} = (f_k k)^4 = \{(fk)^4 k\}^4.$$

In order to accomplish the reduction here we must first secure the expression for the covariant form f_k:

$f_k = (fk)^4$; $k \equiv \gamma\cdot\phi, H_\phi, (f\phi)^4 \equiv (1), (2), (3)$.

f_k1: $(f, \gamma\phi)^4 = \gamma(f\phi)^4$; f_k2: $(fH_\phi)^4 = (fH_\phi)^4$;

f_k3: $\{(f\phi)^4 f\}^4 = (i\phi)^4 + (12/7)(k\phi^2) + (1/5)A\cdot\phi \equiv (1), (2), (3)$.

31: $(i\phi)^4$; $i \equiv (f\phi)^2, \gamma f, \phi^2 \equiv (1), (2), (3)$.

311: $\{(f\phi)^2\phi\}^4 = \{(\phi f)^2\phi\}^4(4, 8, 4; 2, 2, 2)$;

$$\sum_{i=0}^{2} a_i\{(\phi f)^{2+i}\phi\}^{4-i} = \sum_{j=0}^{0} b_j\{(\phi\phi)^{2+j}, f\}^{4-j}$$

where the actual numerical values of the literal coefficients a_i, b_j in these summations can be written down from the general form for Gordan's series given above.

$$\{(\phi f)^2\phi\}^4 = -\{(\phi f)^3\phi\}^3 - (2/7)\{(\phi f)^4\phi\}^2 + (H_\phi f)^4;$$

$$\{(\phi f)^3\phi\}^3(4, 8, 4; 2, 1, 3); \quad \sum_{i=0}^{1} a_i\{(\phi f)^{3+i}\phi\}^{3-i} = \sum_{j=0}^{1} b_j\{(\phi\phi)^{1+j} f\}^{5-j}$$

$$\{(\phi f)^3\phi\}^3 = -(1/2)\{(\phi f)^4\phi\}^2 + (1/2)(H_\phi f)^4$$

$$\{(\phi f)^4\phi\}^2(4, 8, 4; 2, 0, 4); \quad \{(\phi f)^4\phi\}^2 = \sum_{j=0}^{2} b_j\{(\phi\phi)^j f\}^{6-j}$$

---

[*] Grace and Young, p. 92.

$\{(\phi f)^4\phi\}^2 = (\phi^2 f)^6 + (2/7)(H_\phi f)^4$

$\therefore \{(\phi f)^2\phi\}^4 = (3/14)(\phi^2 f)^6 + (55/98)(H_\phi f)^4$

312: $(\phi, \gamma f)^4 = \gamma(f\phi)^4$; 313: $(\phi^2\phi)^4 = (18/35)i_\phi \cdot \phi$.

32: $(k\phi)^2$; $k \equiv (f\phi)^4, H_\phi, \gamma \cdot \phi \equiv (1), (2), (3)$.

321: $\{(\phi f)^4\phi\}^2 = (\phi^2 f)^6 + (2/7)(H_\phi f)^4$; 322: $(H_\phi \phi)^2 = (1/6)i_\phi\phi$;

323: $\gamma(\phi\phi)^2 = \gamma H_\phi$; 33: $A \cdot \phi \equiv \gamma^2\phi, i_\phi \cdot \phi$.

Hence f_k is expressed in terms of

$$\gamma(f\phi)^4,\ (H_\phi f)^4,\ (\phi^2 f)^6,\ i_\phi\phi,\ \gamma H_\phi,\ \gamma^2\phi.$$

In the work to follow there is no reason why we should retain the numerical coefficients which occur in the expansion by Gordan's series, and in fact our work will be very much easier if we do not. Then we can make use of a short cut which will be of very great advantage in saving time and labor. For instance, in reducing the form $\{(f\phi)^2\phi\}^4$, we notice that it is expressed in terms of $\{(f\phi)^3\phi\}^3$ and $\{(f\phi)^4\phi\}^2$. Furthermore when we have reduced these and substituted back, we notice that all the different types of terms which occur in $\{(f\phi)^2\phi\}^4$ also occur in $\{(f\phi)^4\phi\}^2$. We are interested only in the different types of terms which can arise, and so it is much easier to use $\{(f\phi)^4\phi\}^2$ than $\{(f\phi)^2\phi\}^4$ and so avoid the troublesome reduction on the left side of the series and the substitution. For a given form it is always possible to tell by inspection the form by which it can be replaced, but the method must be used with caution not to violate the conditions under which Gordan's series may be used. This device is used continuously in the work to follow.

We are now prepared to reduce the invariant f_{kk} of the fifth degree.

$f_k \equiv \gamma(f\phi)^4, (fH_\phi)^4, i_\phi \cdot \phi, \gamma H_\phi, \gamma^2 \cdot \phi, (\phi^2 f)^6 \equiv (1), (2), (3), (4), (5), (6)$.

$k \equiv (f\phi)^4, H_\phi, \gamma \cdot \phi \equiv (1), (2), (3)$.

f_{kk}11: $\gamma\{(f\phi)^4(f\phi)^4\}^4 \equiv \gamma^3 i_\phi, \gamma^2 j_\phi, \gamma \cdot i_\phi^2, \gamma^2(\phi^2 f)^8, \gamma(\phi H_\phi, f)^8$;

12: $\gamma\{(\phi f)^4 H_\phi\}^4 = \gamma(\phi H_\phi, f)^8$; 13: $\gamma\{(\phi f)^4, \gamma\phi\}^4 = \gamma^2(\phi^2 f)^8$.

21: $\{(fH_\phi)^4(f\phi)^4\}^4 = \{(f(f\phi)^4)^4 H_\phi\}^4$;

$\{f(f\phi)^4\}^4 \equiv (\phi^2 f)^6,\ (H_\phi f)^4,\ \gamma(f\phi)^4,\ i_\phi \cdot \phi,\ \gamma H_\phi,\ \gamma^2 \cdot \phi$

$\equiv (1),\ (2),\ (3),\ (4),\ (5),\ (6)$.

211: $\{(\phi^2 f)^6 H_\phi\}^4 = \{\phi^2 H_\phi)^2 f\}^8 \equiv (H_\phi{}^2 f)^8, i_\phi(\phi^2 f)^8$;

212: $\{(H_\phi f)^4 H_\phi\}^4 = (H_\phi{}^2 f)^8$; 213: $\gamma\{(\phi f)^4 H_\phi\}^4 = \gamma(\phi H_\phi, f)^8$;

214: $i_\phi(\phi H_\phi)^4 = i_\phi j_\phi$; 215: $\gamma(H_\phi H_\phi)^4 = \tfrac{1}{6}\gamma i_\phi{}^2$.

31: $i_\phi\{\phi(f\phi)^4\}^4 = i_\phi(\phi^2 f)^8$; 32: $i_\phi(\phi H_\phi)^4 = i_\phi j_\phi$; 33: $\{i_\phi\phi, \gamma\phi\}^4 = \gamma i_\phi^2$.
41: $\gamma\{H_\phi(f\phi)^4\}^4 = \gamma(\phi H_\phi, f)^8$; 42: $\gamma(H_\phi H_\phi)^4 = \tfrac{1}{6}\gamma i_\phi^2$;
43: $\{\gamma H_\phi, \gamma\phi\}^4 = \gamma^2 j_\phi$.
51: $\gamma^2\{\phi(\phi f)^4\}^4 = \gamma^2(\phi^2 f)^8$; 52: $\gamma^2(\phi H_\phi)^4 = \gamma^2 j_\phi$;
53: $\{\gamma^2\phi, \gamma\phi\}^4 = \gamma^3 i_\phi$.
61: $\{(\phi^2 f)^6 (f\phi)^4\}^4 = \{(f(f\phi)^4)^2\phi^2\}^8$.
$\{(f\phi)^4 f\}^2\ (8, 4, 8; 0, 2, 4) \equiv (H\phi)^4,\ (i\phi)^2,\ k\cdot\phi \equiv (1), (2), (3)$.
611: $\{(\phi H)^4\phi^2\}^8 = (\phi^3, H)^{12}$;
612: $\{(\phi i)^2\phi^2\}^8 = \{(\phi\phi^2)^2 i\}^8 \equiv (\phi H_\phi, i)^8$;
$i \equiv (f\phi)^2,\ \gamma f,\ \phi^2 \equiv (a), (b), (c)$.
612a: $\{(\phi f)^2, \phi H_\phi\}^8 = \{(\phi, \phi H_\phi)^2 f\}^8 \equiv (H_\phi{}^2 f)^8,\ i_\phi(\phi^2 f)^8$;
612b: $\{\gamma f, \phi H_\phi\}^8 = \gamma(f, \phi H_\phi)^8$; 612c. $(\phi^2, \phi H_\phi)^8 \equiv i_\phi j_\phi$.
613: $(\phi k, \phi^2)^8 = \{(\phi\phi^2)^4 k\}^4 \equiv i_\phi(\phi k)^4 \equiv i_\phi j_\phi,\ \gamma i_\phi^2, i_\phi(\phi^2 f)^8$;
62: $\{(\phi^2 f)^6 H_\phi\}^4 = \{(\phi^2 H_\phi)^2 f\}^8 \equiv (H_\phi{}^2 f)^8,\ i_\phi(\phi^2 f)^8$;
63: $\{(\phi^2 f),^6 \gamma\phi\}^4 = \gamma\{(\phi^2\phi)^2 f\}^8 = \gamma(\phi H_\phi, f)^8$.

Conclusion: f_{kk} is reduced rationally and integrally in terms of:

$$\gamma i_\phi^2,\ \gamma^2 j_\phi,\ \gamma^3 i_\phi,\ \gamma(\phi H_\phi, f)^8,\ \gamma^2(\phi^2 f)^8,\ i_\phi(\phi^2 f)^8,\ (H_\phi{}^2 f)^8,\ i_\phi j_\phi,\ (\phi^3 H)^{12}.$$

So far we have as basic invariants γ, i_ϕ, j_ϕ, $(\phi^2 f)^8$, $(\phi H_\phi, f)^8$, $(H_\phi{}^2 f)^8$, $(\phi^3 H)^{12}$. The last two have occurred first in f_{kk}.

We are interested next in the invariant of sixth degree,

$$i_{kk} = (i_k k)^4 = \{(ik)^4\, k\}^4.$$

To reduce it we must first get the forms appearing in the quartic covariant i_k. From the first two syzygies (I) and (II) we have

$$i \equiv (f\phi)^2,\ \gamma f,\ \phi^2; \qquad k \equiv (f\phi)^4,\ H_\phi,\ \gamma\cdot\phi.$$

We find that i_k is reduced in terms of $(t_\phi f)^6$, $i_\phi(\phi f)^4$, $\gamma(H_\phi f)^4$, $i_\phi H_\phi$, $j_\phi\phi$, $(\phi^2 f)^8\cdot\phi$, $(\phi H_\phi, f)^6$, $\gamma i_\phi\phi$, $\gamma^2 H_\phi$, $(H\phi^2)^8$, $\gamma(\phi^2 f)^6$, $\gamma^2(f\phi)^4$, $\gamma^3\phi$.

i_k is a covariant form of order four, degree four.

Then taking the fourth transvectant of each term of i_k with each one of the three forms of k, it can be shown that i_{kk} can be expressed in terms of the following:

$j_\phi(\phi^2f)^8$, $i_\phi(\phi H_\phi, f)^8$, $\gamma(H_\phi{}^2f)^8$, $\gamma i_\phi(\phi^2f)^8$, $i_\phi{}^3$, $j_\phi{}^2$, $\gamma i_\phi j_\phi$, $\gamma^2 i_\phi{}^2$, $\gamma^2(\phi H_\phi, f)^8$, $\gamma^3 j_\phi$, $j_\phi(\phi^2f)^8$, $(\phi^2f)^8\cdot(\phi^2f)^8$, $(\phi^2H_\phi, H)^{12}$, $\gamma(\phi^3H)^{12}$, $\gamma^3(\phi^2f)^8$, $\gamma^4 i_\phi$.

The fundamental invariants are:

$$\gamma,\ i_\phi,\ j_\phi,\ (\phi^2f)^8,\ (\phi H_\phi, f)^8,\ (H_\phi{}^2f)^8,\ (\phi^3H)^{12},\ (\phi^2H_\phi, H)^{12}.$$

The invariant $f_{k\Delta} = (f_k\Delta)^4$ comes next in order. To reduce it we need first the covariant $\Delta = (kk)^2$. We find that Δ may be replaced by:

$$(H\phi^2)^8,\ (\phi^2f)^8\cdot\phi,\ i_\phi(\phi f)^4,\ (t_\phi f)^5,\ \gamma(H_\phi f)^4,\ i_\phi H_\phi,\ j_\phi\phi,\ \gamma i_\phi\phi,\ (\phi H_\phi, f)^6,\ \gamma(\phi^2f)^6,\ \gamma^2H_\phi.$$

At the same time we have

$$f_k \equiv \gamma(f\phi)^4,\ (fH_\phi)^4,\ i_\phi\phi,\ \gamma H_\phi,\ \gamma^2\phi,\ (\phi^2f)^6.$$

We take the fourth transvectant of f_k with Δ. The result is that $f_{k\Delta}$ is reduced in terms of the following transvectants:

$\gamma(\phi^2f)^8(\phi^2f)^8$, $\gamma i_\phi{}^3$, $\gamma i_\phi(\phi H_\phi, f)^8$, $\gamma j_\phi{}^2$, $\gamma j_\phi(\phi^2f)^8$, $\gamma(\phi^2H_\phi, H)^{12}$, $\gamma^2 i_\phi(\phi^2f)^8$, $\gamma^2 i_\phi j_\phi$, $\gamma^2(H_\phi{}^2f)^8$, $\gamma^2(\phi^3H)^{12}$, $\gamma^2 i_\phi j_\phi$, $\gamma^3 i_\phi{}^2$, $\gamma^3(\phi H_\phi, f)^8$, $\gamma^4 j_\phi$, $(\phi^2f)^8(\phi H_\phi, f)^8$, $i_\phi(H_\phi{}^2f)^8$, $i_\phi(\phi^3H)^{12}$, $i_\phi{}^2(\phi^2f)^8$, $i_\phi{}^2j_\phi$, $j_\phi(\phi H_\phi, f)$ $(\phi H_\phi{}^2, H)^{12}$.

Then we have found as fundamental invariants thus far:

$$\gamma,\ i_\phi,\ j_\phi,\ (\phi^2f)^8,\ (\phi H_\phi, f)^8,\ (H_\phi{}^2f)^8,\ (\phi^3H)^{12},\ (\phi^2H_\phi, H)^{12}\ (\phi H_\phi{}^2, H)^{12}.$$

The invariant next to be considered is $i_{k\Delta} = (i_k\Delta)^4$ of eighth degree. i_k is expressed by means of:

$$i_\phi(f\phi)^4,\ \gamma^2(f\phi)^4,\ \gamma(fH_\phi)^4,\ j_\phi\phi,\ (\phi^2f)^8\cdot\phi,\ \gamma i_\phi\phi,\ \gamma^3\phi,$$
$$i_\phi H_\phi,\ \gamma^2H_\phi,\ \gamma(\phi^2f)^6,\ (t_\phi f)^5,\ (\phi H_\phi, f)^6,\ (H\phi^2)^8.$$

Δ is reduced in terms of:

$$(H\phi^2)^8,\ (\phi^2f)^8\phi,\ i_\phi(f\phi)^4,\ (t_\phi f)^5,\ \gamma(H_\phi f)^4,\ i_\phi H_\phi,\ j_\phi\phi,$$
$$\gamma i_\phi\phi,\ (\phi H_\phi, f)^6,\ \gamma(\phi^2f)^6,\ \gamma^2H_\phi.$$

Now if we take the fourth transvectant of all members in one set with all those in the other we can show that $i_{k\Delta}$ is expressed in terms of the following invariants.

$\gamma i_\phi(\phi^3 H)^{12}$, $\gamma i_\phi(H^2{}_\phi f)^8$, $\gamma i_\phi{}^2 j_\phi$, $\gamma i_\phi{}^2(\phi^2 f)^8$, $\gamma j_\phi(\phi H_\phi, f)^8$, $\gamma(\phi^2 f)^8(\phi H_\phi$
$\gamma(\phi H_\phi{}^2, H)^{12}$, $\gamma^2 i_\phi{}^3$, $\gamma^2 i_\phi(\phi H_\phi, f)^8$, $\gamma^2 j_\phi(\phi^2 f)^8$, $\gamma^2 j_\phi{}^2$, $\gamma^2(\phi^2 f)^8(\phi^2 f)$
$\gamma^2(\phi^2 H_\phi, H)^{12}$, $\gamma^3 i_\phi j_\phi$, $\gamma^3 i_\phi(\phi^2 f)^8$, $\gamma^3(\phi^3 H)^{12}$, $\gamma^3(H_\phi{}^2 f)^8$, $\gamma^4 i_\phi{}^2$,
$\gamma^4(\phi H_\phi, f)^8$, $\gamma^5 j_\phi$, $i_\phi{}^4$, $i_\phi{}^2(\phi H_\phi, f)^8$, $i_\phi j_\phi{}^2$, $i_\phi j_\phi(\phi^2 f)^8$,
$i_\phi(\phi^2 f)^8(\phi^2 f)^8$, $i_\phi(\phi^2 H_\phi, H)^{12}$, $j_\phi(\phi^3 H)^{12}$, $j_\phi(H_\phi{}^2 f)^8$, $(\phi^2 f)^8(\phi^3 H)^{12}$, $(\phi^2 f)^8(H_\phi{}^2 f)^8$
$(\phi H_\phi, f)^8(\phi H_\phi, f)^8$, $(H_\phi{}^3 H)^{12}$.

The basic invariants are

$$\gamma,\ i_\phi,\ j_\phi,\ (\phi^2 f)^8,\ (\phi H_\phi, f)^8,\ (H_\phi{}^2 f)^8,\ (\phi^3 H)^{12},\ (\phi^2 H_\phi, H)^{12},\ (\phi H_\phi{}^2, H)^{12},\ (H_\phi{}^3 H)^{12}.$$

We consider in sequence the invariant of ninth degree, $f_{\Delta\Delta} = (f_\Delta \Delta)^4$. Hence we must first reduce the quartic covariant f_Δ.

Δ is reduced in terms of the following covariant forms:

$$(H\phi^2)^8,\ (\phi^2 f)^8\phi,\ i_\phi(\phi f)^4,\ (t_\phi f)^5,\ \gamma(H_\phi f)^4,\ i_\phi H_\phi,\ j_\phi\phi,\ \gamma i_\phi\phi,\ (\phi H_\phi, f)^6,\ \gamma(\phi^2 f)^6,\ \gamma^2 H_\phi.$$

From these we have f_Δ given by

$i_\phi{}^2\phi$, $\gamma j_\phi\phi$, $\gamma^2 i_\phi\phi$, $\gamma(\phi^2 f)^8\phi$, $(\phi H_\phi, f)^8\phi$
$\gamma i_\phi H_\phi$, $\gamma^3 H_\phi$, $j_\phi H_\phi$, $(\phi^2 f)^8 H_\phi$, $(\phi^2 f)^8(\phi f)^4$,
$\gamma i_\phi(\phi f)^4$, $j_\phi(\phi f)^4$, $i_\phi(H_\phi f)^4$, $\gamma^2(H_\phi f)^4$, $\gamma^2(\phi^2 f)^6$,
$i_\phi(\phi^2 f)^6$, $\gamma(\phi^2 H)^8$, $(\phi H_\phi, H)^8$, $\gamma(i H_\phi)^4$, $(i t_\phi)^5$,
$\gamma(i\phi^2)^6$, $(\phi H_\phi, i)^6$, $\gamma\,(k H_\phi)^2$, $i_\phi(k\phi)^2$, $(k t_\phi)^3$,
$(\phi^2 p)^8$, $(\phi^2 f_1)^7$, $(\phi^2 f_2)^6$, $(\phi^2 f_3)^5$, $(H_\phi f_k)^2$,
$i_\phi f_k$.

Then using this set and the set for Δ just preceding, we start to reduce $f_{\Delta\Delta}$. This is a very long and involved process, and due to the style employed, no record has been kept of all the possible combinations of factors which occur. It can be shown definitely however that $f_{\Delta\Delta}$ is composed of invariant terms, every one of which is a product of invariants taken from the group of fundamental forms following:

$$\gamma,\ i_\phi,\ j_\phi,\ (\phi^2 f)^8, (\phi H_\phi, f)^8,\ (H_\phi{}^2 f)^8,\ (\phi^3 H)^{12},\ (\phi^2 H_\phi, H)^{12},\ (\phi H_\phi{}^2, H)^{12},\ (H_\phi{}^3 H)^{12},\ (\phi^3 t_\phi, T)^{18}.$$

The last form is new and occurs for the first time in this ninth degree invariant.

The last invariant is $i_{\Delta\Delta}$ of tenth degree. As above, we need the reduced form of the intermediate covariant form i_Δ, of degree six and order four. We remember that i is given by $(f\phi)^2, \gamma f, \phi^2$ and that

$$\begin{aligned}\Delta \equiv (H\phi^2)^8,\ (\phi^2 f)^8\cdot\phi,\ i_\phi(\phi f)^4,\ (t_\phi f)^5,\ \gamma(H_\phi f)^4,\ i_\phi H_\phi,\\ j_\phi\phi,\ \gamma i_\phi\phi,\ (\phi H_\phi, f)^6,\ \gamma(\phi^2 f)^6,\ \gamma^2 H_\phi.\end{aligned}$$

By taking the cross-transvectants we find that set of forms which we may use for i_Δ. In the reduction followed out there are ninety-one of these forms, and a list of them will not be given here. This does not represent the minimum number of such forms however, for it is often hard to tell which of several covariant forms is the simplest. It is sufficient to say that part of the forms in this list of ninety-one are reducible in terms of the others. But it appears easier to reduce them in a transvectant which is an invariant than before this point. From this point of view we use methods in reducing invariants which would be quite long and involved in the reduction of covariants.

From the list of forms comprising i_Δ and the constituent forms of Δ itself given previously we are ready to undertake the task of reducing the tenth degree invariant. The procedure is so long and tedious that we shall merely state the results. Most of the reductions are accomplished in terms of intermediate forms which have already been treated.

We find that $i_{\Delta\Delta}$ is composed of members which are compounded of the list of basic invariants following:

$$\begin{aligned}\gamma,\ i_\phi,\ j_\phi,\ (\phi^2 f)^8,\ (\phi H_\phi, f)^8,\ (H_\phi{}^2 f)^8,\ (\phi^3 H)^{12},\ (\phi^2 H_\phi, H)^{12},\ (\phi H_\phi{}^2, H)^{12},\\ (H_\phi{}^3 H)^{12},\ (\phi^3 t_\phi, T)^{18},\ (\phi^2 H_\phi t_\phi, T)^{18}.\end{aligned}$$

Now it has been shown that the nine invariants of the Jacobian octavic f, of degree two to ten inclusive, can be replaced by the twelve fundamental invariants given above. In the higher degree invariants the reduction has not actually been carried so far, but has been accomplished in terms of intermediate forms which have already been shown to be equivalent to these twelve forms.

The twelve fundamental forms found are characterized, aside from the first three, by the fact that the transvectant symbol representing them has only forms of the quartic on one side of the symbol, forms of the octavic on the other. In this sense they might be called pure. The invariant γ is a joint invariant of Q and Q' and does not occur in the simultaneous system of f and ϕ. The two invariants i_ϕ and j_ϕ are basic invariants of the quartic ϕ. The others are joint transvectants of the simpler forms of f and ϕ.

It remains for us to reconcile our results with the general theory on which the investigation is based and in this respect the work is incomplete. Let us refer again to the three relations of Stroh, connecting f, ϕ and γ with the covariants of the apolar octavic $F = F_x^8$. We note that f, ϕ, γ, each multiplied by the power of the resultant R, are equated to expressions involving covariant forms of F, and that each one of these three expressions is of degree sixteen in the p's. All our twelve fundamental forms are transvectants of the three forms f, ϕ, γ and hence if we perform the transvection on the right as indicated we can ultimately reduce these to polynomials in the invariants $I_2 \cdots I_{10}$ of F, ranging in degree from eight to forty, for $I_2 \cdots I_{10}$ comprise the complete system of invariants of F.

Theoretically we should get nine invariants for the complete system and we have twelve. From the form of Stroh's three relations we may infer that the resultant R might be added to the list and that three forms of the list are superfluous. But R itself can finally be expressed in terms of these twelve forms, from the form for R given by Gordan on p. 389 of Vol. 3 of the *Mathematische Annalen.* While we are able to get all our twelve forms expressed as polynomials in $I_2 \cdots I_{10}$ of the apolar octavic, we have no obvious way of getting the $I_2 \cdots I_{10}$ in terms of γ, i_ϕ, j_ϕ, $\cdots$, etc., and this prevents us from drawing conclusions as to the number of independent forms in this system.

If we examine our list of twelve fundamental forms more closely we are led to several additional conclusions. In the first place, the invariant γ which is certainly a combinant form of Q and Q' is not found in the joint system of covariants of f and ϕ, for it is of the first degree while a transvectant from this system must be at least of the second degree. Next, while we find the transvectant forms $(\phi^3 t_\phi, T)^{18}$, $(\phi^2 H_\phi t_\phi, T)^{18}$ in our set, we do not meet the remaining two forms of similar composition, $(\phi H_\phi^2 t_\phi, T)^{18}$, $(H_\phi^3 t_\phi, T)^{18}$, as we might expect from symmetry. This might be advanced as an intuitional argument for believing that the first two mentioned are superfluous. Furthermore no additional forms of this type can occur, for T is the highest ordered covariant in the system of f which can be used, and if we had on the right side of our transvectant symbol the product of two forms of f for a transvectant of higher index, the form would be reducible by a criterion due to Gordan.

So we are still faced with the problem of determining how many of these forms furnish a complete system of invariant combinants for our two quintics. This question even for the lower ordered forms has not been treated exhaustively in the literature. For instance the treatment of the case of two quartics by Rowe [*] is not complete. It is hoped that some means of treating this problem in general can be secured. And of course we are still faced with the problem, much larger in scope, of the complete system of covariant combinants of the two quintics, of which the matter of invariants is only a small part.

---

[*] J. E. Rowe, *Transactions of the American Mathematical Society*, Vol. 12, p. 305.

A Boundary Value Problem of Ordinary Self-Adjoint Differential Equations with Singularities.

BY MARION CAMERON GRAY.*

Introduction. The problem of solving the ordinary homogeneous self-adjoint differential equation of the second order with singularities, subject to given boundary conditions, has been discussed by Hilb.* He considers the non-singular equation

$$(d/dx)(xdu/dx) + [-g(x) + \lambda h(x)/x]\, u = 0 \qquad (\epsilon \leqq x \leqq 1)$$

with the boundary conditions $u(\epsilon) = u(1) = 0$, and introduces a singularity by allowing ϵ to become zero. The functions g and h are continuous, and h is positive in $(\epsilon, 1)$. Hilb finds a Green's function for the equation, and the corresponding integral equation; then, in passing to the limiting case $\epsilon = 0$, he applies the Hilbert † theory of quadratic forms in infinitely many variables.

Weyl ‡ has discussed the more general equation

$$(d/dx)(p(x)du/dx) - q(x)u + \lambda u = 0 \qquad (0 \leqq x < \infty)$$

with homogeneous boundary conditions at zero and infinity. The function q is continuous, and p is continuous and positive in the interval $(0, \infty)$, so that the only singularity is at infinity. Weyl also introduces a Green's function, and applies to the resulting integral equation his theory of singular integral equations § which is based on Hilbert's work. He distinguishes two types of singularity, the limit circle type, which leads to solutions for complex values of λ, and the limit point type, which leads to solutions only for real values of λ.

---

* E. Hilb, "Über Integraldarstellungen willkürlicher Funktionen," *Mathematische Annalen*, Vol. 66, pp. 1-66. The work of Wirtinger, *Mathematische Annalen*, Vol. 48, pp. 387-389, may also be cited.

† D. Hilbert, *Grundzüge einer allgemeinen Theorie der linearen Integralgleichungen*, Leipzig (1912).

‡ H. Weyl, "Über gewöhnliche lineare Differentialgleichungen mit singulären Stellen und ihre Eigenfunktionen," *Göttinger Nachrichten*, Mathematische-Physische Klasse, 1909, pp. 37-63, also "Über gewöhnliche Differentialgleichungen mit Singularitäten und die zugehörigen Entwicklungen willkürlicher Funktionen," *Mathematische Annalen*, Vol. 68, pp. 220-269.

§ H. Weyl, "Singuläre Integralgleichungen," *Mathematische Annalen*, Vol. 66, pp. 273-324.

In a later paper [*] Weyl extends his method to the equation

$$(d/dx)(p(x)\,du/dx) - q(x)u + \lambda k(x)u = 0 \qquad (-\infty < x < \infty)$$

where the function k is continuous, and there are singularities at both ends of the interval $(-\infty, \infty)$, first for the case in which k is positive, and then for the polar case in which k may change sign.

In the present paper we consider the equation

$$(d/dx)(p(x)\,du/dx) + q(x)u + \lambda u = 0 \qquad (0 \leq x \leq \pi)$$

with boundary conditions $u(0) = u(\pi) = 0$. We assume that the function q is continuous, and that the function p is continuous with a continuous first derivative, and vanishes a finite number of times in the interval. We introduce a complete normed orthogonal system of functions, $\phi_m(x)$, satisfying the boundary conditions, and use it to pass from the differential equation to a system of infinitely many equations in an infinite number of unknowns.[†] We consider the case in which this system of equations has solutions only for real values of λ, Weyl's limit point type, and we apply the results of the Carleman [‡] theory of integral equations of Class I, for the special case of quadratic forms. From the solutions of the system of infinitely many equations we derive solutions of the given differential equation and discuss their properties, referring in this connexion also to the Hellinger [§] theory of limited quadratic forms (§§ 1 and 2).

In § 3 we introduce a theory of *limited differential systems,* by means of which we derive (§ 4) an expansion theorem for an arbitrary function $f(x)$, subject to the conditions that f be continuous, satisfy the boundary conditions, and have a first derivative which is integrable and of integrable square in $(0, \pi)$. The corresponding expansion theorems of Weyl and Hilb require further the existence of the function

---

[*] H. Weyl, "Über gewöhnliche lineare Differentialgleichungen mit singulären Stellen und ihre Eigenfunktionen," *Göttinger Nachrichten,* Mathematische-Physische Klasse, 1910, pp. 442-467. For further references to the literature of the subject we may refer to those given by Weyl in the introduction to this paper.

[†] Anna Pell Wheeler, "Linear Ordinary Self-Adjoint Differential Equations of the Second Order," *American Journal of Mathematics,* Vol. 49 (1927), p. 310.

[‡] T. Carleman, *Sur les équations intégrales singulières à noyau reél et symétrique.*

[§] E. Hellinger, "Neue Begründung der Theorie quadratischer Formen von unendlichvielen Veränderlichen," *Journal für Mathematik,* Vol. 136, pp. 211-271.

$$(d/dx)(p(x)df/dx),$$

though we may note that Plancherel[*] has obtained an expansion theorem for the Hilb problem in which he requires only that f be continuous and of bounded variation in $(0, 1)$, vanish at each end of the interval, and be such that the integral

$$\int_0^1 |f(t)/t|\, dt$$

converges.

1. *Properties of the Solutions of the System of Infinitely Many Equations Arising from the Differential Equation.* We consider solutions of the linear homogeneous self-adjoint differential equation of the second order

$$(d/dx)(p(x)du/dx) + q(x)u + \lambda u = 0 \tag{1}$$

which are of integrable square in the interval $(0, \pi)$. The function q is continuous in $(0, \pi)$, the function p is continuous with a continuous first derivative, and vanishes a finite number of times in the interval. The boundary conditions are assumed to be

$$u(0) = u(\pi) = 0, \tag{2}$$

if $p(0) \neq 0$ and $p(\pi) \neq 0$, but if for example $p(0) = 0$ the conditions are

$$\lim_{x \to 0} p(x)u(x) = 0 \text{ and } u(\pi) = 0. \tag{2'}$$

An auxiliary system of normed orthogonal functions satisfying the boundary conditions (2) is

$$\phi_m(x) = (2/\pi)^{1/2} \sin mx, \tag{3}$$

and this system is also complete.

To obtain from the system (1) and (2) [or (2′)] a system of infinitely many equations in an infinite number of unknowns we multiply equation (1) by $\phi_m(x)$, and integrate between 0 and π,[†]

$$\int_0^\pi (pu')'\phi_m + \int_0^\pi qu\phi_m + \lambda \int_0^\pi u\phi_m = 0.$$

---

[*] M. Plancherel, "Integraldarstellungen willkürlicher Funktionen," *Mathematische Annalen*, Vol. 67, pp. 519-534.

[†] When there is no ambiguity we shall omit the variable of integration; for definite integrals in $(0, \pi)$ this variable is always x. We use primes throughout to denote derivatives with respect to x.

Now

$$\int_0^\pi (pu')'\phi_m = \int_0^\pi (p\phi_m')'u,$$

since ϕ_m satisfies the boundary conditions (2), while u satisfies the conditions (2) or (2′). Also

$$\int_0^\pi (p\phi_m')'u = \sum_n \int_0^\pi (p\phi_m')'\phi_n \int_0^\pi u\phi_n$$

by the known properties of the complete system (3). A solution u of the system (1) and (2) [or (2′)] therefore satisfies the system of equations

$$\sum_{n=1}^\infty \int_0^\pi (p\phi_m')'\phi_n \int_0^\pi u\phi_n + \sum_{n=1}^\infty \int_0^\pi q\phi_m\phi_n \int_0^\pi u\phi_n + \lambda \int_0^\pi u\phi_m = 0,$$

which may be written

$$\sum_{n=1}^\infty p_{mn}x_n + \sum_{n=1}^\infty q_{mn}x_n + \lambda x_m = 0, \tag{4}$$

where

$$x_m = \int_0^\pi u\phi_m,$$

$$p_{mn} = \int_0^\pi (p\phi_m')'\phi_n,$$

and

$$q_{mn} = \int_0^\pi q\phi_m\phi_n.$$

The matrix $P = (p_{mn})$ is symmetric, the rows are of finite norm, that is Σp_{mn}^2 converges for every value of m, but P is not limited. The matrix $Q = (q_{mn})$ is symmetric and limited.

The matrix $P + Q$ is of the type for which Carleman has given an existence theorem.[*] In this paper we consider only the case in which $P + Q$ is of Class I,[†] that is, the system of equations (4) has no solution of finite norm, not identically zero, for complex values of λ.

The system of equations (4) may, however, have solutions of finite norm, not identically zero, for real values of λ, the characteristic values. Let these values be λ_a, and the corresponding characteristic solutions l_{am}. These characteristic values of λ form the *point spectrum.*

Let $\Delta f(\lambda) = f(\lambda_2) - f(\lambda_1)$ denote the increment of the continuous function $f(\lambda)$ for the interval $\Delta = (\lambda_1, \lambda_2)$ of the real λ axis, and consider the system of equations

---

[*] Carleman, *l. c.*, ch. 1.

[†] Carleman, *l. c.*, p. 187.

$$(5) \qquad \sum_n p_{mn} \Delta\rho_n(\lambda) + \sum_n q_{mn} \Delta\rho_n(\lambda) + \int_\Delta \lambda d\rho_n(\lambda) = 0,$$

which may have solutions of finite norm, and not constant, for certain intervals Δ of the real λ axis. These intervals Δ form the *continuous spectrum,* and the corresponding characteristic differential solutions $\rho_m(\lambda)$ are continuous and of bounded variation in λ. They are determined up to an additive constant, which we fix by the condition $\rho_m(0) = 0$ for every value of m. The sequence $\{\rho_m(\lambda)\}$ defines a continuous monotonic non-decreasing function of bounded variation in λ

$$(6) \qquad \begin{aligned} \rho(\lambda) &= \sum_m [\rho_m(\lambda)]^2 & \lambda > 0 \\ &= 0 & \lambda = 0 \\ &= -\sum_m [\rho_n(\lambda)]^2 & \lambda < 0 \end{aligned}$$

The function $\rho(\lambda)$ is called the basis function of the solutions $\rho_m(\lambda)$.

The term *spectrum* is used to denote the values of λ which belong to the point spectrum, the limiting points of the point spectrum, and the continuous spectrum, and we know by the theory of forms of Class I that such a spectrum exists, that is, there are solutions of finite norm either of the linear system of equations (4), or of the differential system (5) or of both. Further, since the matrix P is not limited, the spectrum may extend over any part of the real λ axis.

We may note that there may be more than one sequence of characteristic differential solutions satisfying all the conditions imposed, but for the present we shall assume that the continuous spectrum is simple, and return later (§ 5) to the more general case.

The sequences $\{l_{\alpha m}\}$ and $\{\rho_m(\lambda)\}$ form an orthogonal system, satisfying the relations

$$(7) \qquad \begin{aligned} \sum_m l_{\alpha m} l_{\beta m} = e_{\alpha\beta} &= 0 & \alpha \neq \beta \\ &= 1 & \alpha = \beta, \end{aligned}$$

$$(8) \qquad \sum_m \Delta_1\rho_m(\lambda)\, \Delta_2\rho_m(\lambda) = \Delta_{12}\rho(\lambda),$$

where Δ_1 and Δ_2 are any two intervals of the real λ axis in which there is a continuous spectrum, and Δ_{12} the interval common to the two intervals Δ_1 and Δ_2. Finally the two types of solutions are orthogonal

$$(9) \qquad \sum_m l_{am}\,\Delta\rho_m(\lambda)=0$$

for every α, and every value of λ.[*]

If the system of equations

$$(10) \qquad \sum_n (p_{mn}+q_{mn})x_n=0$$

has no solution of finite norm other than zero, the solutions l_{am} and $\rho_m(\lambda)$ form a *complete* system, satisfying the relation

$$(11) \qquad \sum_{a} l_{am}l_{an}+\int_{-\infty}^{\infty}\frac{d\rho_m(\lambda)\,d\rho_n(\lambda)}{d\rho(\lambda)}\dagger=e_{mn}.$$

The proof is easily obtained from the following result which is a special case of a theorem given by Carleman.‡

Let the matrix of the system of equations (4) be of Class I, with characteristic numbers λ_a and corresponding characteristic solutions l_{am}. Let the continuous spectrum be simple with characteristic differential forms $\rho_m(\lambda)$ and basis function $\rho(\lambda)$. Write

$$Y_a=\sum_n l_{an}y_n, \qquad X_a=\sum_n l_{an}x_n$$

$$\Delta y(\lambda)=\sum_n \Delta\rho_n(\lambda)y_n, \qquad \Delta x(\lambda)=\sum_n \Delta\rho_n(\lambda)x_n,$$

where $\{x_n\}$ and $\{y_n\}$ are sequences of finite norm, and $\{x_n\}$ is such that $\sum x_m[\sum(p_{mn}+q_{mn})^2]^{1/2}$ converges. Then

$$(12) \qquad -\sum_n (p_{mn}+q_{mn})y_n=\sum_a \lambda_aY_al_{am}+\int_{-\infty}^{\infty}\frac{\lambda\,d\rho_m(\lambda)\,dy(\lambda)}{d\rho(\lambda)},$$

$$(13) \qquad -\sum_{m,n} (p_{mn}+q_{mn})x_my_n=\sum_a \lambda_aX_aY_a+\int_{-\infty}^{\infty}\frac{\lambda\,dx(\lambda)\,dy(\lambda)}{d\rho(\lambda)}.$$

The series and integrals involved are absolutely convergent.

These expansions show at once that there is no sequence of finite norm, not identically zero, orthogonal to all the sequences $\{l_{am}\}$ and $\{\rho_m(\lambda)\}$. For any two sequences $\{x_m\}$ and $\{y_n\}$ of finite norm we obtain expansions

$$(14) \qquad y_m=\sum_a l_{am}\sum_n y_nl_{an}+\int_{-\infty}^{\infty}\frac{d\rho_m(\lambda)\,d\sum_n y_n\rho_n(\lambda)}{d\rho(\lambda)}$$

---

[*] For proofs of these statements about the spectrum we may refer to Hellinger, *l. c.*, pp. 240 et seq., and Carleman, *l. c.*, ch. 3.

† Hellinger, *l. c.*, p. 237, defines this integral in a finite interval, and the definition is extended to an infinite interval by Weyl, *l. c.*, *Mathematische Annalen*, Vol. 66, p. 288.

‡ Carleman, *l. c.*, pp. 101-102.

$$(15) \qquad \sum_m x_m y_m = \sum_a \sum_n l_{am} x_m \sum_n l_{an} y_n + \int_{-\infty}^{\infty} \frac{d \sum_m x_m \rho_m(\lambda)\, d \sum_n y_n \rho_n(\lambda)}{d\rho(\lambda)}^{*}$$

and the relation (11) follows immediately.

2. *The Solutions of the Differential Equation.* From the solutions l_{am} and $\rho_m(\lambda)$ of the last section we can now derive solutions of the differential equation (1). Consider the sequence $\{\rho_m(\lambda)\}$ which is a solution of finite norm for the system of equations (5). By the Riesz-Fischer theorem

$$(16) \qquad \rho_m(\lambda) = \int_0^\pi u(x,\lambda)\phi_m(x)\,dx$$

for a function $u(x, \lambda)$ which is of integrable square in x in $(0, \pi)$. Further the sequence $\{\Sigma\, p_{mn}\rho_n(\lambda)\}$ is of finite norm. Now

$$\begin{aligned}\sum_n p_{mn}\rho_n(\lambda) &= \sum_n \int_0^\pi (p\phi_m')'\phi_n \int_0^\pi u\phi_n \\ &= \int_0^\pi (p\phi_m')'u \\ &= -(2/\pi)^{1/2} m^2 \int_0^\pi pu \sin mx + (2/\pi)^{1/2} m \int_0^\pi p'u \cos mx \\ &= (2/\pi)^{1/2} m \left[\int_0^\pi p'u \cos mx - m \int_0^\pi pu \sin mx\right]\end{aligned}$$

The function u is such that this is of finite norm, and therefore the factor in the bracket is of finite norm. But the sequence $\{\int_0^\pi p'u \cos mx\}$ is itself of finite norm, therefore the sequence $\{m \int_0^\pi pu \sin mx\}$ is likewise. It follows that the function pu has a derivative almost everywhere which is integrable and of integrable square in the interval $(0, \pi)$, and, since p has a derivative, u has a derivative almost everywhere except perhaps for the zeros of the function p. We have also $u(0, \lambda) = u(\pi, \lambda) = 0$, unless $p(0) = 0$ or $p(\pi) = 0$, and in all cases $\lim_{x\to 0} p(x)u(x, \lambda) = \lim_{x\to\pi} p(x)u(x, \lambda) = 0$.

Further, the expression

$$m \int_0^\pi p'u \cos mx - m^2 \int_0^\pi pu \sin mx$$

---

[*] Carleman, *l. c.*, p. 103.

is of finite norm, and

$$-m^2\int_0^\pi pu\sin mx = -m\int_0^\pi (pu)'\cos mx.$$

Hence

$$m\int_0^\pi [p'u-(pu)']\cos mx$$

is of finite norm, and therefore the function $p'u-(pu)'$ has a derivative almost everywhere which is integrable and of integrable square in the interval $(0,\pi)$. Except at the zeros of the function p this function may be written pu', so that at all points for which $p(x)\neq 0$ the functions u' and $(pu')'$ exist almost everywhere. Further we have

$$\int_0^\pi [u(x,\lambda)]^2\,dx = \sum_m\Big[\int_0^\pi u\phi_m\Big]^2 = \rho(\lambda),$$

and therefore $\int_0^\pi [u(x,\lambda)]^2\,dx$ converges to a continuous function of λ.

Now Weyl[*] has defined a *differential solution* $u(x,\lambda)$ of equation (1) as a real continuous function of x and of λ, for which the integral $\int_0^\pi [u(x,\lambda)]^2\,dx$ exists as a continuous function of λ, and which satisfies the boundary conditions (2) or (2′) and the equation

$$(pu')'+qu+\int_0^\lambda \lambda\,du\,\dagger = 0. \tag{17}$$

By the properties we have already found of the function $u(x,\lambda)$ defined by equation (16), we need only show that it satisfies equation (17) in order to prove that it is a differential solution of equation (1). Return to the system of equations (5) satisfied by $\rho_m(\lambda)$, and write it in the form

$$\sum_n\int_0^\pi (p\phi_m')'\phi_n\Delta\rho_n(\lambda)+\sum_n\int_0^\pi q\,\phi_m\phi_n\,\Delta\rho_n(\lambda)+\int_\Delta \lambda d\rho_m(\lambda)=0,$$

Substitute for $\Delta\rho_m(\lambda)$ from equation (16)

$$\int_0^\pi (p\phi_m')'\Delta u(x,\lambda)+\int_0^\pi q\phi_m\Delta u(x,\lambda)+\int_0^\pi \phi_m\int_\Delta \lambda\,du(x,\lambda)=0.$$

---

[*] Weyl. *l. c.*, *Mathematische Annalen*, Vol. 68, p. 239.

[†] In the Stieltjes and Hellinger integrals used throughout the variable of integration is always λ.

Take Δ to be the interval $(0, \lambda)$, then $\Delta\, u(x, \lambda) = u(x, \lambda)$. Further

$$\int_0^\pi (p\phi_m')'u(x, \lambda) = \int_0^\pi [p'u - (pu)']'\phi_m$$

by the boundary properties of the functions $u(x, \lambda)$ and $\phi_m(x)$, and therefore finally

$$\int_0^\pi \phi_m\{[p'u - (pu)']' + qu + \int_0^\lambda \lambda du(x, \lambda)\} = 0. \tag{18}$$

At all points for which $p(x) \neq 0$ this may be written

$$(pu')' + qu + \int_0^\lambda \lambda du(x, \lambda) = 0,$$

so that the function $u(x, \lambda)$ is a differential solution of equation (1), except perhaps at the zeros of the function p.

If we consider the solutions l_{am} of the system of equations (4) we can show in a similar manner that the functions $u_a(x)$ defined by

$$l_{am} = \int_0^\pi u_a(x)\ \phi_m(x)\, dx \tag{19}$$

are solutions of equation (1) except perhaps at the zeros of the function p.[*] The functions $u_a(x)$ are such that the sequence $\{ m \int_0^\pi pu_a\phi_m\}$ is of finite norm, so that the function pu_a has a derivative almost everywhere which is integrable and of integrable square in the interval $(0, \pi)$.

The solutions $u(x, \lambda)$ and $u_a(x)$ have orthogonal properties similar to those of the systems $\rho_m(\lambda)$ and l_{am}. They are derived immediately from the corresponding relations (7), (8) and (9), and have the form

$$\int_0^\pi u_a(x)\ u_\beta(x)\, dx = e_{a\beta}, \tag{20}$$

$$\int_0^\pi \Delta_1 u(x, \lambda)\ \Delta_2 u(x, \lambda)\, dx = \Delta_{12}\rho(\lambda) \tag{21}$$

for any two intervals Δ_1 and Δ_2 of the real λ axis, and

$$\int_0^\pi u_a(x)\ \Delta u(x, \lambda)\, dx = 0 \tag{22}$$

for every value of α, and every interval Δ.

---

[*] These solutions have been obtained for the non-singular differential equation $u'' + qu + \lambda u = 0$ by Anna Pell Wheeler, *l. c.*, *American Journal of Mathematics*, Vol. 49 (1927), p. 312.

3. *Limited Differential Systems.* We now introduce the conception of limited differential systems, and prove some properties of these systems analogous to the known properties of limited matrices. These are used later in obtaining an expansion theorem for an arbitrary function $f(x)$, subject to suitably imposed conditions, in terms of the solutions $u_a(x)$ and $u(x, \lambda)$ of § 2. The functions $\rho_m(\lambda)$ and $\rho(\lambda)$ used throughout this section are the characteristic differential solutions and corresponding basis function of § 1.

Definition I. Let $f(\lambda)$ be a continuous function of λ in the infinite interval $(-\infty, \infty)$, and let $u(\lambda)$ be continuous and of bounded variation in that interval. Then the function $\int_\Delta f(\lambda) du(\lambda)$ is integrable $H(\rho)$ in $(-\infty, \infty)$ if the Hellinger integral

$$\int_{-\infty}^{\infty} [f(\lambda)]^2 \frac{[du(\lambda)]^2}{d\rho(\lambda)}$$

converges.* If $\int_\Delta f(\lambda) du(\lambda)$ is integrable $H(\rho)$, and if, further, $\Delta g(\lambda)$ is any function integrable $H(\rho)$, then the integral

$$\int_{-\infty}^{\infty} f(\lambda) \frac{du(\lambda) dg(\lambda)}{d\rho(\lambda)}$$

is absolutely convergent.

Definition II. The system $\int_\Delta f(\lambda) du_m(\lambda)$ is a *differential system* in $(-\infty, \infty)$ if $\int_\Delta f(\lambda) du_m(\lambda)$ is integrable $H(\rho)$ in $(-\infty, \infty)$ for every value of m, and if, further, the series $\sum [u_m(\lambda)]^2$ converges to a continuous function $u(\lambda)$.

Definition III. The differential system $\int_\Delta f(\lambda) du_m(\lambda)$ is *limited* in $(-\infty, \infty)$ if the sequence

$$\left\{ \int_{-\infty}^{\infty} f(\lambda) \frac{du_m(\lambda) dg(\lambda)}{d\rho(\lambda)} \right\}$$

is of finite norm and norm $< M \int_{-\infty}^{\infty} [f(\lambda)]^2 \frac{[dg(\lambda)]^2}{d\rho(\lambda)}$, for every function $\Delta g(\lambda)$ integrable $H(\rho)$.
A similar definition holds for a finite interval.

THEOREM I. *The system $\Delta\rho_m(\lambda)$ is a limited differential system.*

---

* This definition is similar to that given by A. J. Pell, "Linear Equations with unsymmetric Systems of Coefficients," *Transactions of the American Mathematical Society*, Vol. 20, p. 26.

By the statements of § 1 we see immediately that $\Delta\rho_m(\lambda)$ is a differential system, and it remains to prove that it is limited. The orthogonal relation (8) is equivalent to the relation

$$(23)\qquad \int_{-\infty}^{\infty} \frac{du(\lambda)\,dv(\lambda)}{d\rho(\lambda)} = \sum_m \int_{-\infty}^{\infty} \frac{du(\lambda)\,d\rho_m(\lambda)}{d\rho(\lambda)} \int_{-\infty}^{\infty} \frac{dv(\lambda)\,d\rho_m(\lambda)}{d\rho(\lambda)}$$

for any two functions $u(\lambda)$ and $v(\lambda)$ integrable $H(\rho)$ in $(-\infty, \infty)$.[*] Then, for the differential system in question,

$$\sum_m \left\{ \int_{-\infty}^{\infty} \frac{d\rho_m(\lambda)\,dg(\lambda)}{d\rho(\lambda)} \right\}^2 = \int_{-\infty}^{\infty} \frac{[dg(\lambda)]^2}{d\rho(\lambda)}$$

for every $\Delta g(\lambda)$ integrable $H(\rho)$, therefore the condition of Definition III is satisfied, and the differential system is limited.

Definition IV. The matrix B whose coefficients are

$$(24)\qquad b_{mn} = \int_{-\infty}^{\infty} f(\lambda)\, \frac{du_m(\lambda)\,d\rho_n(\lambda)}{d\rho(\lambda)}$$

is the matrix corresponding to the differential system $\int_\Delta f(\lambda)\,du_m(\lambda)$.

From equation (14) of § 1 we obtain the result that *for every sequence* $\{y_m\}$ *of finite norm there exists a sequence* $\{c_a\}$ *of finite norm, and a function* $\Delta g(\lambda)$ *integrable* $H(\rho)$ *in* $(-\infty, \infty)$ *such that*

$$(25)\qquad y_m = \sum_a l_{am} c_a + \int_{-\infty}^{\infty} \frac{d\rho_m(\lambda)\,dg(\lambda)}{d\rho(\lambda)}$$

For functions $\Delta g(\lambda)$ integrable $H(\rho)$ in $(-\infty, \infty)$ we have a similar result which we prove as the theorem

THEOREM II. *For every function* $\Delta g(\lambda)$ *integrable* $H(\rho)$ *in* $(-\infty, \infty)$ *there exists a sequence* $\{y_n\}$ *of finite norm such that*

$$\Delta g(\lambda) = \sum_n \Delta\rho_n(\lambda)\, y_n.$$

The proof follows immediately from an expansion given by Hellinger,[†]

$$g(\lambda) = \sum_n \rho_n(\lambda) \int_{-\infty}^{\infty} \frac{dg(\lambda)\,d\rho_n(\lambda)}{d\rho(\lambda)}$$

---

[*] Weyl, *l. c.*, *Mathematische Annalen*, Vol. 66, p. 289.

[†] Hellinger, *l. c.*, p. 248. The proof here given is for a finite interval, but it may be extended to include the present case.

which holds for any function $\Delta g(\lambda)$ integrable $H(\rho)$ in $(-\infty, \infty)$. The sequence

$$y_n = \int_{-\infty}^{\infty} \frac{dg(\lambda)\, d\rho_n(\lambda)}{d\rho(\lambda)}$$

is of finite norm by equation (23), and therefore the theorem is established.

We now show that the properties of the differential system may be derived from those of its corresponding matrix by means of the following theorem

THEOREM III. *If the differential system* $\int_{\Delta} f(\lambda)\, du_m(\lambda)$ *is limited the corresponding matrix is limited, and conversely.*

We consider the sequence $\{\sum\limits_{n} b_{mn} y_n\}$ and prove that it is of finite norm for every sequence $\{y_n\}$ of finite norm. Expand the sequence $\{y_n\}$ in the form (25), then

$$\sum_n b_{mn} y_n = \sum_n \int_{-\infty}^{\infty} f(\lambda) \frac{du_m(\lambda)\, d\rho_n(\lambda)}{d\rho(\lambda)} \left[\sum_a l_{an} c_a + \int_{-\infty}^{\infty} \frac{dg(\lambda)\, d\rho_n(\lambda)}{d\rho(\lambda)} \right]$$

and therefore, by equations (9) and (23)

$$\sum_n b_{mn} y_n = \int_{-\infty}^{\infty} f(\lambda) \frac{du_m(\lambda)\, dg(\lambda)}{d\rho(\lambda)} .$$

The function $\Delta g(\lambda)$ is integrable $H(\rho)$, therefore by the definition of a limited differential system the sequence $\{\sum b_{mn} y_n\}$ is of finite norm for every sequence $\{y_n\}$ of finite norm, which proves that the matrix B is limited.*

We now assume that the matrix B is limited and prove that the differential system is also limited. Let $\Delta g(\lambda)$ be any function integrable $H(\rho)$ in $(-\infty, \infty)$. By equation (23) we have

$$\int_{-\infty}^{\infty} f(\lambda) \frac{du_m(\lambda)\, dg(\lambda)}{d\rho(\lambda)} = \sum_n \int_{-\infty}^{\infty} f(\lambda) \frac{du_m(\lambda)\, d\rho_n(\lambda)}{d\rho(\lambda)} \int_{-\infty}^{\infty} \frac{dg(\lambda)\, d\rho_n(\lambda)}{d\rho(\lambda)}$$

and this integral exists for every function $\Delta g(\lambda)$. Hence

$$\int_{-\infty}^{\infty} f(\lambda) \frac{du_m(\lambda)\, dg(\lambda)}{d\rho(\lambda)} = \sum_n b_{mn} y_n$$

---

* Hellinger and Toeplitz, "Grundlagen für eine Theorie der unendlichen Matrizen," *Mathematische Annalen*, Vol. 69, pp. 321-324.

where the sequence $\{y_n\}$ is of finite norm. The sequence $\{\sum b_{mn}y_n\}$ is of finite norm $\leqq M$, since B is limited, and it follows that the differential system is also limited.

THEOREM IV. *If the differential system* $\int_\Delta f(\lambda)\,du_m(\lambda)$ *is limited, the function* $\int_\Delta f(\lambda)\,d\sum x_m u_m(\lambda)$ *is integrable* $H(\rho)$ *in* $(-\infty, \infty)$ *for every sequence* $\{x_m\}$ *of finite norm.*

The differential system is limited and therefore the corresponding matrix B is limited. Also, for any sequence $\{x_m\}$ of finite norm, the function $\int_\Delta f(\lambda)\,d\sum_{m=1}^{N} x_m u_m(\lambda)$ is integrable $H(\rho)$ in $(-\infty, \infty)$, where N has any finite value. By Theorem II there exists a sequence $\{c_p^N\}$ of finite norm such that

$$\int_\Delta f(\lambda)\,d\sum_{m=1}^{N} x_m u_m(\lambda) = \Delta \sum_p c_p^N \rho_p(\lambda)$$

where

$$c_p^N = \int_{-\infty}^{\infty} f(\lambda)\,\frac{d\sum_{m=1}^{N} x_m u_m(\lambda)\,d\rho_p(\lambda)}{d\rho(\lambda)}.$$

By the definition of the matrix B this gives

$$\begin{aligned}\sum_p (c_p^N)^2 &= \sum_p \Big(\sum_{m=1}^{N} x_m b_{mp}\Big)^2 \\ &\leqq M\sum_{m=1}^{N} x_m^2 \\ &\leqq M\sum_{m=1}^{\infty} x_m^2\end{aligned}$$

since B is limited, and hence the series $\sum_p c_p^N \rho_p(\lambda)$ converges uniformly for every finite value of N. Further

$$\lim_{N\to\infty} c_p^N = c_p = \sum_{m=1}^{\infty} x_m b_{mp}$$

and this sequence is of finite norm since B is limited. It follows that

$$\begin{aligned}\lim_{N\to\infty}\int_\Delta f(\lambda)\,d\sum_{m=1}^{N} x_m u_m(\lambda) &= \lim_{N\to\infty}\sum_p c_p^N \rho_p(\lambda) \\ &= \sum_p c_p\rho_p(\lambda)\end{aligned}$$

and this is sufficient to show that the function $\int_{\Delta} f(\lambda)\, d \sum_m x_m u_m(\lambda)$ is integrable $H(\rho)$ in $(-\infty, \infty)$.

By the aid of this theorem we now prove the following important theorem

THEOREM V. *If* $\int_{\Delta} f(\lambda) du_m(\lambda)$ *is a limited differential system in* $(-\infty, \infty)$

$$(26) \qquad \sum_m x_m \int_{-\infty}^{\infty} f(\lambda)\, \frac{du_m(\lambda)\, dg(\lambda)}{d\rho(\lambda)} = \int_{-\infty}^{\infty} f(\lambda)\, \frac{d \sum_m x_m u_m(\lambda)\, dg(\lambda)}{d\rho(\lambda)}$$

for every sequence $\{x_m\}$ *of finite norm, and every function* $\Delta g(\lambda)$ *integrable* $H(\rho)$ in $(-\infty, \infty)$.

Write

$$U(x, \lambda) = \sum_m x_m u_m(\lambda)$$

and consider a finite interval $(-l, l)$ of the real λ axis. Divide this interval into N parts, $\Delta_1 \cdots \Delta_N$, and let λ_i be any point of the interval Δ_i. Then for any sequence $\{x_m\}$ of finite norm, and any function $\Delta g(\lambda)$ integrable $H(\rho)$

$$(27) \qquad \sum_{i=1}^{N} f(\lambda_i)\, \frac{\Delta_i U(x, \lambda) \Delta_i g(\lambda)}{\Delta_i \rho(\lambda)} = \sum_m x_m \sum_{i=1}^{N} f(\lambda_i)\, \frac{\Delta_i u_m(\lambda) \Delta_i g(\lambda)}{\Delta_i \rho(\lambda)}$$

and this is a limited linear form by Theorem IV. The coefficients

$$\sum_{i=1}^{N} f(\lambda_i)\, \frac{\Delta_i u_m(\lambda) \Delta_i g(\lambda)}{\Delta_i \rho(\lambda)}$$

form a sequence of finite norm, and

$$\lim_{N \to \infty} \sum_{i=1}^{N} f(\lambda_i)\, \frac{\Delta_i u_m(\lambda) \Delta_i g(\lambda)}{\Delta_i \rho(\lambda)} = \int_{-l}^{l} f(\lambda)\, \frac{du_m(\lambda)\, dg(\lambda)}{d\rho(\lambda)}$$

exists for every value of m. Then for any fixed system of values of the sequence $\{x_m\}$ we may take the limit of the right hand member of equation (27) term by term, and thus obtain the equation

$$(28) \qquad \int_{-l}^{l} f(\lambda)\, \frac{dU(x, \lambda)\, dg(\lambda)}{d\rho(\lambda)} = \sum_m x_m \int_{-l}^{l} f(\lambda)\, \frac{du_m(\lambda)\, dg(\lambda)}{d\rho(\lambda)}$$

for any finite interval $(-l, l)$. Further, by Theorem IV, the integral

$$\lim_{l \to \infty} \int_{-l}^{l} f(\lambda)\, \frac{dU(x, \lambda)\, dg(\lambda)}{d\rho(\lambda)}$$

converges, and by Definition III the sequence

$$\left\{ \int_{-l}^{l} f(\lambda) \frac{du_m(\lambda)\,dg(\lambda)}{d\rho(\lambda)} \right\}$$

is of finite norm and norm $< M \int_{-\infty}^{\infty} [f(\lambda)]^2 \frac{[dg(\lambda)]^2}{d\rho(\lambda)}$.

The right hand member of equation (28) is therefore uniformly convergent, and again we may take the limit term by term to obtain equation (26).

We now apply the preceding theory to prove the following theorem which is required in the next section:

THEOREM VI. *If the matrix K is limited, the differential system $\Delta \sum_n k_{mn}\rho_n(\lambda)$ is limited in $(-\infty, \infty)$, and the differential system $\int_\Delta (1/\lambda)\, d \sum_n k_{mn}\rho_n(\lambda)$ is limited in $(-\infty, \infty)$ when we exclude an interval $(-\delta, \delta)$ containing the origin.*

By Theorem I the differential system $\Delta\rho_m(\lambda)$ is limited. Further the series $\sum k_{mn}^2$ converges for every value of m since the matrix K is limited, and therefore, by Theorem V,

$$\int_{-\infty}^{\infty} \frac{d \sum_n k_{mn}\rho_n(\lambda)\,dg(\lambda)}{d\rho(\lambda)} = \sum_n k_{mn} \int_{-\infty}^{\infty} \frac{d\rho_n(\lambda)\,dg(\lambda)}{d\rho(\lambda)} \tag{29}$$

for every value of m, and every function $\Delta g(\lambda)$ integrable $H(\rho)$. Write

$$y_n = \int_{-\infty}^{\infty} \frac{dg(\lambda)\,d\rho_n(\lambda)}{d\rho(\lambda)},$$

then, by Theorem II such a sequence $\{y_n\}$ exists and is of finite norm for every function $\Delta g(\lambda)$ integrable $H(\rho)$. Further, the sequence $\{\sum k_{mn}y_n\}$ on the right hand side of equation (29) is of finite norm, and it follows that the integral on the left hand side of equation (29) is also of finite norm for every function $\Delta g(\lambda)$ integrable $H(\rho)$, and therefore finally the differential system is limited.

We now consider the differential system $\int_\Delta (1/\lambda)\, d \sum_n k_{mn}\rho_n(\lambda)$ and prove that it is limited in the intervals $(-\infty, -\delta)$ and (δ, ∞), that is, that the sequence

$$\left\{ \int_{-\infty}^{-\delta} (1/\lambda)\, d \sum_n k_{mn}\rho_n(\lambda)\,dg(\lambda)/d\rho(\lambda) + \int_{\delta}^{\infty} (1/\lambda)\, d \sum k_{mn}\rho_n(\lambda)\,dg(\lambda)/d\rho(\lambda) \right\} \tag{30}$$

is of finite norm for every function $\Delta g(\lambda)$ integrable $H(\rho)$. Now if $\Delta g(\lambda)$ is integrable $H(\rho)$ in $(-\infty, \infty)$ the function $\int_\Delta (1/\lambda)\, dg(\lambda)$ is also integrable $H(\rho)$ in $(-\infty, -\delta)$ and (δ, ∞). Hence if we write

$$\Delta h(\lambda) = \int_\Delta (1/\lambda)\, dg(\lambda)$$

the expression (30) becomes

$$\left\{ \int_{-\infty}^{-\delta} \frac{d \sum\limits_n k_{mn}\rho_n(\lambda)\, dh(\lambda)}{d\rho(\lambda)} + \int_{\delta}^{\infty} \frac{d \sum\limits_n k_{mn}\rho_n(\lambda)\, dh(\lambda)}{d\rho(\lambda)} \right\}$$

which is of finite norm, since we have just proved that the differential system $\Delta \sum k_{mn}\rho_n(\lambda)$ is limited, and the theorem is proved.

A similar method may be used to show that the differential system $\int_\Delta (1/|\lambda|^{1/2})\, d \sum\limits_n k_{mn}\rho_n(\lambda)$ is limited in the intervals $(-\infty, -\delta)$ and (δ, ∞).

4. *Expansion Theorems.* We first derive an expansion theorem for $\int_0^\pi f(x) g(x)\, dx$ in terms of the solutions $u_a(x)$ and $u(x, \lambda)$ of § 2, where f and g are any two functions of integrable square in $(0, \pi)$. We start from the known expansion in terms of the complete normed orthogonal system, $\phi_n(x)$, of equation (3),

$$\int_0^\pi f(x) g(x)\, dx = \sum_m \int_0^\pi f\phi_m \int_0^\pi g\phi_m.$$

Now the sequences $\left\{ \int_0^\pi f\phi_m \right\}$ and $\left\{ \int_0^\pi g\phi_m \right\}$ are of finite norm, therefore by equation (15)

$$\int_0^\pi fg = \sum_a \sum_m l_{am} \int_0^\pi f\phi_m \sum_n l_{an} \int_0^\pi g\phi_n$$

$$+ \int_{-\infty}^{\infty} \frac{d \sum\limits_m \rho_m(\lambda) \int_0^\pi f\phi_m \; d \sum\limits_n \rho_n(\lambda) \int_0^\pi g\phi_n}{d\rho(\lambda)}$$

We now substitute for l_{am} and $\rho_m(\lambda)$ from equations (19) and (16) respectively. This last expansion then becomes

$$\int_0^\pi f(x) g(x)\, dx = \sum_a \int_0^\pi f(x) u_a(x)\, dx \int_0^\pi g(x) u_a(x)\, dx \tag{31}$$

$$+ \int_{-\infty}^{\infty} \frac{d \int_0^\pi f(x) u(x,\lambda)\, dx \; d \int_0^\pi g(x) u(x,\lambda)\, dx}{d\rho(\lambda)},$$

and this is the desired form.

To find a similar expansion for an arbitrary function $f(x)$, subject to suitably imposed conditions we prove some properties of the differential systems that occur in the expansion.

LEMMA I. *The differential system*

$$\int_{\Delta} (1/\lambda) \cdot d \int_0^{\pi} (pu')'\phi_m$$

is limited in $(-\infty, \infty)$ *if we exclude an interval* $(-\delta, \delta)$ *containing the origin.*

We return to the system of equations (5), and let $\Delta = (\delta, \lambda)$ be an interval of the positive real λ axis. $[\lambda > \delta > 0]$. Then, by equation (18)

$$\int_0^{\pi} (pu')'\phi_m + \int_0^{\pi} qu\phi_m + \int_{\delta}^{\lambda} \lambda d\rho_m(\lambda) = 0. \tag{32}$$

We invert the last integral by means of a theorem of Hellinger,[*] namely, that if $F(\lambda) = \int_a^{\lambda} u(\lambda) df(\lambda)$ where $u(\lambda)$ is positive and $f(\lambda)$ is of bounded variation in (a, λ), then

$$f(\lambda) - f(a) = \int_a^{\lambda} \frac{dF(\lambda)}{u(\lambda)} .$$

If we write

$$-F(\lambda) = \int_0^{\pi} (pu')'\phi_m + \int_0^{\pi} qu\phi_m$$

we have

$$\rho_m(\lambda) = \int_{\delta}^{\lambda} \frac{dF(\lambda)}{\lambda} ,$$

or

$$\int_{\Delta} (1/\lambda)\, d \int_0^{\pi} (pu')'\phi_m + \int_{\Delta} (1/\lambda)\, d \int_0^{\pi} qu\phi_m + \Delta\rho_m(\lambda) = 0. \tag{33}$$

This theorem of Hellinger is also applicable if the function $u(\lambda)$ is constantly negative in the interval of integration, hence equation (33) holds for any interval Δ of the real λ axis, outside the interval $(-\delta, \delta)$.

The differential system $\Delta\rho_m(\lambda)$ is limited, by Theorem I. Also

$$\int_{\Delta} (1/\lambda)\, d \int_0^{\pi} qu\phi_m = \int_{\Delta} (1/\lambda)\, d \sum_n q_{mn}\rho_n(\lambda)$$

where Q is the limited matrix defined in § 1, and therefore, by Theorem VI, the differential system $\int_{\Delta} (1/\lambda)\, d \int_0^{\pi} qu\phi_m$ is limited in the intervals

---

[*] Hellinger, *l. c.*, p. 236.

$(-\infty, -\delta)$ and (δ, ∞). It follows immediately from equation (33) that the differential system $\int_\Delta (1/\lambda)\, d \int_0^\pi (pu')'\phi_m$ is limited in the intervals $(-\infty, -\delta)$ and (δ, ∞).

For the characteristic solutions $u_a(x)$ belonging to the characteristic numbers λ_a we have the corresponding result that *the matrix*

$$(1/\lambda_a) \int_0^\pi (pu_a')'\phi_m$$

is limited.[*]

From this result we wish to show that the differential system $\int_\Delta (m/|\lambda|^{1/2})\, d \int_0^\pi pu\phi_m$ is limited in the intervals $(-\infty, -\delta)$ and (δ, ∞), but we first require the following Lemma:

LEMMA II. *The differential system*

$$\int_\Delta (1/m\,|\lambda|^{1/2})\, d \int_0^\pi (pu')'\phi_m \tag{34}$$

is limited in the intervals $(-\infty, -\delta)$ *and* (δ, ∞).

We prove this by introducing corresponding matrices [Definition IV, § 3]. Write

$$v_m(\lambda) = \int_0^\pi (pu')'\phi_m,$$

then the matrix A,

$$a_{mn} = \int_{-\infty}^{\infty} \frac{1}{\lambda} \frac{dv_m(\lambda)\, d\rho_n(\lambda)}{d\rho(\lambda)}$$

is the limited matrix corresponding to the limited differential system $\int_\Delta (1/\lambda)\, dv_m(\lambda)$ of Lemma I (Theorem III). The matrix B defined by the relation

$$b_{mn} = \int_{-\infty}^{\infty} \frac{1}{m\,|\lambda|^{1/2}} \frac{dv_m(\lambda)\, d\rho_n(\lambda)}{d\rho(\lambda)}$$

exists and is of finite norm for every value of m, and we wish to prove that it is limited. Consider the iterated matrix C

---

[*] The results for the characteristic solutions which we state in this section have all been proved for the non-singular equation $u'' + qu + \lambda u = 0$, by Anna Pell Wheeler, *l. c.*, *American Journal of Mathematics*, Vol. 49 (1927), pp. 317-318.

$$c_{mn} = \sum_p b_{mp} b_{np}$$

$$= \sum_p \int_{-\infty}^{\infty} \frac{1}{m\,|\,\lambda\,|^{1/2}} \; \frac{dv_m(\lambda)\,d\rho_p(\lambda)}{d\rho(\lambda)} \int_{-\infty}^{\infty} \frac{1}{n\,|\,\lambda\,|^{1/2}} \; \frac{dv_n(\lambda)\,d\rho_p(\lambda)}{d\rho(\lambda)} .$$

By applying equation (23) we find

$$c_{mn} = \int_{-\infty}^{\infty} \frac{dv_m(\lambda)\,dv_n(\lambda)}{mn\,|\,\lambda\,|\,d\rho(\lambda)} ,$$

and by a rearrangement of terms this becomes

$$c_{mn} = \int_{-\infty}^{\infty} \frac{\dfrac{dv_m(\lambda)}{|\,\lambda\,|} \; \dfrac{dv_n(\lambda)}{n^2} \; \dfrac{n}{m}}{d\rho(\lambda)} .$$

Now consider the matrix D,

$$d_{mn} = \int_{-\infty}^{\infty} \frac{\dfrac{dv_m(\lambda)}{|\,\lambda\,|} \; \dfrac{dv_n(\lambda)}{n^2}}{d\rho(\lambda)}$$

$$= \sum_p \int_{-\infty}^{\infty} \frac{dv_m(\lambda)\,d\rho_p(\lambda)}{|\,\lambda\,|\,d\rho(\lambda)} \int_{-\infty}^{\infty} \frac{dv_n(\lambda)\;d\rho_p(\lambda)}{n^2\,d\rho(\lambda)} .$$

By Lemma I the differential system $\int_{\Delta} (1/|\,\lambda\,|)\; dv_m(\lambda)$ is limited, and

$$v_n(\lambda)/n^2 = (1/n^2) \int_0^{\pi} (pu')'\phi_m$$

$$= \int_0^{\pi} pu \sin nx - (1/n) \int_0^{\pi} p'u \cos nx$$

$$= \sum_p r_{np}\rho_p(\lambda),$$

where R is the limited matrix

$$r_{np} = \int_0^{\pi} p(x) \sin nx \sin px - (1/n) \int_0^{\pi} p'(x) \cos nx \sin px.$$

Hence by Theorem VI $\Delta v_n(\lambda)/n^2$ is a limited differential system. It follows that D is the product of two limited matrices and is therefore limited. Further the matrix C is symmetric, and has been expressed in the form

$$c_{mn} = d_{mn} \cdot (n/m)$$

where D is a limited matrix. Hence C is a limited matrix,* and therefore finally B is a limited matrix, which proves the Lemma.

---

* Cf. A. J. Pell, *l. c.*, *Transactions of the American Mathematical Society*, Vol. 20, p. 35.

We now deduce the required result, namely,

LEMMA III. *The differential system*

$$(m/|\lambda|^{1/2})\, d \int_0^\pi pu\,\phi_m$$

is limited in the intervals $(-\infty, -\delta)$ *and* (δ, ∞).

By the preceding Lemma the differential system (34) is limited. Also

$$\int_\Delta (1/m\,|\lambda|^{1/2})\, d \int_0^\pi (pu')'\phi_m = \int_\Delta (m/|\lambda|^{1/2})\, d \int_0^\pi pu \sin mx$$
$$-\int_\Delta (1/|\lambda|^{1/2})\, d \int_0^\pi p'u \cos mx$$

where $\int_\Delta (1/|\lambda|^{1/2})\, d \int_0^\pi p'u \cos mx$ is a limited differential system since it may be written in the form $\int_\Delta (1/|\lambda|^{1/2})\, d \sum_n t_{mn}\rho_n(\lambda)$ where T is a limited matrix (Theorem VI). We conclude immediately that the differential system $\int_\Delta (1/|\lambda|^{1/2})\, m\, d \int_0^\pi pu\,\phi_m$ is limited in the intervals $(-\infty, -\delta)$ and (δ, ∞).

The corresponding result for the point spectrum is that *the matrix*

$$(m/|\lambda_a|^{1/2}) \int_0^\pi pu_a\phi_m$$

is limited.

We now determine the conditions which must be imposed on an arbitrary function $f(x)$ in order that the function $\int_\Delta |\lambda|^{1/2}\, d \int_0^\pi f(x)\, u(x, \lambda)$ be integrable $H(\rho)$ in $(-\infty, -\delta)$ and (δ, ∞). Return to the equation (33) and apply the theorem given by Carleman[*], that if $f(\lambda)$ and $w(\lambda)$ are continuous functions in the interval (δ, λ) and $\alpha(\lambda)$ a function of bounded variation in that interval, then

$$\int_\delta^\lambda f(\lambda)\, d \int_\delta^\lambda \omega(\lambda) d\alpha(\lambda) = \int_\delta^\lambda f(\lambda)\omega(\lambda) d\alpha(\lambda).$$

For the positive interval (δ, λ) this gives

---

[*] Carleman, *l. c.*, p. 11.

$$-\int_{\delta}^{\lambda} (\lambda)^{1/2}\, d\rho_m(\lambda) = \int_{\delta}^{\lambda} (\lambda)^{1/2}\, d \int_{\delta}^{\lambda} (1/\lambda)\, d \int_{0}^{\pi} (pu')'\phi_m$$

$$+ \int_{\delta}^{\lambda} (\lambda)^{1/2}\, d \int_{\delta}^{\lambda} (1/\lambda)\, d \int_{0}^{\pi} qu\phi_m$$

$$= \int_{\delta}^{\lambda} (1/(\lambda)^{1/2})\, d \int_{0}^{\pi} (pu')'\phi_m + \int_{\delta}^{\lambda} (1/(\lambda)^{1/2})\, d \int_{0}^{\pi} qu\phi_m.$$

For the negative interval $(-\delta, -\lambda)$ we obtain similarly

$$-\int_{\delta}^{\lambda} |\lambda|^{1/2}\, d\rho_m(\lambda) = \int_{\delta}^{\lambda} (\lambda/|\lambda|)\, (1/|\lambda|^{1/2})\, d \int_{0}^{\pi} (pu')'\phi_m$$

$$+ \int_{\delta}^{\lambda} (\lambda/|\lambda|)\, (1/|\lambda|^{1/2})\, d \int_{0}^{\pi} qu\phi_m,$$

so that for any interval Δ outside $(-\delta, \delta)$ we have

$$\int_{\Delta} [1/m(|\lambda|)^{1/2}]\, d \int_{0}^{\pi} (pu')'\phi_m + \int_{\Delta} [1/m(|\lambda|)^{1/2}]\, d \int_{0}^{\pi} qu\phi_m$$

$$+ \int_{\Delta} (\lambda/m\,|\lambda|)\, |\lambda|^{1/2}\, d\rho_m(\lambda) = 0.$$

Now let $f(x)$ be a function of x such that the sequence $\{m \int_{0}^{\pi} f\phi_m\}$ is of finite norm, multiply this last equation by $m \int_{0}^{\pi} f\phi_m$ and sum for m

$$\sum_{m} m \int_{0}^{\pi} f\phi_m \int_{\Delta} [1/m(|\lambda|)^{1/2}]\, d \int_{0}^{\pi} (pu')'\phi_m$$

$$+ \sum_{m} m \int_{0}^{\pi} f\phi_m \int_{\Delta} [1/m(|\lambda|)^{1/2}]\, d \int_{0}^{\pi} qu\phi_m$$

$$+ \sum_{m} m \int_{0}^{\pi} f\phi_m \int_{\Delta} (\lambda/|\lambda|)\, (|\lambda|^{1/2}/m)\, d\rho_m(\lambda) = 0.$$

In each of the first two terms of this equation we have the product of a limited differential system and a sequence of finite norm, and they therefore represent functions which are integrable $H(\rho)$ in the intervals $(-\infty, -\delta)$ and (δ, ∞) (Theorem IV). It follows that *the function*

$$(\lambda/|\lambda|)\, |\lambda|^{1/2}\, d \int_{0}^{\pi} f(x) u(x, \lambda)\, dx$$

is integrable $H(\rho)$ *in* $(-\infty, -\delta)$ *and* (δ, ∞) *for every function* $f(x)$ *such that the sequence* $\{m \int_{0}^{\pi} f\phi_m\}$ *is of finite norm.*

For the point spectrum we have the similar result that *the sequence* $\{ | \lambda_a |^{1/2} \int_0^\pi f(x) u_a(x) \}$ *is of finite norm for every function* $f(x)$ *such that the sequence* $\{m \int_0^\pi f\phi_m\}$ *is of finite norm.*

An expansion theorem for the function $p(x)f(x)$, where f is any function satisfying the above condition, may now be obtained, starting from the known expansion

$$p(x)f(x) = \sum_m \phi_m(x) \int_0^\pi p(x)f(x)\phi_m(x)\,dx.$$

We divide the real λ axis into three intervals $(-\infty, -\delta)$, $(-\delta, \delta)$ and (δ, ∞), and denote the characteristic values of λ belonging to these intervals by λ_{a_1}, λ_{a_2} and λ_{a_3}, respectively. Then by the expansion (31)

$$\begin{aligned} p(x)f(x) = {} & \sum_m \phi_m(x) \sum_{a_1} \int_0^\pi p\phi_m u_{a_1} \int_0^\pi f u_{a_1} \\ & \qquad + \sum_m \phi_m(x) \int_{-\infty}^{-\delta} \frac{d\int_0^\pi p\phi_m u \, d\int_0^\pi fu}{d\rho(\lambda)} \\ & + \sum_m \phi_m(x) \sum_{a_2} \int_0^\pi p\phi_m u_{a_2} \int_0^\pi f u_{a_2} \\ & \qquad + \sum_m \phi_m(x) \int_{-\delta}^{\delta} \frac{d\int_0^\pi p\phi_m u \, d\int_0^\pi fu}{d\rho(\lambda)} \\ & + \sum_m \phi_m(x) \sum_{a_3} \int_0^\pi p\phi_m u_{a_3} \int_0^\pi f u_{a_3} \\ & \qquad + \sum_m \phi_m(x) \int_{\delta}^{\infty} \frac{d\int_0^\pi p\phi_m u \, d\int_0^\pi fu}{d\rho(\lambda)} \\ = {} & A_1 + A_2 + A_3. \end{aligned}$$

Consider first the positive interval (δ, ∞), the terms in the expansion belonging to this interval being denoted by A_3. Then

$$\begin{aligned} A_3 = {} & \sum_m \frac{\phi_m(x)}{m} \sum_{a_3} \frac{m\int_0^\pi p u_{a_3}\phi_m}{(\lambda_{a_3})^{1/2}} (\lambda_{a_3})^{1/2} \int_0^\pi f u_{a_3} \\ & + \sum_m \frac{\phi_m(x)}{m} \int_\delta^\infty \frac{\dfrac{m\,d\int_0^\pi pu\phi_m}{(\lambda)^{1/2}} (\lambda)^{1/2}\, d\int_0^\pi fu}{d\rho(\lambda)} \end{aligned}$$

and, by the properties of the differential system $\int_\Delta (1/\lambda^{1/2})\, m\, d \int_0^\pi pu\phi_m$ and of the matrix $(1/(\lambda_{a_3})^{1/2})\, m \int_0^\pi pu_{a_3}\phi_m$ (Lemma III and Theorem V) we see that this may be written

$$(35)\qquad A_3 = \sum_{a_3} p(x) u_{a_3}(x) \int_0^\pi f u_{a_3} + \int_\delta^\infty \frac{p(x)\, du(x,\lambda)\, d\int_0^\pi f(x) u(x,\lambda)\, dx}{d\rho(\lambda)}.$$

For the negative interval $(-\infty, -\delta)$ we find the corresponding expansion

$$(36)\qquad A_1 = \sum_{a_1} p(x) u_{a_1}(x) \int_0^\pi f u_{a_1} + \int_{-\infty}^{-\delta} \frac{p(x)\, du(x,\lambda)\, d\int_0^\pi f(x) u(x,\lambda)\, dx}{d\rho(\lambda)}$$

by the same method.

We now consider the interval $(-\delta, \delta)$ in which the method of proof has to be modified since the differential system $\int_\Delta (1/|\lambda|^{1/2})\, md \int_0^\pi pu\phi_m$ is not limited in this interval. The proof is obtained from the following Lemmas.

LEMMA IV. *The differential system* $m \int_0^\pi pu\phi_m$ *is limited in the interval* $(-\delta, \delta)$.

We return to equation (18) satisfied by $u(x,\lambda)$, and write it in the form

$$\Delta \int_0^\pi (p\phi_m')' u(x,\lambda) + \Delta \int_0^\pi q\phi_m u(x,\lambda) + \int_0^\pi \phi_m \int_\Delta \lambda du(x,\lambda) = 0.$$

In this we substitute the value of $\phi_m(x)$, and obtain the equation

$$-m\Delta \int_0^\pi pu \sin mx + \Delta \int_0^\pi p'u \cos mx + (1/m)\Delta \int_0^\pi qu \sin mx + (1/m) \int_0^\pi \sin mx \int_\Delta \lambda\, du(x,\lambda) = 0.$$

The differential systems $\Delta \int_0^\pi p'u \cos mx$ and $(1/m)\, \Delta \int_0^\pi qu \sin mx$ are limited in $(-\delta, \delta)$. Also

$$(1/m) \int_0^\pi \sin mx \int_\Delta \lambda \, du(x, \lambda) = (1/m) \int_\Delta \lambda \, d\rho_m(\lambda)$$

and this differential system is limited in $(-\delta, \delta)$. It follows immediately that the differential system $m\, \Delta \int_0^\pi pu \sin mx$ is limited in $(-\delta, \delta)$.

LEMMA V. *The function* $\Delta \int_0^\pi f(x) u(x, \lambda) dx$ *is integrable* $H(\rho)$ *in the interval* $(-\delta, \delta)$ *for every function* $f(x)$ *which is of integrable square in* $(0, \pi)$. For

$$\Delta \int_0^\pi f(x) u(x, \lambda) = \Delta \sum_m x_m \rho_m(\lambda),$$

where $\{x_m\}$ is a sequence of finite norm, and $\Delta\rho_m(\lambda)$ is a limited differential system. The truth of the Lemma follows immediately by Theorem IV.

For the characteristic values λ_{a_2} we have similarly that *the matrix* $m \int_0^\pi u_{a_2} \phi_m$ *is limited,* and that *the sequence* $\left[\int_0^\pi f u_{a_2} \right]$ *is of finite norm for every function* $f(x)$ *which is of integrable square in* $(0, \pi)$.

In this interval we have then

$$A_2 = \sum_m \frac{\phi_m(x)}{m} \sum_{a_2} m \int_0^\pi p u_{a_2} \phi_m \int_0^\pi f u_{a_2}$$

$$+ \sum_m \frac{\phi_m(x)}{m} \int_{-\delta}^{\delta} \frac{m d \int_0^\pi p u \phi_m \, d \int_0^\pi f u}{d\rho(\lambda)}$$

from which, by the properties we have just proved,

$$A_2 = \sum_{a_2} p(x) u_{a_2}(x) \int_0^\pi f u_{a_2} \tag{37}$$

$$+ \int_{-\delta}^{\delta} \frac{p(x) du(x, \lambda) \, d \int_0^\pi f(x) u(x, \lambda) \, dx}{d\rho(\lambda)}$$

By adding the results (35), (36) and (37) we see that the complete expansion may be expressed as the theorem.

THEOREM VII. *Let $f(x)$ be a function of x such that the sequence $\left[m \int_0^\pi f(x) \sin mx \right]$ is of finite norm, and let $u_a(x)$ and $u(x,\lambda)$ be the functions defined by equations (19) and (17), respectively. Then the function $p(x)f(x)$ may be expanded in the interval $0 \leqq x \leqq \pi$ as the uniformly convergent series and integral*

$$p(x)\,f(x) = \sum_a p(x)u_a(x) \int_0^\pi fu_a + \int_{-\infty}^{\infty} \frac{p(x)\,du(x,\lambda)\; d\int_0^\pi f(x)u(x,\lambda)\,dx}{d\rho(\lambda)} \tag{38}$$

where the summation and integration extend over every part of the real λ axis in which there is a spectrum.

5. *Extension of the preceding theory to a continuous spectrum which is not simple.* We now state briefly the generalizations of the preceding theory which are required when we assume the existence of a number (at most denumerable) of continuous spectra. Let $\rho_m{}^{(\alpha)}(\lambda)$ be the characteristic differential forms with corresponding basis functions $\rho^{(\alpha)}(\lambda) \cdot (\alpha = 1, 2, \cdots)$. The orthogonality relations now are

$$\sum_m l_{am} l_{\beta m} = e_{a\beta},$$

$$\begin{aligned} \sum_m \Delta_1\rho_m{}^{(\alpha)}(\lambda)\;\Delta_2\rho_m{}^{(\beta)}(\lambda) &= 0 & \alpha \neq \beta \\ &= \Delta_{12}\rho^{(\alpha)}(\lambda) & \alpha = \beta, \end{aligned}$$

$$\sum_m l_{am}\rho_m{}^{(\beta)}(\lambda) = 0.$$

For the matrix expansions (12) and (13) we find the more general forms

$$-\sum_n (p_{mn} + q_{mn})y_n = \sum_a \lambda_a Y_a l_{am} + \sum_\alpha \int_{-\infty}^{\infty} \lambda \; \frac{dy^{(\alpha)}(\lambda)\; d\rho_m{}^{(\alpha)}(\lambda)}{d\rho^{(\alpha)}(\lambda)}$$

$$-\sum_{m,n} (p_{mn} + q_{mn})x_m y_n = \sum_a \lambda_a X_a Y_a + \sum_\alpha \int_{-\infty}^{\infty} \lambda \; \frac{dx^{(\alpha)}(\lambda)\; dy^{(\alpha)}(\lambda)}{d\rho^{(\alpha)}(\lambda)}$$

where X and Y have the same values as before, while

$$y^{(\alpha)}(\lambda) = \sum_m \rho_m{}^{(\alpha)}(\lambda) y_m, \qquad x^{(\alpha)}(\lambda) = \sum_m \rho_m{}^{(\alpha)}(\lambda) x_m.$$

From these expansions it follows that the systems $l_{\alpha m}$ and $\rho_m^{(\alpha)}(\lambda)$ are complete, and therefore

$$\sum_\alpha l_{\alpha m} l_{\alpha n} + \sum_\alpha \int_{-\infty}^{\infty} \frac{d\rho_m^{(\alpha)}(\lambda)\, d\rho_n^{(\alpha)}(\lambda)}{d\rho^{(\alpha)}(\lambda)} = e_{mn}.$$

We now have a sequence of differential solutions $u^{(\alpha)}(x, \lambda)$, defined by

$$\rho_m^{(\alpha)}(\lambda) = \int_0^\pi u^{(\alpha)}(x, \lambda) \phi_m(x)\, dx \tag{39}$$

satisfying equation (18) and the other conditions imposed on the solutions $u(x, \lambda)$ of § 2. The orthogonality relations in this case are

$$\int_0^\pi u_\alpha(x) u_\beta(x)\, dx = e_{\alpha\beta},$$

$$\begin{aligned} \int_0^\pi \Delta_1 u^{(\alpha)}(x, \lambda)\ \Delta_2 u^{(\beta)}(x, \lambda)\, dx &= 0 && \alpha \neq \beta \\ &= \Delta_{12}\rho^{(\alpha)}(\lambda) && \alpha = \beta, \end{aligned}$$

$$\int_0^\pi u_\alpha(x) \Delta u^{(\beta)}(x, \lambda) = 0.$$

The definitions of § 3 are modified as follows. We say that the sequence $\left\{ \int_\Delta f(\lambda)\, du^{(\alpha)}(\lambda) \right\}$ is *integrable* $H(\rho^{(\alpha)})$ if the integral

$$\int_{-\infty}^{\infty} [f(\lambda)]^2 \frac{[du^{(\alpha)}(\lambda)]^2}{d\rho^{(\alpha)}(\lambda)}$$

exists for every value of α, and if further the series

$$\sum_\alpha \int_{-\infty}^{\infty} [f(\lambda)]^2 \frac{[du^{(\alpha)}(\lambda)]^2}{d\rho^{(\alpha)}(\lambda)}$$

converges.

The system of functions $\int_\Delta f(\lambda)\, du_m^{(\alpha)}(\lambda)$ is a *limited differential system* if the functions $\int_\Delta f(\lambda)\, du_m^{(\alpha)}(\lambda)$ are integrable $H(\rho^{(\alpha)})$ for every value of m, and if further the sequence

$$\left\{ \int_{-\infty}^{\infty} f(\lambda) \frac{du_m^{(\alpha)}(\lambda)\, dg^{(\alpha)}(\lambda)}{d\rho^{(\alpha)}(\lambda)} \right\}$$

is of finite norm and norm

$$\leq M \sum_{(\alpha)} \int_{-\infty}^{\infty} [f(\lambda)]^2 \frac{[dg^{(\alpha)}(\lambda)]^2}{d\rho^{(\alpha)}(\lambda)}$$

for every value of α, and every $\Delta g^{(\alpha)}(\lambda)$ integrable $H(\rho^{(\alpha)})$.

The *corresponding matrix B* is defined by

$$b_{mn} = \sum_a \int_{-\infty}^{\infty} f(\lambda) \frac{du_m{}^{(a)}(\lambda)\, d\rho_n{}^{(a)}(\lambda)}{d\rho^{(a)}(\lambda)} .$$

With these definitions the theorems of § 3 are still true, and the expansions of § 4 may be summed up as the theorem.

THEOREM VIII. *Let $u_a(x)$ and $u^{(a)}(x, \lambda)$ be the functions defined by equations (19) and (39) respectively. Then for any two functions $f(x)$ and $g(x)$ of integrable square in the interval $(0, \pi)$*

$$\int_0^\pi f(x)g(x)dx = \sum_a \int_0^\pi f(x)u_a(x)dx \int_0^\pi g(x)u_a(x)dx$$
$$+ \sum_a \int_{-\infty}^{\infty} \frac{d\int_0^\pi f(x)u^{(a)}(x,\lambda)dx\, d\int_0^\pi g(x)u^{(a)}(x,\lambda)dx}{d\rho^{(a)}(\lambda)} .$$

Further, if $f(x)$ is a continuous function of x satisfying the boundary conditions (2) or (2′) and having a first derivative which is integrable and of integrable square in the interval $(0, \pi)$, then

$$p(x)f(x) = \sum_a p(x)u_a(x) \int_0^\pi f(x)u_a(x)dx$$
$$+ \sum_a \int_{-\infty}^{\infty} \frac{p(x)du^{(a)}(x,\lambda)\, d\int_0^\pi f(x)u^{(a)}(x,\lambda)dx}{d\rho^{(a)}(\lambda)} ,$$

where the series and integrals involved are uniformly and absolutely convergent.

Note. Instead of equation (1) we might consider the more general equation

(1′) $$(d/dx)[p(x)du/dx] + q(x)u + \lambda k(x)u = 0,$$

where k is continuous and positive in the interval $(0, \pi)$, the boundary conditions being given by equation (2) or (2′). If we assume that the function

$$\frac{1}{(k)^{1/2}} \frac{d}{dx}\left(p(x) \frac{d}{dx} \frac{1}{(k)^{1/2}}\right)$$

exists and is continuous, equation (1′) may be reduced to the form of the original equation (1)*. The solutions of equation (1′) are therefore found from the solutions of the corresponding equation (1).

---

[*] Cf. Hilbert, *l. c.*, p. 51.

In this case we find the general expansion

$$p(x)f(x)=\sum_a p(x)u_a(x)\int_0^\pi kfu_a+\sum_a\int_{-\infty}^{\infty}\frac{p(x)\,du^{(a)}(x,\lambda)\;d\int_0^\pi kfu^{(a)}}{d\rho^{(a)}(\lambda)}$$

where the series and integrals converge uniformly and absolutely for every function $f(x)$ satisfying the conditions previously imposed, namely, that f be continuous, satisfy the boundary conditions, and have a first derivative which is integrable and of integrable square in the interval $(0,\pi)$.

Nets of Conics in the Real Domain.

By Alan D. Campbell.

The nets of conics in the complex domain were reduced to canonical types by C. Jordan,[*] and for the Galois Fields $GF(p^n)$ $p > 2$ by A. H. Wilson.[†] The purpose of this paper is to solve the corresponding problem for the real domain. The nets are reduced to twenty-six independent types.

Denoting the net by

$$\text{(a)} \qquad \lambda C_1 + \mu C_2 + \nu C_3 = 0$$

where

$$C_i = a_i x^2 + b_i y^2 + c_i z^2 + 2f_i yz + 2g_i zx + 2h_i xy$$

$(i = 1, 2, 3)$ we apply linear transformations to x, y, z and λ, μ, ν to reduce the net to a typical form. The reduction to types is aided by considering the cubic curve $\Delta = 0$ in the plane (λ, μ, ν) whose equation is obtained by setting the discriminant of (a) equal to zero. To a line in the (λ, μ, ν) plane corresponds a pencil of the net (a) whose intersections with $\Delta = 0$ give values of λ, μ, ν for which the corresponding conic in (a) is composite. Nets of conics may therefore be divided into two main categories according as they do or do not possess a pencil of composite conics. In the former case $\Delta = 0$ is also composite. In the latter case the net may be reduced to a standard form by taking for $\lambda = 0$ the tangent at a real inflection on $\Delta = 0$, for $\mu = 0$ the harmonic polar of the point of contact of $\lambda = 0$, and for $\nu = 0$ a line through the inflection and tangent to the cubic (at a point on $\mu = 0$) or through a node or cusp (on $\mu = 0$) if the cubic have a node or cusp.

The types of pencils of conics in the real domain are known.[‡] The types or pencils with $\Delta = 0$ are

$$\text{(b)} \quad \lambda x^2 + \mu y^2 = 0 \qquad\qquad \text{(c)} \quad \lambda x^2 + 2\mu xy = 0$$

$$\text{(d)} \quad 2\lambda xy + 2\mu xz = 0 \qquad\qquad \text{(e)} \quad 2\lambda xy + \mu(x^2 - y^2) = 0.$$

---

[*] "Réduction d'un Réseau de Formes Quadratiques," *Journal de Mathématiques*, Vol. 6, Series 2 (1906), pp. 403-438.

[†] "The Canonical Types of Nets of Modular Conics," *American Journal of Mathematics*, Vol. 36 (1914), pp. 187-210.

[‡] L. E. Dickson, "On Families of Quadratic Forms in a General Field," *Quarterly Journal of Mathematics*, Vol. 39, pp. 316-333.

For a net containing the pencil (b) we have the general form

$$\lambda x^2 + \mu y^2 + \nu(cz^2 + 2fyz + 2gzx + 2hxy) = 0.$$

If $c \neq 0$ we can put

$$T: \quad x = x', \quad y = y', \quad z = -(g/c)x' - (f/c)y' + z';$$
$$\lambda = \lambda' + (g^2/c)\nu', \quad \mu = \mu' + (f^2/c)\nu', \quad \nu = \nu',$$

and get (after dropping the primes on the variables) the form

$$\lambda x^2 + \mu y^2 + \nu(cz^2 + 2h'xy) = 0$$

from which we derive the typical nets

(1) $\lambda x^2 + \mu y^2 + \nu z^2 = 0$

(2) $\lambda x^2 + \mu y^2 + 2\nu xy = 0$

(3) $\lambda x^2 + \mu y^2 + \nu(z^2 + 2xy) = 0.$

If $c = 0$ and f and g are not both zero, we can choose a new z and ν so as to get the form

$$\lambda x^2 + \mu y^2 + 2\nu(fyz + gzx) = 0$$

from which we derive the types

(4) $\lambda x^2 + \mu y^2 + 2\nu yz = 0$

(5) $\lambda x^2 + \mu y^2 + 2\nu(yz + zx) = 0.$

For a net containing the pencil (c) we have the general form

$$\lambda x^2 + 2\mu xy + \nu(by^2 + cz^2 + 2fyz + 2gzx) = 0.$$

If $c \neq 0$ we can subject this net to a transformation similar to the above transformation T and get the form

$$\lambda x^2 + 2\mu xy + \nu(b'y^2 + cz^2) = 0$$

and derive the new types

(6) $\lambda x^2 + 2\mu xy + \nu(z^2 - y^2) = 0$

(7) $\lambda x^2 + 2\mu xy + \nu(z^2 + y^2) = 0.$

If $c = 0$ and $f \neq 0$ in the above general form of net we can put

$$x = x', \quad y = -(g/f)x' + y', \quad z = z';$$
$$\lambda = \lambda' + 2(g/f)\mu' + b(g^2/f^2)\nu', \quad \mu = \mu' + b(g/f)\nu', \quad \nu = \nu',$$

and we get the form

$$\lambda x^2 + 2\mu xy + \nu(by^2 + 2fyz) = 0$$

and then a new z gives the type

$$(8) \qquad \lambda x^2 + 2\mu xy + 2\nu yz = 0.$$

If $c = f = 0$ we have the form

$$\lambda x^2 + 2\mu xy + \nu(by^2 + 2gzx) = 0$$

and derive the new types

$$(9) \qquad \lambda x^2 + 2\mu xy + 2\nu zx = 0$$

$$(10) \qquad \lambda x^2 + 2\mu xy + \nu(y^2 + 2xz) = 0.$$

For a net containing the pencil (d) we have the general form

$$2\lambda xy + 2\mu xz + \nu(ax^2 + by^2 + cz^2 + 2fyz) = 0.$$

A binary transformation on y and z and an obvious transformation on λ, μ, ν gives if $a \neq 0$ the new types

$$(11) \qquad 2\lambda xy + 2\mu xz + \nu(x^2 + y^2 + z^2) = 0$$

$$(12) \qquad 2\lambda xy + 2\mu xz + \nu(x^2 - y^2 - z^2) = 0$$

$$(13) \qquad 2\lambda xy + 2\mu xz + \nu(x^2 + 2yz) = 0$$

$$(14) \qquad 2\lambda xy + 2\mu xz + \nu(x^2 - y^2) = 0$$

and if $a = 0$

$$(15) \qquad 2\lambda xy + 2\mu xz + \nu(y^2 + z^2) = 0$$

$$(16) \qquad 2\lambda xy + 2\mu xz + 2\nu yz = 0.$$

For a net containing the pencil (e) we have the general form

$$2\lambda xy + \mu(x^2 - y^2) + \nu(ax^2 + cz^2 + 2fyz + 2gzx) = 0.$$

If $c \neq 0$ a new z and ν give the form

$$2\lambda xy + \mu(x^2 - y^2) + \nu(a'x^2 + cz^2) = 0.$$

Hence we have the new types

$$(17) \qquad 2\lambda xy + \mu(x^2 - y^2) + \nu(z^2 + x^2) = 0$$

$$(18) \qquad 2\lambda xy + \mu(x^2 - y^2) + \nu(z^2 - x^2) = 0$$

$$(19) \qquad 2\lambda xy + \mu(x^2 - y^2) + \nu z^2 = 0.$$

If $c=0$ we have the form

$$2\lambda xy+\mu(x^2-y^2)+\nu(ax^2+2fyz+2gzx)=0.$$

We can put (if $g\neq 0$)*

$$x=x'-(f/g)y', \quad y=y', \quad z=z';$$
$$\lambda=\lambda', \quad \mu=(2fg/f^2-g^2)\lambda'+\mu', \quad \nu=\nu',$$

and we get the form

$$\lambda(2\alpha x'y'+a'x'^2)+\mu(x'^2+b'y'^2+2h'x'y')+\nu(a''x'^2+b''y'^2 \\ +2gz'x'+2h''x'y')=0.$$

If we now put $x'=x''$, $y'=-(a'/\alpha)x''+y''$, $z'=z''$, follow this by a transformation in λ, μ, ν that will rid the new C_2 and C_3 of $x''y''$ and rid C_3 of y''^2, we arrive (after an obvious transformation of the variables) at the form

$$2\lambda xy+\mu(x^2-y^2)+\nu(a'''x^2+2zx)=0$$

which reduces to (14) by choosing a new z.

When the cubic $\Delta=0$ is non-composite it has at least one real inflectional tangent say $\lambda=0$. The harmonic polar of the point of contact of $\lambda=0$ may be taken to be $\mu=0$. If one of the tangents from the point of contact, or the line joining the point of contact to a singular point, is $\nu=0$ the equation of the cubic is now of the form

$$\Delta=\mu^2\lambda-\nu(a\lambda^2+b\lambda\nu+c\nu^2)=0.$$

There is no $\lambda\mu\nu$ term in Δ since every line $\lambda=\alpha\nu$ cuts the cubic and $\mu=0$ in a harmonic set. If $\nu=0$, then $\Delta=\lambda\mu^2$, hence the pencil $\lambda C_1+\mu C_2=0$ must have $\lambda\mu^2$ as discriminant. From Dickson (*loc. cit.*, p. 318) we see that all such pencils belong to classes with typical forms reducible to

$$\lambda x^2+2\mu yz=0, \quad \lambda x^2+\mu(y^2+z^2)=0, \quad \text{or} \quad \lambda(x^2\pm z^2)+2\mu xy=0.$$

On the other hand $\lambda=0$ in Δ gives $\Delta=-c\nu^3$ and there are just two classes of pencils with such a discriminant, namely those with typical forms reducible to

$$\mu x^2+\nu(y^2+2zx)=0 \quad \text{or} \quad 2\mu xy+\nu(y^2+2zx)=0.$$

---

* If $g=0$, $f\neq 0$, we change λ, μ, ν so as to replace x^2 by y^2 in C_3, put $x=y'$, $y=x'$, $z=z'$, and we get the above result. If $g=f=0$, $a\neq 0$ we have a pencil $2\lambda xy+\nu x^2=0$, contrary to hypothesis.

From these considerations we see that a net of conics with the above Δ can be put in one of the three following forms:

$$\lambda x^2 + 2\mu yz + \nu C_3 = 0$$
$$\lambda(x^2 - z^2) + 2\mu xy + \nu C_3 = 0$$
$$\lambda(x^2 + z^2) + 2\mu xy + \nu C_3 = 0.$$

In the first case we have the general form

$$\lambda x^2 + 2\mu yz + \nu(ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy) = 0.$$

Here

$$\Delta \equiv | a, b, c | \nu^3 + 2(gh - af)\mu\nu^2 - a\mu^2\nu + (bc - f^2)\lambda\nu^2 - \lambda\mu^2 - 2f\lambda\mu\nu = 0,$$

so that $a = f = 0$ and either g or h is zero and we have the form

$$\lambda x^2 + 2\mu yz + \nu(by^2 + cz^2 + 2hxy) = 0.$$

We therefore have the new types

(20) $$\lambda x^2 + 2\mu yz + \nu(y^2 + z^2 + 2xy) = 0$$
(21) $$\lambda x^2 + 2\mu yz + \nu(y^2 - z^2 + 2xy) = 0$$
(22) $$\lambda x^2 + 2\mu yx + \nu(z^2 + 2xy) = 0.$$

In the second case we have the general form

$$\lambda(x^2 - z^2) + 2\mu xy + \nu(ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy) = 0$$

whose discriminant is

$$\Delta \equiv | a, b, c | \nu^3 + 2(fg - ch)\mu\nu^2 - c\mu^2\nu$$
$$+ (-ab + bc - f^2 + h^2)\lambda\nu^2 - b\lambda^2\nu + \lambda\mu^2 + 2h\lambda\mu\nu = 0.$$

Hence $h = c = 0$ and either f or g is zero. If $f = 0$ we have the form

$$\lambda(x^2 - z^2) + 2\mu xy + \nu(ax^2 + by^2 + 2gzx) = 0$$

and therefore the new type

(23) $$\lambda(x^2 - z^2) + 2\mu xy + \nu(2\alpha x^2 + y^2 + 2zx) = 0, \quad \alpha \neq 0,$$

where 2α is used instead of α to avoid fractions when solving $\mu = 0$ with $\Delta = 0$.

If $g = 0$ we have similarly the new types

(24) $$\lambda(x^2 - z^2) + 2\mu xy + \nu(\beta x^2 + y^2 + 2yz) = 0, \quad \beta \neq 0,$$

and

(25) $$\lambda(x^2 - z^2) + 2\mu xy + \nu(x^2 + 2yz) = 0,$$

according as b is not or is zero.

In the third case above we have the general form for the net

$$\lambda(x^2+z^2)+2\mu xy+\nu(ax^2+by^2+cz^2+2fyz+2gzx+2hxy)=0$$

whose discriminant is

$$\Delta\equiv|\,a,b,c\,|\,\nu^3+2(fg-ch)\mu\nu^2-c\mu^2\nu \\ +(ab+bc-h^2-f^2)\lambda\nu^2+b\lambda^2\nu-\lambda\mu^2-2h\lambda\mu\nu=0.$$

Hence $h=c=0$ and either f or g is zero. If $f=0$ we have the form

$$\lambda(x^3+z^2)+2\mu xy+\nu(2\alpha x^2y+y^2+2zx)=0$$

with discriminant $\Delta\equiv\mu^2\lambda-\nu(\lambda^2+2\alpha\lambda\nu-\nu^2)$. The pencil $\lambda C_1+\nu C_3=0$ has three distinct degenerate conics. From Dickson we see that every such pencil belongs to one of three classes whose typical pencils are readily reducible to:

$$\lambda(x^2-y^2)+\mu(x^2-z^2)=0,\quad\text{or}\quad\lambda(x^2+y^2)+\mu(x^2-z^2)=0,$$

or $2\lambda xy+\mu(x^2-y^2+z^2)=0$, respectively. Each one of these pencils contains at least one conic that is a pair of real lines. If we take this conic for $C_1=0$ we can transform the net into (23) or (24). If $g=0$ and $b\neq 0$ we have the form

$$\lambda(x^2+z^2)+2\mu xy+\nu(\beta x^2+y^2+2yz)=0$$

with discriminant $\Delta\equiv-\lambda\mu^2+\nu(\lambda-\nu)(\lambda+\beta\nu)$. Here again the pencil $\lambda C_1+\nu C_3=0$ has one pair of real lines, and so the net can be transformed into (23) or (24). Finally if $g=b=0$ we have the new type

$$(26)\qquad \lambda(x^2+z^2)+2\mu xy+\nu(x^2+2yz)=0.$$

We shall now discuss the equivalence or non-equivalence of these twenty-six typical nets. Nets with non-equivalent cubics are not equivalent, but some non-equivalent nets have the same (or equivalent) cubics. The nets (1) to (19) inclusive all have composite cubics and so are not equivalent to the nets (20) through (26). These nets with composite cubics fall into four distinct classes as follows: Class A of nets (1) to (5) inclusive where each net has a pencil (b) in it; Class B of nets (6) to (10) inclusive where each net has no pencil (b) but has a pencil (c); Class C of nets (11) to (16) inclusive where each net has no pencil (b) or (c) but has a pencil (d); Class D of nets (17), (18), (19) where each net has no pencil (b) or (c) or (d) but has a pencil (e). The cubics $\Delta=0$ of the nets in Class A serve to prove the non-equivalence of these nets to one another, except the two nets (4) and (5);

but the pencil $\lambda = 0$ in (4) cannot be transformed into the pencil $\lambda + \mu = 0$ in (5), so these two nets are non-equivalent. In Class B the nets (6) and (7) have equivalent cubics but there is no real transformation that will send (7) into (6) for any such transformation must send the pencil $\lambda x^2 + 2\mu xy = 0$ of (7) into the same pencil of (6) and so must be of the form:

$$T: \quad x = a_1 x', \quad y = a_2 x' + b_2 y', \quad z = a_3 x' + b_3 y' + c_3 z',$$

where $a_1 b_2 c_3 \neq 0$. But T cannot send the conic $z^2 + y^2 = 0$ of (7) into a conic whose y^2 and z^2 terms have opposite signs, such a conic as occurs in (6). The other nets of Class B are distinguished by their cubics. In Class C the nets all have non-equivalent cubics. In Class D the only nets with equivalent cubics are (17) and (18); but in (18) the sum of the coefficients of x^2, y^2, z^2 is identically zero in λ, μ, ν (hence we must have this sum of coefficients in any equivalent net identically zero in λ, μ, ν) also any transformation must send the pencil $2\lambda xy + \mu(x^2 - y^2) = 0$ of (17) into the same pencil of (18). There is no real non-singular transformation that will satisfy all these conditions, hence these nets are non-equivalent.

The nets (20), (21), and (22) have each a double line and so are not equivalent to the nets (23) to (26) inclusive; the cubics of these three nets with double lines have respectively a crunode, an acnode, and a cusp and so distinguish these nets. The cubic of (25) has an acnode and the cubic of (26) a crunode, so these two nets are not equivalent to each other or to the nets (23) and (24). The cubic of (23) for $\alpha = \pm 1$ has a node. If $\alpha = +1$ we can reduce (23) to (26) by the transformation

$$\lambda = -\lambda' + 2\nu', \quad \mu = -\mu', \quad \nu = \lambda' - \nu'; \quad x = -y', \quad y = x', \quad z = y' - z'$$

By a similar transformation we can reduce (23) for $\alpha = -1$ to (25). If $|\alpha| < 1$ in (23) the harmonic polar $\mu = 0$ of (0, 1, 0) cuts the cubic in one real and two conjugate imaginary points, hence for these values of α the net is not equivalent to (24). If $\beta = 1$ in (24) this net has a double line and is transformable to (21) by

$$\lambda = -\lambda', \quad \mu = \mu', \quad \nu = -\lambda' - \nu'; \quad x = z', \quad y = y', \quad z = -x' - y'.$$

If $\beta < 1$ in (24) we put

$$\lambda = \lambda', \quad \mu = [(1 - \beta)^{3/4}/\beta]\mu', \quad \nu = (-1/\beta)\lambda' + [(1 - \beta)^{1/2}/\beta]\nu';$$
$$x = [1/(1 - \beta)^{1/4}]y', \quad y = [\beta/(1 - \beta)^{1/2}]x', \quad z = -[1/(1 - \beta)^{1/2}]x' + z'$$

and we get (23) for $2\alpha = (\beta - 2)/(1 - \beta)^{1/2}$.

To show that for $\beta > 1$ the net (24) is not equivalent to (23) we use a projectively invariant relation connecting a line conic $Au^2 + Bv^2 + Cw^2 + 2Fvw + 2Gwu + 2Huv = 0$ and a point conic $a'x^2 + b'y^2 + c'z^2 + 2f'yz + 2g'zx + 2h'xy = 0$ and given by the equation

$$R: \quad a'A + b'B + c'C + 2f'F + 2g'G + 2h'H = 0.^{*}$$

To a net of line conics dual to (24) for $\beta > 1$, namely

$$(24') \qquad \lambda(u^2 - w^2) + 2\mu uv + \nu(\beta u^2 + v^2 + 2vw) = 0,$$

there corresponds by R a net of point conics

$$N: \quad \lambda(-y^2 + yz) + \mu xz + \nu[(1/\beta)x^2 + (1/\beta)z^2 - y^2] = 0.$$

We can reduce N to (23) for $|\alpha| < 1$ namely $\alpha = (2 - \beta)/\beta$ by the transformation

$$x = y', \quad y = x' + z', \quad z = 2x'.$$

But the cubic $\Delta = 0$ of (23) for $|\alpha| < 1$ cuts the harmonic polar $\mu = 0$ in one real and two conjugate imaginary points and therefore $\Delta = 0$ shows the non-equivalence of (23) for $|\alpha| < 1$ to (23) for $\alpha < -1$ or $\alpha > 1$. On the other hand it is easy to show that a net of line conics (23′) dual to (23) for $\alpha < -1$ or > 1 is apolar to a net of point conics (23) with $\alpha > 1$ and $\alpha < -1$ respectively. Hence (24) for $\beta > 1$ cannot be reduced to (23) for $\alpha > 1$ or $\alpha < -1$, and its cubic $\Delta = 0$ shows it is not equivalent to (23) for $|\alpha| < 1$.

---

[*] H. J. S. Smith, *Collected Works*, Vol. II, pp. 524-540.

Rational Tacnodal and Oscnodal Quartic Curves Considered as Plane Sections of Quartic Surfaces.[*]

By L. T. Moore and J. H. Neelley.

Introduction. It is well known that the plane sections of the Steiner Quartic Surface include all types to within projection of the rational plane quartic curve with simple singularities and that the invariants of any curve cut from this surface may be expressed as symmetric functions of the coefficients of the cutting plane.[†] If we vary this surface so that two of its double lines become coincident the plane sections are rational quartic curves with tacnodes;[‡] if the three double lines are made coincident the resulting surface has as plane sections rational quartic curves with oscnodes.[§]

It is the purpose of this paper to express the invariants of the plane sections of these two surfaces in terms of the coefficients of the cutting plane.

1. *Invariants of the Rational Tacnodal Quartic.* The equation of the tacnodal surface may be taken as

$$x_0^4 + x_1^2x_2^2 + x_0^2x_1x_3 = 0 \tag{1}$$

where $x_0x_1 = 0$ is the tacnodal line and $x_0x_2 = 0$ is the distinct double line. Two tropes of the surface are $2x_2 \pm x_3 = 0$ and the other is $x_1 = 0$ which is tangent to the surface along the tacnodal line.

The plane

$$(\alpha x) \equiv \alpha_0x_0 + \alpha_1x_1 + \alpha_2x_2 + \alpha_3x_3 = 0 \tag{2}$$

cuts the surface in a quartic with a tacnode which projected upon the plane $x_3 = 0$ is

$$\alpha_3x_0^4 - \alpha_0x_0^3x_1 - \alpha_1x_0^2x_1^2 - \alpha_2x_0^2x_1x_2 + \alpha_3x_1^2x_2^2 = 0. \tag{3}$$

---

[*] Presented to the American Mathematical Society, May 7, 1927.

[†] J. E. Rowe, *Transactions of the American Mathematical Society*, Vol. 12, pp. 295-310.

[‡] Salmon-Rodgers, *Analytic Geometry of Three Dimensions*, Vol. 2, p. 214; models and sections of these surfaces are given by J. H. Neelley, "Compound Singularities of the Rational Plane Quartic Curve," *American Journal of Mathematics*, Vol. 49, pp. 389-400.

[§] Salmon-Rogers, *l. c.*

This quartic can also be derived from the conic

$$(4) \qquad \alpha_1 x_0^2 - \alpha_3 x_1^2 - \alpha_3 x_2^2 + \alpha_0 x_0 x_1 + \alpha_2 x_1 x_2 = 0$$

by means of the quadratic transformation[*]

$$(5) \qquad x_0 = x_0' x_1', \quad x_1 = x_0'^2, \quad x_2 = x_1' x_2'$$

which has the inverse transformation

$$x_0' = x_0 x_1, \quad x_1' = x_0^2, \quad x_2' = x_1 x_2.$$

The transformation (5) sends (4) into a quartic with a tacnode at (0, 0, 1) and the distinct node at (0, 1, 0).

The invariants of (3) may be obtained from observation of the surface and the position of the cutting plane or from the conic (4) by consideration of its intersections with the reference lines. Both methods are employed in this paper.

A plane on the intersection of the nodal lines (0, 0, 0, 1) cuts out a curve with a triple point and the condition is $\alpha_3 = 0$.

The three tropes meet in the point (1, 0, 0, 0) and a plane on this point gives a curve which has three double lines on a point. This condition is

$$\alpha_0 = 0.$$

But a plane on either of the pinch points at the ends of the tacnodal line cuts out a ramphoid cusp and the curve with this singularity has three double lines on a point.[†] So the ramphoid cusp condition $\alpha_2^2 - 4\alpha_3^2 = 0$ and $\alpha_0 = 0$ form the condition for three double lines to meet in a point.

The section has an undulation when (2) touches a trope conic, that is, when

$$(\alpha_0^2 + 4\alpha_1\alpha_2 + 8\alpha_1\alpha_3) \quad (\alpha_0^2 - 4\alpha_1\alpha_2 + 8\alpha_1\alpha_3)(\alpha_2^2 - 4\alpha_3^2) = 0.$$

The condition that the quartic (3) breaks up into two conics is the vanishing of the discriminant of conic (4), or

$$\tfrac{1}{4}(\alpha_1\alpha_2^2 - 4\alpha_1\alpha_3^2 - \alpha_0^2\alpha_3) = 0.$$

The conic in lines on the six flex lines of (3) is[‡]

---

[*] Salmon-Fiedler, *Analytische Geometrie der höheren ebenen Kurven*, p. 343.

[†] J. H. Neelley, *l. c.*

[‡] Hilton, *Plane Algebraic Curves*, p. 275.

$$\begin{aligned}&(4\alpha_0^2\alpha_3^2 + 12\alpha_1\alpha_2^2\alpha_3 - 48\alpha_1\alpha_3^3)\xi_0^2 + (4\alpha_3^2 - \alpha_2^2)^2\,\xi_1^2\\ &\quad + (4\alpha_1^2\alpha_2^2 + 12\alpha_0^2\alpha_1\alpha_3 + 48\alpha_1^2\alpha_3^2)\xi_2^2\\ &\quad + 2(8\alpha_1\alpha_2\alpha_3^2 - 2\alpha_1\alpha_2^3 + 6\alpha_0^2\alpha_2\alpha_3)\xi_1\xi_2\\ &\quad - 32\alpha_0\alpha_1\alpha_2\alpha_3\xi_0\xi_2 + 2(16\alpha_0\alpha_3^2 - 4\alpha_0\alpha_2^2\alpha_3)\xi_0\xi_1 = 0,\end{aligned}$$

which has the discriminant

$$144\alpha_3^2\,\Delta^2\ (4\alpha_1\alpha_2^2\alpha_3 - 16\alpha_1\alpha_3^3 - \alpha_0^2\alpha_2^2)$$

where Δ is the discriminant of (4). If degenerate curves are excluded neither α_3 nor Δ can vanish and so the condition that three flex lines meet in a point is

$$4\alpha_1\alpha_2^2\alpha_3 - 16\alpha_1\alpha_3^3 - \alpha_0^2\alpha_2^2 = 0.$$

The conic on the points of inflection is [*]

$$\begin{aligned}&(6\alpha_0^2\alpha_2^2 + 12\alpha_0^2\alpha_3^2 - 4\alpha_1\alpha_2^2\alpha_3 + 16\alpha_1\alpha_3^3)x_0^2 + (8\alpha_1^2\alpha_2^2 - 8\alpha_0^2\alpha_1\alpha_3)x_1^2\\ &\quad + (2\alpha_2^4 - 16\alpha_2^2\alpha_3^2 + 32\alpha_3^4)x_2^2\\ &\quad + 2(7\alpha_0\alpha_1\alpha_2^2 + 4\alpha_0\alpha_1\alpha_3^2 - 3\alpha_0^3\alpha_3)x_0x_1\\ &\quad + 2(4\alpha_0\alpha_2^3 - 16\alpha_0\alpha_2\alpha_3^2)x_0x_2\\ &\quad + 2(5\alpha_1\alpha_2^3 - 20\alpha_1\alpha_2\alpha_3^2 - \alpha_0^2\alpha_2\alpha_3)x_1x_2 = 0.\end{aligned}$$

which has the discriminant

$$\begin{aligned}&36\alpha_3\,\Delta(\alpha_1^2\alpha_2^6 - 8\alpha_1^2\alpha_2^4\alpha_3^2 + 16\alpha_1^2\alpha_2^2\alpha_3^4 - 8\alpha_0^2\alpha_1\alpha_2^2\alpha_3^3\\ &\quad + 32\alpha_0^2\alpha_1\alpha_3^5 - \alpha_0^4\alpha_2^2\alpha_3^2 + 8\alpha_0^4\alpha_3^4).\end{aligned}$$

The condition that four flexes lie on a line is therefore

$$\begin{aligned}&\alpha_1^2\alpha_2^6 - 8\alpha_1^2\alpha_2^4\alpha_3^2 + 16\alpha_1^2\alpha_2^2\alpha_3^4 - 8\alpha_0^2\alpha_1\alpha_2^2\alpha_3^3\\ &\quad + 32\alpha_0^2\alpha_1\alpha_3^5 - \alpha_0^4\alpha_2^2\alpha_3^2 + 8\alpha_0^4\alpha_3^4 = 0.\end{aligned}$$

The curve has a cusp if the plane is on one of the four pinch points which gives as the cusp condition

$$\alpha_1\alpha_3(\alpha_2^2 - 4\alpha_3^2) = 0.$$

The curve has two cusps if

$$\alpha_1 = 0 \quad\text{and}\quad \alpha_2^2 - 4\alpha_3^2 = 0.$$

These invariants are sufficient to set up a correspondence between them and the invariants of the fundamental involution of the rational plane quartic curve. Since Δ does not vanish for a proper quartic it is used to raise the invariants derived from the surface to the proper degree.

---

[*] Richmond and Stuart, *Proceedings of the London Mathematical Society,* Vol. 2, pp. 131-133; Hilton, *l. c.,* pp. 275-276.

This correspondence is given in the table below:

Singularity	*Invts. of Fund. Involution*	*Invts. from Surface*
Triple Point	I_2	$\alpha_3(\alpha_1\alpha_2^2 - 4\alpha_1\alpha_3^2 - \alpha_0^2\alpha_3)$.
Disc. of q	I_2'	$3\alpha_0^2\alpha_3^2 + \alpha_1\alpha_2^2\alpha_3 - 4\alpha_1\alpha_3^3 - \alpha_0^2\alpha_2^2$
Disc. of $N = 0$	$I_2' - 4I_2$	$4\alpha_0^2\alpha_3^2 - \alpha_0^2\alpha_2^2$
Disc. of $g_2 = 0$	$I_2' + 12I_2$	$4\alpha_1\alpha_2^2\alpha_3 - 16\alpha_1\alpha_3^3 - \alpha_0^2\alpha_2^2$
Undulation	I_4	$(\alpha_0^2 + 4\alpha_1\alpha_2 + 8\alpha_1\alpha_3)(\alpha_0^2 - 4\alpha_1\alpha_2 + 8\alpha_1\alpha_3)(\alpha_2^2 - 4\alpha$
Disc. of $F = 0$	$I_4 - (I_2' - 36I_2)^2$	$(\alpha_1^2\alpha_2^6 - 8\alpha_1^2\alpha_2^4\alpha_3^2 + 16\alpha_1^2\alpha_2^2\alpha_3^4 - 8\alpha_0^2\alpha_1\alpha_2^2\alpha_3^3$ $+ 32\alpha_0^2\alpha_1\alpha_3^5 - \alpha_0^4\alpha_2^2\alpha_3^2 + 8\alpha_0^4\alpha_3^4)$
Cusp	I_6	$\alpha_1\alpha_3(\alpha_2^2 - 4\alpha_3^2)^2(\alpha_1\alpha_2^2 - 4\alpha_1\alpha_3^2 - \alpha_0^2\alpha_3)^2$

The two sets of invariants are connected by the relations

$$\begin{aligned} 4I_2 &= M(\alpha_1\alpha_2^2\alpha_3 - 4\alpha_1\alpha_3^3 - \alpha_0^2\alpha_3^2). \; . \\ (6) \qquad I_2' &= M(3\alpha_0^2\alpha_3^2 + \alpha_1\alpha_2^2\alpha_3 - 4\alpha_1\alpha_3^3 - \alpha_0^2\alpha_2^2). \\ I_4 &= M^2(\alpha_0^2 + 4\alpha_1\alpha_2 + 8\alpha_1\alpha_3)(\alpha_0^2 - 4\alpha_1\alpha_2 + 8\alpha_1\alpha_3)(\alpha_2^2 - 4\alpha_3^2)^2. \\ 16I_6 &= -M^3\alpha_1\alpha_3(\alpha_2^2 - 4\alpha_3^2)^2(\alpha_1\alpha_2^2 - 4\alpha_1\alpha_3^2 - \alpha_0^2\alpha_3)^2. \end{aligned}$$

where M is a positive constant and the other constant factors are such that the tacnode condition $I_6' = 0$ is satisfied.

2. *An Invariant Classification of Tacnodal Quartics.* There are two types of tacnodes, the embrassement and the opposition. At an embrassement both branches of the curve lie on the same side of the tangent at the tacnode; at an opposition the branches lie on opposite sides of the tangent. Each proper quartic cut from surface (1) has an embrassement.

However, every section of the surface

$$(7) \qquad x_0^4 + x_0^2x_1x_3 - x_1^2x_2^2 = 0$$

by the plane $(\beta x) = 0$ is a quartic with an opposition.* This surface may be derived from (1) by the transformation

$$x_0 = ix_0', \quad x_1 = ix_1', \quad x_2 = x_2', \quad x_3 = ix_3'$$

which gives

$$(8) \qquad \alpha_0 = i\beta_0, \quad \alpha_1 = i\beta_1, \quad \alpha_2 = \beta_2, \quad \alpha_3 = i\beta_3$$

---

* R. Gentry, *A Dissertation on the Forms of Plane Quartic Curves*, (Bryn Mawr, 1896), pp. 31-35.

From equations (6) it can be shown that

$$M\,\alpha_0^2\alpha_2^2=\frac{(I_2'+12I_2)(I_2'-4I_2)I_2^2-I_6(I_2'-4I_2)}{I_6-(I_2'-4I_2)I_2^2},$$

and

$$M\,\alpha_0^2\alpha_3^2=\frac{(I_2'+12I_2)(I_2'-4I_2)I_2^2-I_2^2(I_2'-4I_2)^2}{4\{I_6-(I_2'-4I_2)I_2^2\}},$$

Hence

$$\alpha_2^2/\alpha_3^2=\frac{(I_2'+12I_2)I_2^2-I_6}{4I_2^3}.$$

This invariant by virtue of the relations (8) is greater than zero for the quartic with an embrassement and less than zero for the quartic with an opposition.

3. *Invariants of the Rational Oscnodal Quartic.* The oscnodal quartic surface has the equation [*]

$$(9) \qquad x_0^3x_3-(x_1^2-x_0x_2)^2=0.$$

The projection upon the plane $x_3=0$ of its section by the plane $(\alpha x)=0$ gives

$$(10) \qquad \alpha_0x_0^4+\alpha_1x_0^3x_1+\alpha_2x_0^3x_2+\alpha_3x_0^2x_2^2-2\alpha_3x_0x_1^2x_2+\alpha_3x_1^4=0.$$

This quartic may also be derived from the conic

$$(11) \qquad \alpha_2x_0^2+\alpha_0x_1^2+\alpha_3x_2^2+\alpha_1x_0x_1+\alpha_2x_1x_2=0$$

by the transformation [†]

$$(12) \qquad x_0=x_0'x_1',\quad x_1=x_0'^2,\quad x_2=x_0'x_2'-x_1'^2$$

whence

$$x_0'=x_1^2,\quad x_1'=x_0x_1,\quad x_2'=x_1x_2+x_0^2.$$

The surface (9) has three coincident double lines which form the oscnodal line

$$x_0x_1=0$$

and one trope

$$x_3=0.$$

The quartic (10) accordingly has an oscnode at $(0,0,1)$.

The methods of the first section of this paper give the correspondence of the two sets of invariants as tabulated below.

---

[*] Salmon-Rogers, *l. c.*

[†] Salmon-Fiedler, *l. c.*

Invariants of Fund. Involution	*Invariants from Surface* (9)
I_2	$\alpha_1{}^2\alpha_3{}^2+\alpha_2{}^3\alpha_3-4\alpha_0\alpha_2\alpha_3{}^2$
I_2'	$\alpha_1{}^2\alpha_3{}^2-15\alpha_2{}^3\alpha_3-4\alpha_0\alpha_2\alpha_3{}^2$
$I_2'-4I_2$	$\alpha_2{}^3\alpha_3$
$I_2'+12I_2$	$\alpha_1{}^2\alpha_3{}^2-4\alpha_0\alpha_2\alpha_3{}^2-3\alpha_2{}^3\alpha_3$
I_4	$\alpha_2{}^3\alpha_3{}^3(4\alpha_0\alpha_2-\alpha_1{}^2)$
$I_4-(I_2'-36I_2)^2$	$40\alpha_0\alpha_2{}^4\alpha_3{}^3-10\alpha_1{}^2\alpha_2{}^3\alpha_3{}^3-\alpha_1{}^4\alpha_3{}^4-9\alpha_2{}^6\alpha_3{}^2$ $-16\alpha_0{}^2\alpha_2{}^2\alpha_3{}^4+8\alpha_0\alpha_1{}^2\alpha_2\alpha_3{}^4$
I_6	$\alpha_2{}^3\alpha_3{}^3(4\alpha_0\alpha_2\alpha_3-\alpha_1{}^2\alpha_3-\alpha_2{}^3)^2.$

The relations connecting the invariants are

$$\begin{aligned} 16I_2 &= M(\alpha_1{}^2\alpha_3{}^2+\alpha_2{}^3\alpha_3-4\alpha_0\alpha_2\alpha_3{}^2) \\ 4I_2' &= M(\alpha_1{}^2\alpha_3{}^2-15\alpha_2{}^3\alpha_3-4\alpha_0\alpha_2\alpha_3{}^2) \\ I_4 &= 16M^2\,\alpha_2{}^3\alpha_3{}^3(4\alpha_0\alpha_2-\alpha_1{}^2) \\ 64I_6 &= -M^3\alpha_2{}^3\alpha_3{}^3\,(4\alpha_0\alpha_2\alpha_3-\alpha_1{}^2\alpha_3-\alpha_2{}^3)^2 \end{aligned} \tag{13}$$

where the constants have been chosen so that the invariants satisfy the oscnodal conditions[*]

$$\begin{cases} I_2I_4-I_2(I_2'-4I_2)^2-64I_6=0. \\ (I_2'-4I_2)(I_2'+60I_2)-I_4=0. \end{cases}$$

M is a negative constant here.

YALE UNIVERSITY.
CARNEGIE INSTITUTE OF TECHNOLOGY.

---

[*] J. H. Neelley, *l. c.*

Affinely Connected Function Space Manifolds.

BY ARISTOTLE D. MICHAL.

Introduction.

The year 1887 marked the beginning of systematic research by Volterra along the line of *functions of curves or functionals.*[*] At approximately the same time Ricci[†] began his systematic investigations on the *absolute calculus* by taking as a starting point the researches of Gauss, Riemann and Christoffel. The advent of Einstein's General Relativity Theory gave impetus to the study of differential invariants and differential geometry. The researches of Levi-Civita, Weyl, Veblen, Eisenhart, Cartan, Schouten and of numerous other authors are well known in this relativistic age.[‡]

The object of this paper is to develop a theory of affinely connected manifolds in the function space of real continuous functions of a single independent variable. In § 1 some fundamental notions on functional transformations and functionals are summarized. The connection between differential invariants and algebraic invariants is well known in n-dimensional theories. In the general algebraic invariant theory in n variables, the underlying group of transformations is the linear homogeneous group with non-vanishing determinant. The analogue of this group in function space is the *Fredholm group* of functional transformations with non-vanishing Fredholm determinants. The theory of the Fredholm integral equation is therefore used in § 2 to develop a *functional invariant theory of functional forms.* The discussion is for the most part limited to the quadratic functional form (2.13). The infinite group of transformations (2.15) induced in the coefficients of this form by the Fredholm group has a *non-homogeneous* linear character. The theory of functional vectors and functional tensors in function space is given in § 3. It is to be observed that the inversion of the functional tensor law of transformation is effected by solving mixed linear integral equations. Next § 4 is devoted to the generalization of the theory of affine connection and infinitesimal parallelism. The generalized theory of infinitesimal parallelism leads naturally to *functional equations with functional derivatives* and to *integro-differential equations* of static type. The functional curvature tensor and its successive covariant functional derivatives are deduced in § 5 as integrability conditions of *functional equations with functional derivatives.* In connection with this subject the theory of functional operations such as covariant functional differentiation, is developed in such a manner as to make clear the rôle of the functionals that possess differentials of the Volterra form.

---

[*] Volterra: "Sopra le funzioni che dipendono da altre funzioni; Sopra le funzioni dipendenti da linee," *Rendiconti della Reale Accademia dei Lincei,* Vol. 3 (1887), five papers. For additional references on the subject of functionals, see the footnotes of our paper.

[†] Ricci, *Bulletin des Sciences Mathématiques,* Vol. 16 (1892). See also Ricci and Levi-Civita, *Mathematische Annalen,* Vol. 54 (1901); Wright, *Invariants of Quadratic Differential Forms* (Cambridge University Press, 1908); and the Cambridge Tract by O. Veblen.

[‡] The literature on these recent researches in differential geometry and differential invariants is extensive. The following books and papers, in addition to the ones already cited, form a fairly complete n-dimensional background for the generalizations of this memoir:

Weyl, *Space—Time—Matter* (English translation of fourth edition), London (1922);

Levi-Civita, *The Absolute Differential Calculus* (English translation), London (1927);

Eisenhart, *Riemmannian Geometry,* Princeton (1926);

Cartan, *La Géométrie des espaces de Riemann,* Paris (1925);

Schouten, *Der Ricci-Kalkül,* Berlin (1924);

Veblen and T. Y. Thomas, "The Geometry of Paths," *Transactions of the American Mathematical Society,* Vol. 25 (1923), pp. 551-608;

T. Y. Thomas and A. D. Michal, "Differential Invariants of Affinely Connected Manifolds," *Annals of Mathematics,* Vol. 28 (1927), pp. 196-236; "Differential Invariants of Relative Quadratic Differential Forms," *Annals of Mathematics,* Vol. 28 (1927), pp. 631-688.

The following four paragraphs of this paper are devoted to the development of the function space analogues of various methods and results given by Veblen, Eisenhart, T. Y. Thomas and the author in the cited books and papers. The center of attention is, in the main, the functional invariant theory of the *integro-differential equation* (6.9). Theorem (7.I) shows the connection between *Fredholm integral equations and normal functional coordinates.* The reader's attention is called to the important remarks made at the end of § 9 in connection with replacement theorems for *affine functional invariants that are not functional tensors.*

A functional invariant theory of quadratic differential functional forms (10.3) is developed to an extent sufficient for the purposes of an introductory theory of *Riemannian* function space manifolds. *The fundamental functional coefficients* $g_{\alpha\beta}[y^\lambda]$ *of the form* (10.3) *are not functional tensors.* Minima of functionals, of course, enter into the discussion of § 11.

Because of the extension of tensor notation to integration [see statement immediately following expression (2.2)] the formulae of this paper attain a form which, it is hoped, is at once both compact and intelligible.

1. *Fundamental notions on function space and functionals.* Let y^x be a real continuous function of a real variable x defined over the interval

$$a \leq x \leq b.$$

The numerical values that y^x assumes while x ranges continuously from a to b may be thought of as the coördinates of a point P in an ideal space which we hall call *function space.* There is of course a certain amount of redundancy n the coördinate system y since the values of y^x at the rational points on the nterval (a, b) completely determine the function y^x. This redundancy, however, is of a useful sort as it permits the employment of the remarkable theories of *functions of curves* initiated by Vito Volterra.*

A functional transformation

$$(1.1) \qquad y^x = y^x[\bar{y}^t] \qquad a \leq x \leq b,$$

where $y^x[\bar{y}^t]$ is a functional of $\bar{y}^t$ that depends in particular on the variable x and is continuous as a function of x, may be interpreted in two ways. If y^x and $\bar{y}^x$ are the coördinates of two points P and $\bar{P}$ of function space in the *same coördinate system, then* (1, 1) is a point transformation of function space that takes the point $\bar{P}$ into the point P. However, if we interpret y^x and $\bar{y}^x$ to mean the coördinates of the same point P in two different coördinate ystems, then (1. 1) is a transformation of coördinates that gives the coördinates y^x of a point P in terms of the coördinates $\bar{y}^x$ of P in another coördinate system.

We shall have occasion to use only functional transformations (1. 1) in which $y^x[\bar{y}^t]$ is a continuous functional of $\bar{y}^t$ in a region of function space. We shall say that $y^x[\bar{y}^t]$ is a *continuous functional* of $\bar{y}^t$ in a region R of function space if for every $\epsilon > 0$ and every function ϕ^t of R, there exists a positive number δ such that

$$(1.2) \qquad | y^x[\bar{y}^t] - y^x[\phi^t] | < \epsilon$$

for all x in (a, b) whenever

$$| \bar{y}^t - \phi^t | < \delta$$

for all t in (a, b), $\bar{y}^t$ being a function that is in R.

Functionals of y^t that depend in particular on several continuous variables will occur frequently throughout this paper. In n-dimensional tensor analysis the convention of letting a subscript stand for a covariant index and a superscript for a contravariant index is widely used. We shall make an analogous distinction between the continuous variables that are written as subscripts and those that appear as superscripts (see § 3). A functional $f^{\alpha\beta\cdots\gamma}_{\lambda\mu\cdots\nu}[y^t]$ of y^t that depends in particular on the continuous real variables $\alpha\beta\cdots\gamma$, $\lambda\mu\cdots\nu$ will be said to be a continuous functional of y^t in a region R of our function space if for every $\epsilon > 0$ and every function ϕ^t of R, there exists a positive number δ such that

---

* "Sopra le funzioni che dipendono da altre funzioni; Sopra le funzioni dipendenti da linee," *Rendiconti della Reale Accademia dei Lincei*, Vol. 3 (1887). See also his *Leçons sur les équations intégrales* (1913), Chapter I and his *Leçons sur les fonctions de lignes* (1913), Chapters I, II, III, IV.

$$| f^{\alpha\beta\cdots\gamma}_{\lambda\mu\cdots\nu}[y^t] - f^{\alpha\beta\cdots\gamma}_{\lambda\mu\cdots\nu}[\phi^t] | < \epsilon$$

for all $\alpha\beta\cdots\gamma$, $\lambda\mu\cdots\nu$ in (a, b) whenever

$$| y^t - \phi^t | < \delta$$

for all t in (a, b), y^t being in R.

Throughout our paper we shall consider functionals which have differentials in the Fréchét sense.* These Fréchét differentials will be assumed in general to be of the Volterra form.

2. *Algebraic functional forms and the Fredholm group.* A special transformation of the form (1. 1) is the Fredholm transformation

$$y^i = \bar{y}^i + K_a^{\ i}\bar{y}^a \tag{2. 1}$$

where the quantity $K_a^{\ i}$ is a continuous real function of the continuous real variables i and α, and the expression $K_a^{\ i}\bar{y}^a$ stands for

$$\int_a^b K_a^{\ i}\bar{y}^a \, d\alpha. \tag{2. 2}$$

The above convention of letting the repetition of an index in a term, once as a subscript and once as a superscript, stand for a Riemann integration over the interval (a, b) with respect to this index will be used throughout our paper.

Let us assume now that the Fredholm determinant of the kernel $K_a^{\ i}$ is not equal to zero. Under this assumption we have

$$\bar{y}^i = y^i + k_a^{\ i}y^a \tag{2. 3}$$

where $k_a^{\ i}$ is the resolvent kernel of K^{ai} which satisfies the well known relations

$$k_a^{\ i} + K_a^{\ i} + k_\beta^{\ i}K_a^{\ \beta} = 0; \quad k_a^{\ i} + K_a^{\ i} + K_\beta^{\ i}k_a^{\ \beta} = 0. \tag{2. 4}$$

---

[*] Cf. Fréchét's paper in the *Transactions of the American Mathematical Society*, Vol. 15 (1914), p. 139. See also Evans, *The Cambridge Colloquium*, p. 69 and P. Lévy, *Leçons D'Analyse Fonctionnelle*, pp. 48-105.

The explicit expression for the resolvent kernel $k_\alpha{}^i$ is

$$k_\alpha{}^i = -D_\alpha{}^i[K_\mu{}^\lambda]/D[K_\tau{}^\sigma].$$

In this formula, the functionals $D[K_\mu{}^\lambda]$ and $D_\alpha{}^i[K_\mu{}^\lambda]$ denote the Fredholm determinant of $K_\mu{}^\lambda$ and its first minor respectively. We have the well-known expansions

$$D[K_\mu{}^\lambda] = 1 + K_\alpha{}^\alpha + (1/2!)K^{\alpha_1\alpha_2}_{\alpha_1\alpha_2} + \cdots + (1/p!)K^{\alpha_1\alpha_2\cdots\alpha_p}_{\alpha_1\alpha_2\cdots\alpha_p} + \cdots$$
$$D_j{}^i[K_\mu{}^\lambda] = K_j{}^i + K^{i\alpha_1}_{j\alpha_1} + (1/2!)K^{i\alpha_1\alpha_2}_{j\alpha_1\alpha_2} + \cdots + (1/p!)K^{i\alpha_1\alpha_2\cdots\alpha_p}_{j\alpha_1\alpha_2\cdots\alpha_p} + \cdots$$

in which

$$K^{\alpha_1\alpha_2\cdots\alpha_p}_{\beta_1\beta_2\cdots\beta_p} = \begin{vmatrix} K_{\beta_1}{}^{\alpha_1}, & K_{\beta_2}{}^{\alpha_1}, & \cdots, & K_{\beta_p}{}^{\alpha_1} \\ \cdot & \cdot & \cdot & \cdot \\ K_{\beta_1}{}^{\alpha_p}, & K_{\beta_2}{}^{\alpha_p}, & \cdots, & K_{\beta_p}{}^{\alpha_p} \end{vmatrix}.$$

With these facts in mind one can readily verify that the class of Fredholm transformations (2.1) with non-vanishing Fredholm determinants possesses the group property.*

The differential invariant theories that arise in connection with n-dimensional geometries have a close relation to the classical theory of algebraic forms and algebraic invariants. An analogous situation exists in the functional invariant theories of function space. With this remark in mind we turn now to some fundamental ideas in a theory of functional algebraic forms and functional algebraic invariants.

By a linear functional form we shall understand a linear functional

$$l_\alpha y^\alpha \tag{2.5}$$

which has the law of transformation

$$l_\alpha y^\alpha = \bar{l}_\alpha \bar{y}^\alpha \tag{2.6}$$

under the group of Fredholm transformations (2.1) with non-vanishing determinants. In (2.5), l_α is assumed to be a continuous function of α in the fundamental interval (a, b). As a consequence of (2.1) and (2.6) we obtain

$$\bar{l}_i = l_i + l_\beta K_i{}^\beta. \tag{2.7}$$

Thus the group of Fredholm transformations (2.1) with non-vanishing Fredholm determinants induces in the coefficients l_α an associated group of Fredholm transformations (2.7) with non-vanishing Fredholm determinants.

---

* I. Fredholm, *Sur une classe d'équations fonctionnelles*, pp. 372-373.

The coefficients $q_{\alpha\beta}$ of a quadratic functional form

$$q_{\alpha\beta}y^{\alpha}y^{\beta}; \quad q_{\alpha\beta}=q_{\beta\alpha} \tag{2.8}$$

transform in accordance with the law

$$\bar{q}_{\alpha\beta}=q_{\alpha\beta}+q_{\alpha\gamma}K_{\beta}^{\gamma}+q_{\gamma\beta}K_{\alpha}^{\gamma}+q_{\gamma\delta}K_{\alpha}^{\gamma}K_{\beta}^{\delta}. \tag{2.9}$$

This can be proved quite simply by observing that the condition

$$q_{\alpha\beta}y^{\alpha}y^{\beta}=\bar{q}_{\alpha\beta}\bar{y}^{\alpha}\bar{y}^{\beta} \tag{2.10}$$

leads to a functional identity in $\bar{y}^{i}$

$$\phi_{\alpha\beta}\bar{y}^{\alpha}\bar{y}^{\beta}=0; \quad \phi_{\alpha\beta}=\phi_{\beta\alpha} \tag{2.11}$$

when we substitute in (2.10) the expression for y^{i} as given by (2.1). A necessary and sufficient condition that (2.11) be true is that

$$\phi_{\alpha\beta}=0, \tag{2.12}$$

which is condition (2.9).

The functional form (2.8), however, is not the type of quadratic functional that we wish to consider in detail. Further developments in this paper (particularly with reference to the generalization of n-dimensional Riemannian manifolds) suggest that we consider quadratic functional forms

$$g_{\alpha\beta}y^{\alpha}y^{\beta}+\int_{a}^{b}(y^{\alpha})^{2}\,d\alpha; \quad g_{\alpha\beta}=g_{\beta\alpha}. \tag{2.13}$$

With reference to (2.13) we have the following theorem:

THEOREM 2.I. *A necessary and sufficient condition that*

$$g_{\alpha\beta}y^{\alpha}y^{\beta}+\int_{a}^{b}(y^{\alpha})^{2}\,d\alpha=\bar{g}_{\alpha\beta}\bar{y}^{\alpha}\bar{y}^{\beta}+\int_{a}^{b}(\bar{y}^{\alpha})^{2}\,d\alpha \tag{2.14}$$

under the group of Fredholm transformations (2.1) *with non-vanishing Fredholm determinants is that*

$$\bar{g}_{\alpha\beta}=g_{\alpha\beta}+g_{\gamma\beta}K_{\alpha}^{\gamma}+g_{\alpha\gamma}K_{\beta}^{\gamma}+g_{\gamma\delta}K_{\alpha}^{\gamma}K_{\beta}^{\delta}+K_{\alpha}^{\beta}+K_{\beta}^{\alpha}+\int_{a}^{b}K_{\alpha}^{\gamma}K_{\beta}^{\gamma}\,d\gamma. \tag{2.15}$$

In order to obtain $g_{\alpha\beta}$ as a functional of $\bar{g}_{\alpha\beta}$ we have to solve the seemingly complicated integral equation (2.15). The existence, however, of a resolvent kernel k_{β}^{α} to K_{β}^{α} insures the unicity of the continuous solution

$$g_{\alpha\beta}=\bar{g}_{\alpha\beta}+\bar{g}_{\gamma\beta}k_{\alpha}^{\gamma}+\bar{g}_{\alpha\gamma}k_{\beta}^{\gamma}+\bar{g}_{\gamma\delta}k_{\alpha}^{\gamma}k_{\beta}^{\delta}+k_{\alpha}^{\beta}+k_{\beta}^{\alpha}+\int_{a}^{b}k_{\alpha}^{\gamma}k_{\beta}^{\gamma}\,d\gamma. \tag{2.16}$$

of (2.15). To establish this, one need only follow a method of proof similar to that used by Volterra* in his treatment of equation (2.1).

DEFINITION OF A RELATIVE SCALAR FUNCTIONAL INVARIANT. *A continuous functional* $f[g_{\alpha\beta}]$ *of* $g_{\alpha\beta}$ *will be called a relative scalar functional invariant of weight* w *of the quadratic functional form* (2.13) *if its law of transformation is*

$$f[\bar{g}_{\alpha\beta}] = (D[K_\nu{}^\mu])^w f[g_{\alpha\beta}] \qquad (w \text{ a constant number}) \tag{2.17}$$

under the group of transformations (2.1).

In (2.17) the quantity $D[K_\nu{}^\mu]$ stands for the Fredholm determinant of the function $K_\nu{}^\mu$ which occurs in (2.15).

We shall establish the following theorem.

THEOREM 2.II. *The Fredholm determinant* $D[g_{\alpha\beta}]$ *is a relative scalar functional invariant of weight* 2; i.e.,

$$D[\bar{g}_{\alpha\beta}] = (D[K_\nu{}^\mu])^2 D[g_{\alpha\beta}]. \tag{2.18}$$

To prove this theorem we define the quantity $M_{\alpha\beta}$ by

$$M_{\alpha\beta} = g_{\alpha\beta} + K_\beta{}^\alpha + g_{\alpha\delta} K_\beta{}^\delta \tag{2.19}$$

In terms of the quantities $M_{\alpha\beta}$ and $K_\alpha{}^\beta$, the relation (2.15) takes the form

$$\bar{g}_{\alpha\beta} = K_\alpha{}^\beta + M_{\alpha\beta} + K_\alpha{}^\gamma M_{\gamma\beta}. \tag{2.20}$$

By the theorem on the multiplication of Fredholm determinants† we obtain the relations

$$D[\bar{g}_{\alpha\beta}] = D[K_\nu{}^\mu]\, D[M_{\alpha\beta}]. \tag{2.21}$$

But by (2.19) and the same theorem on Fredholm determinants we obtain

$$D[M_{\alpha\beta}] = D[g_{\alpha\beta}]\, D[K_\nu{}^\mu]. \tag{2.22}$$

Our theorem follows immediately on observing that

$$D[K_\beta{}^\alpha] = D[K_\alpha{}^\beta]. \tag{2.23}$$

A quadratic functional form (2.13) has an infinite number of functional invariants when we extend the notion of a functional invariant in the following manner.

---

[*] Cf. M. Bôcher, *An Introduction to the Study of Integral Equations*, Cambridge Tract No. 10, pp. 21-22.

[†] I. Fredholm, *Sur une classe d'équations fonctionnelles*, pp. 381-383. See also G. C. Evans, *Cambridge Colloquium*, "Functionals and Their Applications," pp. 122-125.

DEFINITION OF A FUNCTIONAL INVARIANT. *A continuous functional* $f^{\gamma\delta\cdots\epsilon}_{\lambda\mu\cdots\sigma}[g_{\alpha\beta}]$ *of the coefficients* $g_{\alpha\beta}$ *of the quadratic functional form* (2. 13) *that depends in particular on the variables* $\gamma\delta\cdots\epsilon$, $\lambda\mu\cdots\sigma$ *will be called a relative functional invariant of weight* w *if under the group of Fredholm transformations* (2. 1) *with non-vanishing Fredholm determinants we have*

$$f^{\prime\delta\cdots\epsilon}_{\lambda\mu\cdots\sigma}[\bar{g}_{\alpha\beta}] = \phi^{\gamma\delta\cdots\epsilon}_{\lambda\mu\cdots\sigma}\left[f^{cd\cdots e}_{lm\cdots s}[g_{\alpha\beta}], g_{ij}, K_q{}^p, k_v{}^u\right](D[K_\beta{}^a])^w \tag{2.24}$$

where $\phi^{\gamma\delta\cdots\epsilon}_{\lambda\mu\cdots\sigma}$ *is a fixed functional of its four functional arguments* $f^{cd\cdots e}_{lm\cdots s}$, g_{ij}, $K_q{}^p$ *and* $k_v{}^u$ *that eliminates the variables* $cd\cdots e$, $lm\cdots s$, ij, p, q, u, v, *and* $k_v{}^u$ *is the resolvent kernel of* $K_v{}^u$.

Clearly the Fredholm determinant $D[g_{\alpha\beta}]$ is a relative functional invariant of weight two. A relative functional invariant of weight zero will be spoken of as an absolute functional invariant. An important absolute functional invariant of (2. 13) is the resolvent kernel $g^{\alpha\beta}$ of the coefficients $g_{\alpha\beta}$ of (2. 13). In fact we have the theorem.

THEOREM 2. III. *If the Fredholm determinant* $D[g_{\alpha\beta}]$ *of a quadratic functional form*

$$g_{\alpha\beta}y^\alpha y^\beta + \int_a^b (y^\alpha)^2\, d\alpha \tag{2.25}$$

is not zero, then the resolvent kernel $g^{\alpha\beta}[g_{ab}]$ *of* $g_{\alpha\beta}$

$$g^{\alpha\beta}[g_{ab}] = -D^{\alpha\beta}[g_{ab}]/D[g_{ab}] \tag{2.26}$$

is an absolute functional invariant of the quadratic form (2. 25) *while the first Fredholm minor* $D^{\alpha\beta}[g_{ab}]$ *of the Fredholm determinant* $D[g_{ab}]$ *is a relative functional invariant of weight two. In fact we have*

$$\begin{aligned}
&\text{(a)}\quad g^{\alpha\beta}[\bar{g}_{ab}] = g^{\alpha\beta}[g_{ab}] + g^{\gamma\beta}[g_{ab}]k_\gamma{}^\alpha + g^{\alpha\gamma}[g_{ab}]k_\gamma{}^\beta + g^{\gamma\delta}[g_{ab}]k_\gamma{}^\alpha k_\delta{}^\beta \\
&(2.27)\qquad\qquad + k_\beta{}^\alpha + k_\alpha{}^\beta + \int_a^b k_\gamma{}^\alpha k_\gamma{}^\beta\, d\gamma \\
&\text{(b)}\quad D^{\alpha\beta}[\bar{g}_{ab}] = (D[K_\mu{}^\lambda])^2\{D^{\alpha\beta}[g_{ab}] + D^{\gamma\beta}[g_{ab}]k_\gamma{}^\alpha \\
&\qquad\qquad + D^{\alpha\gamma}[g_{ab}]k_\gamma{}^\beta + D^{\gamma\delta}[g_{ab}]k_\gamma{}^\alpha k_\delta{}^\beta \\
&\qquad\qquad - D[g_{\sigma\tau}]\,(k_\beta{}^\alpha + k_\alpha{}^\beta + \int_a^b k_\gamma{}^\alpha k_\gamma{}^\beta\, d\gamma)\}.
\end{aligned}$$

To prove this theorem we first observe that

$$D[\bar{g}_{\alpha\beta}] \neq 0.$$

This is a direct consequence of (2. 18) and of our hypothesis about $D[g_{\alpha\beta}]$.

Thus $g^{\alpha\beta}[\bar{g}_{ab}]$ exists. Let $m_{\alpha\beta}$ be the resolvent kernel of $M_{\alpha\beta}$ and $k_{\beta}{}^{\alpha}$ that of $K_{\beta}{}^{\alpha}$, then it follows that the resolvent kernel* of

$$K_{\alpha}{}^{\beta} + M_{\alpha\beta} + K_{\alpha}{}^{\gamma} M_{\gamma\beta}$$

is

$$k_{\alpha}{}^{\beta} + m_{\alpha\beta} + \int_a^b m_{\alpha\gamma} k_{\gamma}{}^{\beta}\, d\gamma.$$

Hence it is a consequence of (2. 20) that

$$g^{\alpha\beta}[\bar{g}_{ab}] = k_{\alpha}{}^{\beta} + m_{\alpha\beta}[g_{ab}] + \int_a^b m_{\alpha\gamma}[g_{ab}] k_{\gamma}{}^{\beta}\, d\gamma. \tag{2. 28}$$

By a similar argument we obtain from (2. 19) the relation

$$m_{\alpha\beta}[g_{ab}] = g^{\alpha\beta}[g_{ab}] + k_{\beta}{}^{\alpha} + k_{\delta}{}^{\alpha}\, g^{\delta\beta}[g_{ab}]. \tag{2. 29}$$

Finally on eliminating the functional $m_{\alpha\beta}[g_{ab}]$ in (2. 28) by means of (2. 29) we obtain equation (a) of (2. 27). The relation (b) of (2. 27) is readily deduced by multiplying each member of equation (a) of (2. 27) by the corresponding member of equation (2. 18). This completes the proof of our theorem.

One can proceed now with a study of the invariant nature of the Fredholm minors of any order of the Fredholm determinant $D[g_{ab}]$. Let us, however, turn to certain other questions. First let us inquire into the meaning of the condition

$$K_{\alpha}{}^{\beta} + K_{\beta}{}^{\alpha} + \int_a^b K_{\alpha}{}^{\gamma} K_{\beta}{}^{\gamma}\, d\gamma = 0. \tag{2. 30}$$

One can readily verify the fact that (2. 30) is the necessary and sufficient condition that the functional

$$\int_a^b (y^{\alpha})^2\, d\alpha$$

be invariant under a Fredholm transformation (2. 1). Furthermore it is evident from (2. 30) that $K_{\alpha}{}^{\beta}$ is the resolvent kernel of $K_{\beta}{}^{\alpha}$ and that the Fredholm determinant

$$D[K_{\beta}{}^{\alpha}] = \pm 1. \tag{2. 31}$$

The family of Fredholm transformations (2. 1) with non-vanishing Fredholm determinants that satisfy condition (2. 30) form a sub-group of the group of all Fredholm transformations with non-vanishing Fredholm determinants.

---

* Cf. G. C. Evans, *loc. cit.*, pp. 121-122.

We shall call this sub-group the *orthogonal Fredholm group of transformations.* As consequences of (2. 30) we have the following formulae

$$
\begin{aligned}
&K_{\beta}^{a} + K_{a}^{\beta} + \int_{a}^{b} K_{\gamma}^{a} K_{\gamma}^{\beta}\, d\gamma = 0 \\
(2.\,32) \qquad &k_{\beta}^{a} + k_{a}^{\beta} + \int_{a}^{b} k_{\gamma}^{a} k_{\gamma}^{\beta}\, d\gamma = 0 \\
&k_{\beta}^{a} + k_{a}^{\beta} + \int_{a}^{b} k_{a}^{\gamma} k_{\beta}^{\gamma}\, d\gamma = 0.
\end{aligned}
$$

With the aid of these formulae and of (2. 30) itself a multitude of interesting results flow forth. For example, *under the orthogonal Fredholm group and under that alone do we have*

$$
\begin{aligned}
&D[\bar{g}_{a\beta}] = D[g_{a\beta}] \\
&\bar{g}_{a\beta} = g_{a\beta} + g_{\gamma\beta} K_{a}^{\gamma} + g_{a\gamma} K_{\beta}^{\gamma} + g_{\gamma\delta} K_{a}^{\gamma} K_{\beta}^{\delta} \\
(2.\,33) \qquad &g^{a\beta}[\bar{g}_{ab}] = g^{a\beta}[g_{ab}] + g^{\gamma\beta}[g_{ab}]\, k_{\gamma}^{a} + g^{a\gamma}[g_{ab}]\, k_{\gamma}^{\beta} + g^{\gamma\delta}[g_{ab}]\, k_{\gamma}^{a} k_{\delta}^{\beta} \\
&D^{a\beta}[\bar{g}_{ab}] = D^{a\beta}[g_{ab}] + D^{\gamma\beta}[g_{ab}]\, k_{\gamma}^{a} + D^{a\gamma}[g_{ab}]\, k_{\gamma}^{\beta} + D^{\gamma\delta}[g_{ab}]\, k_{\gamma}^{a} k_{\delta}^{\beta}.
\end{aligned}
$$

3. *Tensor algebra in function space.* In order to develop a theory of tensors in function space, we must first specify the underlying group of functional transformations. Consider a functional transformation

$$(3.\,1) \qquad y^{i} = y^{i}[\bar{y}^{a}]$$

which takes the functional region

$$(3.\,1a) \qquad |\, \bar{y}^{a} - \bar{y}_{0}^{a} \,| < M$$

of a fixed continuous function $\bar{y}_{0}^{a}$ into the functional region

$$(3.\,1b) \qquad |\, y^{i} - y_{0}^{i} \,| < N$$

of the fixed continuous function y_{0}^{i} that corresponds to $\bar{y}_{0}^{i}$. We shall assume that (3. 1) satisfies the following conditions.

(a) *The functional* $y^{i}[\bar{y}^{a}]$ *is single-valued, continuous as a function of* i *for* $a \leq i \leq b$, *and continuous as a functional of* $\bar{y}^{a}$ *in the functional region* (3. 1a)

(b) *the functional* y^{i} *has a Fréchét differential of the form*

$$(3.\,2) \qquad \delta y^{i} = \delta \bar{y}^{i} + {}'y_{a}^{i}\, [\bar{y}]\, \delta \bar{y}^{a}$$

where

$$|\, \delta y^{i} - \Delta y^{i} \,| < \epsilon \max |\, \delta \bar{y}^{i} \,|,$$

ϵ *being a quantity that approaches zero with* $\max |\, \delta \bar{y}^{i} \,|$

(c) *in* (3.2) *the functional* $'y_{\alpha}^{i}[\bar{y}]$ *is assumed to be continuous in* i, α, $\bar{y}^{\alpha}$ for $a \leqq i$, $\alpha \leqq b$ and $|\bar{y}^{\alpha} - \bar{y}_0^{\alpha}| < M$, *and to have a non-vanishing Fredholm determinant* $D['y_{\alpha}^{i}]$ *for* $|\bar{y}^{\alpha} - \bar{y}_0^{\alpha}| < M$.

The totality of functional transformations (3.1) which satisfy the above conditions form a group. In fact if

$$\bar{y}^i = \bar{y}^i\,[\bar{\bar{y}}^{\alpha}]$$

is a functional transformation that takes the region

$$|\bar{\bar{y}}^i - \bar{\bar{y}}_0^i| < L$$

into the region (3.1a) then

$$y^i = \psi^i[\bar{\bar{y}}]\ \left(y^i[\bar{y}[\bar{\bar{y}}]] \equiv \psi^i[\bar{\bar{y}}]\right)$$

is a functional transformation that takes the region

$$|\bar{\bar{y}}^i - \bar{\bar{y}}_0^i| < L$$

into the region (3.1b). Such a transformation obviously satisfies condition (a). Clearly

$$\delta y^i = \delta\bar{\bar{y}}^i + '\psi_{\alpha}^{i}[\bar{\bar{y}}]\,\delta\bar{\bar{y}}^{\alpha}$$

where

$$'\psi_{\alpha}^{i}[\bar{\bar{y}}] = 'y_{\alpha}^{i}[\bar{y}] + '\bar{y}_{\alpha}^{i}[\bar{\bar{y}}] + 'y_{\beta}^{i}[\bar{y}]\,'\bar{y}_{\alpha}^{\beta}[\bar{\bar{y}}].$$

Hence

$$|\delta y^i - \Delta\psi^i[\bar{\bar{y}}]| < \epsilon_1 \max|\delta\bar{\bar{y}}| + |\Delta\bar{y}^i[\bar{\bar{y}}] + 'y_{\alpha}^{i}[\bar{y}]\,\delta\bar{\bar{y}}^{\alpha} + 'y_{\beta}^{i}[\bar{y}]'\,\bar{y}_{\alpha}^{\beta}[\bar{\bar{y}}]\,\delta\bar{\bar{y}}^{\alpha} - \Delta\psi^i[\bar{\bar{y}}]|.$$

Since

$$|\Delta\bar{y}^i[\bar{\bar{y}}] + 'y_{\alpha}^{i}[\bar{y}]\,\Delta\bar{y}^{\alpha}[\bar{\bar{y}}] - \Delta\psi^i[\bar{\bar{y}}]| < \epsilon_3 \max|\delta\bar{\bar{y}}|$$

we obtain the inequality

$$|\delta y^i - \Delta\psi^i[\bar{\bar{y}}]| < (\epsilon_1 + \epsilon_2 + \epsilon_3)\max|\delta\bar{\bar{y}}|$$

in which ϵ_1, ϵ_2 and ϵ_3 are quantities that approach zero with $\max|\delta\bar{\bar{y}}^i|$. Thus the transformation $y^i = \psi^i[\bar{\bar{y}}]$ satisfies condition (b). By Fredholm's theorem on the multiplication of Fredholm determinants we have

$$D['\psi_{\alpha}^{i}] = D['y_{\alpha}^{i}]\,D['\bar{y}_{\mu}^{\lambda}]$$

and hence $D['\psi_{\alpha}^{i}] \neq 0$ in the region $|\bar{\bar{y}}^i - \bar{\bar{y}}_0^i| < L$. The group property of the totality of transformations (3.1) follows readily with the aid of the above observations.

In order to have a group of transformations with single-valued inverses throughout a functional region, we shall restrict ourselves to transformations (3. 1) that satisfy the following additional condition:

(d) *the functional transformation* (3. 1) *possesses a unique single-valued inverse*

$$\bar{y}^i = \bar{y}^i\,[y^\alpha] \tag{3. 3}$$

that takes the region (3. 1b) *into the region* (3. 1a) *and that satisfies conditions* (a), (b) *and* (c) *in the region* (3. 1b).

The class of all admitted transformations (3. 1) with their inverses forms a group of functional transformations.

The symbol $\mathfrak{G}$ *will be used to denote an arbitrary transformation of this group of functional transformations.*

Clearly we have

$$\delta\bar{y}^i = \delta y^i + {'\bar{y}_\alpha{}^i}\,[y]\,\delta y^\alpha \tag{3. 4}$$

where $'\bar{y}_\alpha{}^i$ is the resolvent kernel of $'y_\alpha{}^i$.

The functionals $'y_\alpha{}^i[\bar{y}^\beta]$ and $'\bar{y}_\alpha{}^i[y^\beta]$ will be by definition the *functional derivatives* of the functionals $y^i[\bar{y}^\beta]$ and $\bar{y}^i[y^\beta]$ respectively.

If we eliminate $\delta\bar{y}^i$ in (3. 2) by means of (3. 4) we obtain the identity

$$'y_\alpha{}^i[\bar{y}]\delta y^\alpha + {'\bar{y}_\alpha{}^i}[y]\delta y^\alpha + {'y_\beta{}^i}[\bar{y}]\,{'\bar{y}_\alpha{}^\beta}[y]\delta y^\alpha = 0.$$

From the arbitrariness of δy^α we have the fundamental relation

$$'y_\alpha{}^i[\bar{y}] + {'\bar{y}_\alpha{}^i}[y] + {'y_\beta{}^i}[\bar{y}]{'\bar{y}_\alpha{}^\beta}[y] = 0. \tag{3. 5}$$

By a similar argument we obtain the second fundamental relation

$$'\bar{y}_\alpha{}^i[y] + {'y_\alpha{}^i}[\bar{y}] + {'\bar{y}_\beta{}^i}[y]{'y_\alpha{}^\beta}[\bar{y}] = 0. \tag{3. 6}$$

on first eliminating δy^i in (3. 4) by means of (3. 2). Clearly the identities (3. 5) and (3. 6) are the well known relations that hold between a kernel of an integral equation and its resolvent kernel.

Let $f[y^\alpha]$ be a continuous functional of y^α that is a *scalar;* i. e.,

$$f[y^\alpha] = \bar{f}[\bar{y}^\alpha] \tag{3. 7}$$

under a transformation $\mathfrak{G}$. Furthermore let it possess a differential of the Volterra form

$$\delta f[y] = {'f_\alpha}[y]\,\delta y^\alpha \tag{3. 8}$$

where $'f_\alpha$ is continuous both as a functional of y^α and as a function of α. By definition, $'f_\alpha[y]$ shall also be called the *functional derivative* of the functional $f[y]$ with respect to the argument y^β.

From (3. 7) we obtain

$$'\bar{f}_\alpha[\bar{y}]\,\delta\bar{y}^\alpha = 'f_\beta[y]\,'y_\alpha^{\beta}[\bar{y}]\,\delta\bar{y}^\alpha + 'f_\alpha[y]\,\delta\bar{y}^\alpha.$$

Hence the functional derivative $'\bar{f}_\alpha[\bar{y}]$ in the variable $\bar{y}^\gamma$ is given in terms of the functional derivative $'f_\alpha[y]$ in the variable y^γ by the formula

$$'\bar{f}_\alpha[\bar{y}] = 'f_\alpha[y] + 'f_\beta[y]\,'y_\alpha^{\beta}[\bar{y}]. \tag{3. 9}$$

The functional derivative $'f_\alpha[y]$ of a scalar functional $f[y]$ is therefore a covariant functional vector. This result is in accordance with the following definition.

DEFINITION OF A COVARIANT FUNCTIONAL VECTOR:[*] *A functional $\xi_i[y^\alpha]$ will be said to be a covariant functional vector if its law of transformation under the group of transformations $\mathfrak{G}$ is given by*

$$\bar{\xi}_\alpha[\bar{y}] = \xi_\alpha[y] + \xi_\beta[y]\,'y_\alpha^{\beta}[\bar{y}]. \tag{3. 10}$$

The formula (3. 10) has the three important properties:

(a) $\bar{\xi}_\alpha$ is a linear homogeneous functional of ξ_α
(b) If ξ_α vanishes in one coördinate system y^i it vanishes in every coördinate system
(c) the group property.

To prove property (c), and incidentally derive an interesting formula, let us consider the transformations

$$\begin{aligned} \bar{\bar{\xi}}_\alpha[\bar{\bar{y}}] &= \bar{\xi}_\alpha[\bar{y}] + \bar{\xi}_\beta[\bar{y}]\,'\bar{y}_\alpha^{\beta}[\bar{\bar{y}}] \\ \bar{\xi}_\alpha[\bar{y}] &= \xi_\alpha[y] + \xi_\gamma[y]\,'y_\alpha^{\gamma}[\bar{y}]. \end{aligned} \tag{3. 11}$$

From (3. 11) we obviously obtain the relation

$$\bar{\bar{\xi}}_\alpha[\bar{\bar{y}}] = \xi_\alpha[y] + \xi_\beta[y]\,\big('y_\alpha^{\beta}[\bar{y}] + '\bar{y}_\alpha^{\beta}[\bar{\bar{y}}] + 'y_\gamma^{\beta}[\bar{y}]\,'\bar{y}_\alpha^{\gamma}[\bar{\bar{y}}]\big). \tag{3. 12}$$

But

$$'z_\alpha^{\beta}[\bar{\bar{y}}] = 'y_\alpha^{\beta}[\bar{y}] + '\bar{y}_\alpha^{\beta}[\bar{\bar{y}}] + 'y_\gamma^{\beta}[\bar{y}]\,'\bar{y}_\alpha^{\gamma}[\bar{\bar{y}}],\ z^\beta[\bar{\bar{y}}^\alpha] = y^\beta[\bar{y}^\gamma[\bar{\bar{y}}^\alpha]] \tag{3. 13}$$

---

[*] If we use the geometrical terminology we may think of (3.10) as a formula which gives the infinitely many components of the functional vector ξ_α at a point P in the coördinate system $\bar{y}^i$ in terms of its components at the same point P but in a different coördinate system y^i.

as is shown most easily by observing that ${}'z_\alpha{}^\beta[\bar y]$ as given by (3.13) is the kernel of the product of two Fredholm transformations of the type (3.2). Furthermore, the unique continuous solution of the Fredholm integral equation (3.10) is

$$\xi_\alpha[y] = \bar\xi_\alpha[\bar y] + \bar\xi_\beta[\bar y]\,{}'\bar y_\alpha{}^\beta[y] \tag{3.14}$$

since ${}'\bar y_\alpha{}^\beta[y]$ is the resolvent kernel of ${}'y_\alpha{}^\beta[\bar y]$. Finally the identical transformation

$$y^i = \bar y^i$$

implies

$${}'y_\alpha{}^i = 0; \quad {}'\bar y_\alpha{}^i = 0.$$

But these conditions reduce (3.10) and (3.14) to the forms

$$\bar\xi_\alpha[\bar y] = \xi_\alpha[y], \quad \xi_\alpha[y] = \bar\xi_\alpha[\bar y]$$

respectively. The proof of (c) is therefore complete.

Formula (3.13) shows that the functional derivative of a transformation $\mathfrak{G}$ does not transform like a covariant functional vector. In fact let us consider any scalar functional $s_i[y^\alpha]$ that depends in particular on a variable i and has a differential of the form

$$\delta s_i = \delta y^i + {}'s_{i,\alpha}[y]\,\delta \bar y^\alpha \tag{3.15}$$

where ${}'s_{i,\alpha}[y]$ denotes the functional derivative of s_i. Throughout our paper we shall understand that *subscripts following a comma denote variables that arise by the process of functional differentiation.* By a reasoning similar to the one used in obtaining (3.9) we find that

$${}'\bar s_{i,\alpha}[\bar y] = {}'s_{i,\alpha}[y] + {}'s_{i,\beta}\,{}'y_\alpha{}^\beta[\bar y] + {}'y_\alpha{}^i[\bar y]. \tag{3.16}$$

The law of transformation (3.16) possesses the group property. However, if the scalar functional has a differential of the Volterra form

$$\delta s_i = {}'s_{i,\alpha}[y]\,\delta y^\alpha$$

${}'s_{i,\alpha}$ will be a covariant functional vector for each value of i.

DEFINITION OF A CONTRAVARIANT FUNCTIONAL VECTOR: *A functional* $\xi^i[y^\alpha]$ *will be said to be a contravariant functional vector if its law of transformation under the group of transformations* $\mathfrak{G}$ *is given by*

$$\bar\xi^\alpha[\bar y] = \xi^\alpha[y] + \xi^\beta[y]\,{}'\bar y_\beta{}^\alpha[y]. \tag{3.17}$$

The unique continuous solution of the Fredholm integral equation (3.17) is

$$\xi^\alpha[y] = \bar\xi^\alpha[\bar y] + \bar\xi^\beta[\bar y]\,{}'y_\beta{}^\alpha[\bar y]. \tag{3.18}$$

It is evident from (3. 2) and (3. 4) that the differential δy^i of a transformation $\mathfrak{G}$ transforms according to the contravariant vector law.

If $\xi_i[y]$ and $\eta^i[y]$ are covariant and contravariant functional vectors then

$$\xi_i \, \eta^i$$

is a scalar functional, i. e.,

$$\bar{\xi}_i[\bar{y}] \, \bar{\eta}^i[\bar{y}] = \xi_i[y] \, \eta^i[y]. \tag{3. 19}$$

This can be shown by a direct calculation by means of the defining relations (3. 10) and (3. 17), and the functional identity (3. 5). Conversely if (3. 19) holds for an arbitrary contravariant functional vector η^i then ξ_i is of necessity a covariant functional vector. Also if (3. 19) holds for an arbitrary covariant functional vector ξ_i then η^i is necessarily a contravariant functional vector.

The above methods suggest a way of obtaining the law of transformation of more complicated functional quantities. Suppose that

$$T_\beta{}^a[\bar{y}] \, \bar{\xi}_a[\bar{y}] \, \bar{\eta}^\beta[\bar{y}] = T_\beta{}^a[y] \, \xi_a[y] \, \eta^\beta[y] \tag{3. 20}$$

under the group of transformations $\mathfrak{G}$ for arbitrary covariant functional vectors ξ_a and for arbitrary contravariant functional vectors η^a. If in (3. 20) we substitute for $\xi_a[y]$ and $\eta^\beta[y]$ their expressions in terms of $\bar{\xi}_a[\bar{y}]$ and $\bar{\eta}^\beta[\bar{y}]$ respectively we obtain a functional identity of the form

$$\phi_\beta{}^a \, \bar{\xi}_a[\bar{y}] \, \bar{\eta}^\beta[\bar{y}] = 0 \tag{3. 21}$$

where $\phi_\beta{}^a$ is independent of $\bar{\xi}_a$ and $\bar{\eta}^\beta$. Hence by a double use of the fundamental lemma of the calculus of variations we obtain the condition

$$\phi_\beta{}^a = 0. \tag{3. 22}$$

The explicit form of this condition is

$$\begin{aligned} \overline{T}_\beta{}^a[\bar{y}] = T_\gamma{}^a[y] \, {}'y_\beta{}^\gamma[\bar{y}] + T_\beta{}^a[y] + {}'\bar{y}_\delta{}^a[y] \\ (T_\gamma{}^\delta[y] \, {}'y_\beta{}^\gamma[\bar{y}] + T_\beta{}^\delta[y]). \end{aligned} \tag{3. 23}$$

On multiplying both sides of (3. 23) by ${}'y_\alpha{}^i[\bar{y}]$ and integrating with respect to α, we obtain an equivalent but more compact form of (3. 23)

$$\overline{T}_\beta{}^a[\bar{y}] \, {}'y_a{}^i = - {}'\bar{y}_\delta{}^i \, (T_\gamma{}^\delta[y] \, {}'y_\beta{}^\gamma + T_\beta{}^\delta[y]). \tag{3. 24}$$

This result was obtained by making use of the fundamental functional identity (3. 5).

A functional $T_\beta{}^a[y]$ whose law of transformation under the group of transformations $\mathfrak{G}$ is (3. 24) will be called a *functional tensor* (or merely tensor) of order two. This particular tensor $T_\beta{}^a$ is covariant of order one and contravariant of order one.

GENERAL DEFINITION OF A FUNCTIONAL TENSOR: *A functional* $T^{ijkl\cdots q}_{abc\cdots g}\,[y^\sigma]$ *that depends in particular on the variables* $i, j, \cdots, q$; $a, b, \cdots, g$ *will be by definition a functional tensor if its law of transformation under the group of transformations* $\mathfrak{G}$ *is given by*

$$
\begin{aligned}
T^{ijkl\cdots pq}_{abc\cdots g}\,[\bar{y}] &= \binom{ijkl\cdots pq}{abc\cdots g} + {}'\bar{y}_a{}^i[y]\binom{ajkl\cdots pq}{abc\cdots g} + {}'y_a{}^j\binom{iakl\cdots pq}{abc\cdots g} \\
(3.25)\qquad & + \cdots + {}'\bar{y}_a{}^q\binom{ijkl\cdots pa}{abc\cdots g} + {}'\bar{y}_a{}^i\,{}'\bar{y}_\beta{}^j\binom{a\beta kl\cdots pq}{abc\cdots g} + {}'\bar{y}_a{}^i\,{}'\bar{y}_\beta{}^k\left(\begin{matrix}aj\beta l\cdots pq\\ abc\cdots g\end{matrix}\right. \\
& + \cdots + {}'\bar{y}_a{}^p\,{}'\bar{y}_\beta{}^q\binom{ijkl\cdots a\beta}{abc\cdots g} + \cdots + \cdots \\
& \qquad + {}'\bar{y}_a{}^i\,{}'\bar{y}_\beta{}^j\,{}'\bar{y}_\gamma{}^k \cdots {}'\bar{y}_\pi{}^p\,{}'\bar{y}_\rho{}^q\binom{a\beta\gamma\cdots\pi\rho}{abc\cdots g}
\end{aligned}
$$

where

$$
\begin{aligned}
\binom{ijkl\cdots pq}{abc\cdots fg} &= T^{ij\cdots q}_{ab\cdots g}\,[y] + {}'y_a{}^a[\bar{y}]\,T^{ij\cdots q}_{ab\cdots g} + {}'y_b{}^a\,T^{ij\cdots q}_{aa\cdots g} \\
&\qquad + \cdots + {}'y_g{}^a\,T^{ij\cdots q}_{ab\cdots a} \\
(3.26)\qquad & + {}'y_a{}^a\,{}'y_b{}^\beta\,T^{ij\cdots q}_{a\beta c\cdots g} + {}'y_a{}^a\,{}'y_c{}^\beta\,T^{ij\cdots q}_{ab\beta\cdots g} + \cdots + {}'y_f{}^a\,{}'y_g{}^\beta\,T^{ij\cdots q}_{abc\cdots a\beta} \\
& + \cdots + \cdots + {}'y_a{}^a\,{}'y_b{}^\beta\,{}'y_c{}^\gamma \cdots {}'y_f{}^\zeta\,{}'y_g{}^\eta\,T^{ij\cdots q}_{ab\gamma\cdots\zeta\eta}\,.
\end{aligned}
$$

The functional tensor T will be referred to as one of order $\kappa + \nu$, covariant of order κ and contravariant of order ν when κ is the number of variables $abc\cdots fg$ and ν the number of variables $ijk\cdots pq$. Thus there are $2^{\kappa+\nu}$ terms on the right side of (3.25).

The law of transformation of a functional tensor enjoys the properties (a), (b) and (c) of page 485. Furthermore, symmetry or skew-symmetry relations in the continuous variables $ij\cdots q$, $ab\cdots g$ are not altered by a change of the variable y^i.

Let $H_a[y]$ be a covariant functional vector that possesses a functional derivative $'H_{a,\beta}[y]$. Then as Volterra * has shown, the integrability conditions of the functional equation

$$(3.27)\qquad \delta f[y] = H_a[y]\,\delta y^a,$$

with the functional $f[y]$ as the unknown, are

$$(3.28)\qquad {}'H_{a,\beta} - {}'H_{\beta,a} = 0.$$

---

* Cf. his *Leçons sur les fonctions de lignes*, Paris (1913), pp. 27-28.

It is a consequence of the law of transformation of the vector H_α that $'H_{\alpha,\beta} - 'H_{\beta,\alpha}$ is a covariant functional tensor of order two. Hence (3.28) is a tensor functional equation; i. e., if it holds in a variable y^i, it continues to hold in any other variable $\bar{y}^i$.

From the functional tensor $\phi_{\alpha\beta}[y]$ where

$$\phi_{\alpha\beta} = 'H_{\alpha,\beta} - 'H_{\beta,\alpha}$$

we can form the covariant functional tensor $\phi_{\alpha\beta\gamma}[y]$ of order three where

$$\phi_{\alpha\beta\gamma}[y] = '\phi_{\alpha\beta,\gamma}[y] - '\phi_{\gamma\beta,\alpha}[y] - '\phi_{\alpha\gamma,\beta}[y].$$

Obviously

$$\phi_{\alpha\beta\gamma}[y] \equiv 0.$$

More generally, let $\underset{\alpha\beta\gamma\ldots\kappa\lambda}{\psi}[y]$ be an alternating covariant functional tensor, i. e.,

$$\psi_{fgh\ldots mn} = \psi_{\alpha\beta\gamma\ldots\kappa\lambda} \quad \text{or} \quad -\psi_{\alpha\beta\gamma\ldots\kappa\lambda}$$

according as $fgh \cdots mn$ is an even or an odd permutation of $\alpha\beta\gamma \cdots \kappa\lambda$. The functional $\underset{\alpha\beta\gamma\ldots\kappa\lambda}{\psi}[y]$ defined by

$$(3.29) \qquad \underset{\alpha\beta\gamma\ldots\kappa\lambda\mu}{\psi}[y] = '\underset{\alpha\ldots\lambda,\mu}{\psi}[y] - \underset{\mu\beta\gamma\ldots\kappa\lambda,\alpha}{\psi}[y] - '\underset{\alpha\mu\gamma\ldots\kappa\lambda,\beta}{\psi}[y] - \cdots - '\underset{\alpha\beta\gamma\ldots\kappa\mu,\lambda}{\psi}[y]$$

is an alternating covariant functional tensor of one higher order than $\underset{\alpha\beta\gamma\ldots\kappa\lambda}{\psi}[y]$. If now we form the functional tensor $\underset{\alpha\beta\gamma\ldots\kappa\lambda\mu\nu}{\psi}[y]$ from $\underset{\alpha\beta\gamma\ldots\kappa\lambda\mu}{\psi}[y]$ according to the rule (3.29) we shall find that *

$$\underset{\alpha\beta\gamma\ldots\kappa\lambda\mu\nu}{\psi}[y] \equiv 0.$$

4. *Affinely connected function space manifolds.* Levi-Cività's † notion of infinitesimal parallelism in n-dimensional Riemannian geometry and its subsequent extension by Weyl ‡ to more general affinely connected n-dimensional manifolds has played a significant part in the recent developments of

---

[*] These results suggest the development of a theory of invariant functions of r-dimensional manifolds in *function space* under continuous groups of *functional* transformations. I have already developed a theory of invariant functions of r-dimensional manifolds in n-dimensional space under continuous groups of n-dimensional point transformations. See my paper in *The Transactions of the American Mathematical Society*, July (1927), pp. 612-646.

[†] Nozione di parallelismo in una varietà qualunque, *Rendiconti Circolo Mathematico Palermo*, Vol. 42 (1927), pp. 173-205.

[‡] *Raum, Zeit, Materie* (4th edition, p. 100).

new differential geometries. In this section of our paper we shall develop a theory of affine connection and infinitesimal parallelism in our function space.

Let $y^i(s)$ stand for a function of the two variables i and s where s is to be considered as a parameter that plays a different rôle from the variable i. The function $y^i(s)$ is assumed to be a continuous function of i and an *analytic function* of the parameter s for $a \leqq i \leqq b$ and $s_0 \leqq s \leqq s_1$. We may think then of such a function $y^i(s)$ as giving rise to a one dimensional variety or curve in our function space. For a fixed value of s we have a continuous function of i which gives the coördinates of a point P_s on our curve. As s ranges over the interval (s_0, s_1) the point P_s traces the curve in question, where, of course, $y^i(s_0)$ and $y^i(s_1)$ are the coördinates of the initial and final points respectively of our curve. An equation

$$y^i = y^i(s) \tag{4.1}$$

is then the equation of an analytic *parameterized curve* in function space. The same curve will be represented by

$$y^i = y^i(s(t)) \tag{4.2}$$

when $s = s(t)$ is the analytic inverse of an arbitrary analytic substitution $t = t(s)$.

More generally a parameterized r-dimensional variety in our function space will be determined by the equation

$$y^i = y^i\,(u_1, u_2, \cdots, u_r) \tag{4.3}$$

where $y^i(u_1, u_2, \cdots, u_r)$ is continuous as a function of i and analytic as a function of the parameters $u_1, u_2, \cdots, u_r$. The r-dimensional variety (the real geometrical object) may have *any one* of the infinitude of equations

$$y^i = y^i(u_1(v_1, v_2, \cdots, v_r), \cdots, u_r(v_1, \cdots, v_r)) \tag{4.4}$$

as its defining equations. In (4.4), the functions $u_1(v_1, \cdots, v_r), \cdots, u_r(v_1, \cdots, v_r)$ are r arbitrary analytic functions of $v_1, \cdots, v_r$ that are functionally independent.

Consider a functional $H^i_{\alpha\beta}[y^k]$ of y^k that depends in particular on the variables i, α, β and has differentials

$$\delta^r H^i_{\alpha\beta}[y^k] \qquad (r = 1, 2, 3, \cdots) \tag{4.5}$$

of the Volterra form

$${}^{(r)}H^i_{\alpha\beta,\gamma_1\gamma_2\cdots\gamma_r}[y^k] \quad \delta y^{\gamma_1}\,\delta y^{\gamma_2}\cdots\delta y^{\gamma_r}. \tag{4.6}$$

In (4. 6), the functional ${}^{(r)}H^i_{\alpha\beta,\gamma_1\gamma_2\cdots\gamma_r}[y^k]$ is assumed to be continuous as a functional of y^k and as a function of $i\alpha\beta\gamma_1\gamma_2\cdots\gamma_r$, and, without loss of generality, symmetric in the variables $\gamma_1\gamma_2\cdots\gamma_r$. We shall speak of ${}^{(r)}H^i_{\alpha\beta,\gamma_1\gamma_2\cdots\gamma_r}$ as the rth functional derivative of $H^i_{\alpha\beta}$.

We shall say that a contravariant functional vector $\xi^i[y]$ undergoes a *general linear displacement* from the point with coördinates y^i to a point with coördinates $y^i+\delta y^i$ if the vector ξ^i at the point y^i becomes the vector $\xi^i+\Delta\xi^i$ at the point $y^i+\delta y^i$ where

$$\Delta\xi^i=\delta\xi^i[y]+H^i_{\alpha\beta}\,\xi^\alpha\,\delta y^\beta \tag{4. 7}$$

as far as terms of the first order in δy^α. In (4. 7), $\delta\xi^i$ is the differential of the functional ξ^i

$$\delta\xi^i[y]={}'\xi_\alpha{}^i[y]\,\delta y^\alpha. \tag{4. 8}$$

A functional $H^i_{\alpha\beta}[y]$ will be called an *affine connection.* We have thus endowed our amorphous function space with certain geometrical properties by the introduction of an affine connection $H^i_{\alpha\beta}[y]$. A function space possessing an affine connection will be called an *affinely connected function space manifold.*

We shall see presently that the vector law of transformation of $\Delta\xi^i[y]$ determines the law of transformation of an affine connection $H^i_{\alpha\beta}[y]$. In order to derive this law of transformation, it is necessary to employ a more restricted underlying group of functional transformations. We shall deal with a group of functional transformations in which each transformation satisfies conditions (a), (b), (c) and (d) of page — as well as the following additional requirement:

(e) *in the functional neighborhood* $|\bar{y}^i-\bar{y}_0{}^i|<M$, *the functional derivative* ${}'y_\alpha{}^i[\bar{y}]$ *possesses Fréchet differentials*

$$\delta^p\,{}'y_\alpha{}^i[\bar{y}] \qquad (p=1,2,\cdots,s) \tag{4. 9}$$

of the Volterra form

$${}^{(p+1)}y^i_{\alpha\beta_1\cdots\beta_p}[\bar{y}]\quad\delta\bar{y}^{\beta_1}\cdots\delta\bar{y}^{\beta_p}, \tag{4. 10}$$

where the functional ${}^{(p+1)}y^i_{\alpha\beta_1\cdots\beta_p}[\bar{y}]$, *the* $(p+1)$*st functional derivative of* $y^i[\bar{y}]$, *is a symmetric function of the variables* $\alpha\beta_1\cdots\beta_p$, *and is continuous as a functional of* $\bar{y}^\gamma$ *and as a function of* $i\alpha\beta\cdots\beta_p$.

If s is the highest order of the differentials in (4. 9), the symbol $\mathfrak{G}_s$ will be used to denote an arbitrary functional transformation that belongs to such a group.

Under the group of transformations $\mathfrak{G}_1$ we obviously have for a contravariant vector $\xi^i[y]$

$$\begin{aligned} \xi^i[y] &= \bar{\xi}^i[\bar{y}] + {}'y_\alpha{}^i\,[\bar{y}]\,\bar{\xi}^\alpha[\bar{y}] \\ \delta\,\xi^i[y] &= \delta\,\bar{\xi}^i + {}'y_\alpha{}^i\,\delta\bar{\xi}^\alpha + {}''y^i{}_{jk}[\bar{y}]\,\bar{\xi}^j\,\delta\bar{y}^k. \end{aligned} \tag{4.11}$$

Hence from (4.7) we have

$$\begin{aligned} \Delta\xi^i = \delta\bar{\xi}^i + {}'y_\alpha{}^i\,\delta\bar{\xi}^\alpha + ({}''y^i{}_{jk} + H^i{}_{jk} + H^i{}_{\alpha k}\,{}'y_j{}^\alpha \\ + H^i{}_{j\alpha}\,{}'y_k{}^\alpha + H^i{}_{\alpha\beta}\,{}'y_j{}^\alpha\,{}'y_k{}^\beta)\,\bar{\xi}^j\,\delta\bar{y}^k. \end{aligned} \tag{4.12}$$

A necessary and sufficient condition that (4.12) be equivalent to

$$\Delta\xi^i = \Delta\bar{\xi}^i + {}'y_\alpha{}^i\,\Delta\bar{\xi}^\alpha,$$

where

$$\Delta\bar{\xi}^i = \delta\bar{\xi}^i[y] + \bar{H}_{\alpha\beta}{}^i[\bar{y}]\;\bar{\xi}^\alpha\,\delta\bar{y}^\beta,$$

for arbitrary vectors ξ^i and differentials $\delta\bar{y}^k$ is that

$$\begin{aligned} H^i{}_{jk}[\bar{y}] + H^\alpha{}_{jk}\,{}'y_\alpha{}^i = {}''y^i{}_{jk} + H^i{}_{jk}[y] + H^i{}_{\alpha k}\,{}'y_j{}^\alpha \\ + H^i{}_{j\alpha}\,{}'y_k{}^\alpha + H^i{}_{\alpha\beta}\,{}'y_j{}^\alpha\,{}'y_k{}^\beta. \end{aligned} \tag{4.13}$$

Multiplying both sides of (4.13) by ${}'\bar{y}_i{}^\beta[y]$ and integrating with respect to the variable i we obtain the relation

$$\begin{aligned} \bar{H}^\gamma{}_{jk}\,{}'y_\gamma{}^i = -{}'\bar{y}_\gamma{}^i\;({}''y^\gamma{}_{jk} + H^\gamma{}_{jk} + H^\gamma{}_{\alpha k}\,{}'y_j{}^\alpha + H^\gamma{}_{j\alpha}\,{}'y_k{}^\alpha \\ + H^\gamma{}_{\alpha\beta}\,{}'y_j{}^\alpha\,{}'y_k{}^\beta). \end{aligned} \tag{4.14}$$

Another important equivalent form of (4.13) can be obtained by solving the Fredholm integral equation (4.13) with kernel ${}'y_\alpha{}^i$. The unique continuous solution (since the Fredholm determinant $D[{}'y_\alpha{}^i] \neq 0$) is

$$\begin{aligned} \bar{H}_{jk}{}^i = {}''y_{jk}{}^i + H_{jk}{}^i + H^i{}_{\alpha k}\,{}'y_j{}^\alpha + H^i{}_{j\alpha}\,{}'y_k{}^\alpha + H^i{}_{\alpha\beta}\,{}'y_j{}^\alpha\,{}'y_k{}^\beta \\ + {}'\bar{y}_\gamma{}^i\,({}''y^\gamma{}_{jk} + H^\gamma{}_{jk} + H^\gamma{}_{\alpha k}\,{}'y_j{}^\alpha + H^\gamma{}_{j\alpha}\,{}'y_k{}^\alpha + H^\gamma{}_{\alpha\beta}\,{}'y_j{}^\alpha\,{}'y_k{}^\beta). \end{aligned} \tag{4.15}$$

The law of transformation of an affine connection $H^i{}_{jk}[y]$ can thus be given in any one of the three equivalent forms (4.13), (4.14), (4.15). It is clear from (4.15) that an affine connection $H^i{}_{jk}$ transforms like a functional tensor when and only when

$$ {}''y^\gamma{}_{jk}[\bar{y}] = 0. \tag{4.16}$$

For example, a linear functional transformation

$$y^i = \bar{y}^i + a^i + \beta^i{}_k\bar{y}^k$$

satisfies condition (4.16).

Any affine connection $H^i{}_{jk}$ can be written as

$$H^i{}_{jk} = \Gamma^i{}_{jk}[y] + \Omega^i{}_{jk}[y] \tag{4.17}$$

where

$$\Gamma^i{}_{jk}[y] = \Gamma^i{}_{kj}[y] \tag{4.18}$$

and

$$\Omega^i{}_{jk}[y] = -\Omega^i{}_{kj}[y]. \tag{4.19}$$

The functional *tensor* $\Omega^i{}_{jk}$ will be called the *torsion* of our affinely connected manifold. From now on in this paper, we shall deal with affinely connected function space manifolds whose torsion tensor $\Omega^i{}_{jk}$ is identically zero. This then means that our affine connection $\Gamma^i{}_{jk}[y]$ is a symmetric function of the variables j and k.

We shall say that a functional contravariant vector $\xi[y]$ undergoes a *parallel displacement* from the point with coördinates y^i to a neighboring point with coördinates $y^i + \delta y^i$ if $\Delta\xi^i = 0$, i. e., if

$$\delta\xi^i[y] = -\Gamma^i{}_{\alpha\beta}[y]\xi^\alpha\delta y^\beta, \tag{4.20}$$

where $\Gamma^i{}_{\alpha\beta}[y]$ is the symmetric affine connection of our affinely connected function space manifold.

The functional total differential equation (4.20) is equivalent to the functional equation with functional derivatives

$${}'\xi_\beta{}^i[y] = -\Gamma^i{}_{\alpha\beta}[y]\xi^\alpha[y]. \tag{4.21}$$

The integrability conditions [*] of equation (4.21) are deduced by imposing the condition that the differential $\delta\,{}'\xi_\beta{}^i[y]$ be a self-adjoint linear functional [*] of δy^i. A necessary and sufficient condition that $\delta'\xi_\beta{}^i[y]$ be a self-adjoint functional of δy^i is that

$$\xi^\alpha({}'\Gamma^i{}_{\alpha\beta,\gamma}\;[y] - {}'\Gamma^i{}_{\alpha\gamma,\beta} + \Gamma^\sigma{}_{\alpha\beta}\Gamma^i{}_{\sigma\gamma} - \Gamma^\sigma{}_{\alpha\gamma}\Gamma^i{}_{\sigma\beta})z^\alpha w^\beta = 0 \tag{4.22}$$

for all continuous functions z^α and w^β of sufficiently small moduli. Hence a necessary and sufficient condition that (4.22) hold is that

$$\xi^\alpha[y]B^i{}_{\alpha\beta\gamma}[y] = 0 \tag{4.23}$$

where

$$B^i{}_{\alpha\beta\gamma}[y] = {}'\Gamma^i{}_{\alpha\beta,\gamma}[y] - {}'\Gamma_{\alpha\gamma,\beta} + \Gamma^\sigma{}_{\alpha\beta}\Gamma^i{}_{\sigma\gamma} - \Gamma^\sigma{}_{\alpha\gamma}\Gamma^i{}_{\sigma\beta}. \tag{4.24}$$

---

[*] For the theory of the integrability conditions of functional equations and for the concept of a self-adjoint functional see P. Lévy, *loc. cit.*, pp. 91-94, 151-170 and and G. C. Evans, *loc. cit.*, pp. 17-25.

The functional $B^i_{\alpha\beta\gamma}$ plays an important rôle in the theory of affinely connected function space manifolds. We shall see in the next paragraph that $B^i_{\alpha\beta\gamma}$ is a functional tensor. We may use the terminology of n-dimensional geometry and call the functional $B^i_{\alpha\beta\gamma}[y]$ the *curvature tensor* of the affinely connected manifold determined by the affine connection $\Gamma^i_{\alpha\beta}[y]$.

Let

$$y^i = y^i(s)$$

be the equation of an arbitrarily given parameterized curve with $s = s_0$ and $s = s_1$, as the values of the parameters corresponding to the initial and final points of the curve. It is clear from (4. 20) that for a parallel displacement of the vector $\xi^i[y]$ along the given curve we must have*

$$\frac{\partial \xi^i[y^a(s)]}{\partial s} = -\Gamma^i_{\alpha\beta}[y^\gamma(s)]\, \xi^\alpha[y^\lambda(s)]\, \frac{\partial y^\beta(s)}{\partial s} . \tag{4. 25}$$

Since the curve is fixed we see that (4. 25) is a linear *integro-differential equation* of static type

$$\frac{\partial \phi^i(s)}{\partial s} = \psi_\alpha{}^i(s)\, \phi^\alpha(s) \tag{4. 26}$$

with $\phi^i(s)$ as the unknown function of the two variables i and s, and $\psi^i{}_\alpha(s)$ as the known function of the three variables i, α and s. Let

$$\xi^i[y^a(s)] = \phi^i(s)$$

for the given curve. Under sufficiently restrictive conditions, the unique solution $\phi^i(s)$, continuous in i and analytic in s, of the integro-differential equation (4. 25) which takes on the initial value

$$\eta^i = \phi^i(s_0) \tag{4. 27}$$

will be

$$\begin{aligned}\phi^i(s) = \eta^i + \eta^\omega \Big[(s - s_0)\, {}_{(1)}\lambda^i{}_\omega(s_0) + \frac{(s-s_0)^2}{2\,!}\, {}_{(2)}\lambda^i{}_\omega(s_0) \\ + \frac{(s-s_0)^3}{3\,!}\, {}_{(3)}\lambda^i{}_\omega(s_0) + \cdots \Big]\end{aligned} \tag{4. 28}$$

where

$$ {}_{(1)}\lambda^i{}_\omega(s) = -\Gamma^i{}_{j\omega}\, \frac{\partial y^j(s)}{\partial s} \tag{4. 29}$$

$$ {}_{(2)}\lambda^i{}_\omega(s) = \Gamma^i{}_{ja}\, \Gamma^a{}_{k\omega} \frac{\partial y^j}{\partial s}\, \frac{\partial y^k}{\partial s} - {}'\Gamma^i{}_{j\omega,k}\, \frac{\partial y^j}{\partial s}\, \frac{\partial y^k}{\partial s} - \Gamma^i{}_{j\omega}\, \frac{\partial^2 y^j}{\partial s^2}$$

and in general

$$ {}_{(m)}\lambda^i{}_\omega(s) = \frac{\partial\, {}_{(m-1)}\lambda^i{}_\omega(s)}{\partial s} - {}_{(m-1)}\lambda^i{}_a(s)\, \Gamma^a{}_{j\omega,}\frac{\partial y^j(s)}{\partial s} . \tag{4. 30}$$

---

[*] Cf. § 6 with particular reference to statements about formula (6, 3).

The solution (4.28) will depend in general on the choice of the function $y^i(s)$ although it will be independent of the parameterization s. This then means that the parallely displaced vector $\xi^i[y]$ will be a functional of function space curves. If the affinely connected manifold is such that

$$B^i{}_{\alpha\beta\gamma}[y]=0 \tag{4.31}$$

then the integrability conditions (4.23) will be satisfied identically and the solution (4.28) of the functional equation (4.20) or (4.21) will be independent of the choice of the curve.

5. *The functional curvature tensor and other affine functional invariants.* The law of transformation of an affine connection can obviously be written in the form

$$\begin{aligned} ''y^i{}_{\alpha\beta}[\bar{y}]=\bar{\Gamma}^i{}_{\alpha\beta}[\bar{y}] + \bar{\Gamma}^\sigma{}_{\alpha\beta}\,'y_\sigma{}^i - \Gamma^i{}_{\alpha\beta}[y] - \Gamma^i{}_{\sigma\beta}\,'y_\alpha{}^\sigma \\ - \Gamma^i{}_{\alpha\sigma}\,'y_\beta{}^\sigma - \Gamma^i{}_{\sigma\tau}\,'y_\alpha{}^\sigma\,'y_\beta{}^\tau \end{aligned} \tag{5.1}$$

as is evident by a reference to formula (4.13). Since $''y_{\alpha\beta}{}^i$ is indeed a functional of $\bar{y}^\gamma$ we must have

$$'''y^i{}_{\alpha\beta\gamma}[\bar{y}] - '''y^i{}_{\alpha\beta\gamma}[\bar{y}] = 0. \tag{5.2}$$

To obtain this condition (which is obviously the condition for the self-adjointness of the differential $\delta''y_{\alpha\beta}{}^i$) in terms of the first functional derivatives $'y_\alpha{}^i[\bar{y}]$, we first calculate the third functional derivatives $'''y^i{}_{\alpha\beta\gamma}$ from (5.1) by taking the differential of both sides of (5.1) and equating the coefficients of $\delta\bar{y}^\gamma$. We thus shall have the condition (5.2) expressed in terms of $''y_{\alpha\beta}{}^i$ and $'y_\alpha{}^i$. If then we eliminate $''y_{\alpha\beta}{}^i$ from this final expression by means of (5.1) itself, we shall obtain the equations

$$\begin{aligned} \bar{B}^i{}_{\alpha\beta\gamma}[\bar{y}] + \bar{B}^\sigma{}_{\alpha\beta\gamma}\,'y_\sigma{}^i[\bar{y}] = B^i{}_{\alpha\beta\gamma}[y] + B^i{}_{\sigma\beta\gamma}\,'y_\alpha{}^\sigma \\ + B^i{}_{\alpha\sigma\gamma}\,'y_\beta{}^\sigma + B^i{}_{\alpha\beta\sigma}\,'y_\gamma{}^\sigma \\ + B^i{}_{\sigma\tau\gamma}\,'y_\alpha{}^\sigma\,'y_\beta{}^\tau + B^i{}_{\sigma\beta\tau}\,'y_\alpha{}^\sigma\,'y_\gamma{}^\tau + B^i{}_{\alpha\sigma\tau}\,'y_\beta{}^\sigma\,'y_\gamma{}^\tau \\ + B^i{}_{\sigma\tau\nu}\,'y_\alpha{}^\sigma\,'y_\beta{}^\tau\,'y_\gamma{}^\nu \end{aligned} \tag{5.3}$$

where $B^i{}_{\alpha\beta\gamma}[y]$ is the functional that was defined by (4.24). The unqiue continuous solution $\bar{B}^i{}_{\alpha\beta\gamma}[\bar{y}]$ of the Fredholm integral equation (5.3) is

$$\bar{B}^i{}_{\alpha\beta\gamma}[\bar{y}] = \left({}^{\,i}_{\alpha\beta\gamma}\right) + '\bar{y}_\sigma{}^i[y]\,\left({}^{\,\sigma}_{\alpha\beta\gamma}\right), \tag{5.4}$$

where

$$\begin{aligned} \left({}^{i}{}_{\alpha\beta\gamma}\right) = B^i{}_{\alpha\beta\gamma}[y] + B^i{}_{\sigma\beta\gamma}\,'y_\alpha{}^\sigma[\bar{y}] + B^i{}_{\alpha\sigma\gamma}\,'y_\beta{}^\sigma + B^i{}_{\alpha\beta\sigma}\,'y_\gamma{}^\sigma \\ B^i{}_{\sigma\tau\gamma}\,'y_\alpha{}^\sigma\,'y_\beta{}^\tau + B^i{}_{\sigma\beta\tau}\,'y_\alpha{}^\sigma\,'y_\gamma{}^\tau + B^i{}_{\alpha\sigma\tau}\,'y_\beta{}^\sigma\,'y_\gamma{}^\tau + B^i{}_{\sigma\tau\nu}\,'y_\alpha{}^\sigma\,'y_\beta{}^\tau\,'y_\gamma{}^\nu. \end{aligned} \tag{5.5}$$

Another useful form of the law of transformation of $B^i{}_{\alpha\beta\gamma}$ is obtained by multiplying both sides of (5.3) by $\bar{y}_i{}^\omega[y]$ and integrating over the variable i. If we simplify the resultant expression by means of (3. 6) we obtain

$$\bar{B}^i{}_{\alpha\beta\gamma}[\bar{y}]\ {}'y_i{}^\omega[\bar{y}] = -{}'\bar{y}_\sigma{}^\omega[y] \left(\begin{smallmatrix}\sigma\\ \alpha\beta\gamma\end{smallmatrix}\right). \tag{5. 6}$$

Formula (5. 3) shows that $B^i{}_{\alpha\beta\gamma}[y]$ is a functional tensor which is covariant of order three and contravariant of order one.

By a process similar to the one by means of which we deduced (5. 3) from (5. 1), we can obtain the law of transformation of another functional tensor. In fact on taking the differential of both sides of (5. 3), equating the coefficient of the differential $\delta\bar{y}^\delta$, and eliminating $''y^i{}_{\alpha\beta}[\bar{y}]$ by means of (5. 1) we obtain the law of transformation of the functional tensor $B^i_{\alpha\beta\gamma;\delta}[y]$ defined by

$$\begin{aligned} B^i_{\alpha\beta\gamma;\delta}[y] = {}'B^i_{\alpha\beta\gamma,\delta}[y] + B^\omega{}_{\alpha\beta\gamma}\,\Gamma^i{}_{\omega\delta} - B^i{}_{\omega\beta\gamma}\,\Gamma^\omega{}_{\alpha\delta} - B^i{}_{\alpha\omega\gamma}\,\Gamma^\omega{}_{\beta\delta} \\ - B^i{}_{\alpha\beta\omega}\,\Gamma^\omega{}_{\gamma\delta}. \end{aligned} \tag{5. 7}$$

We shall call $B^i{}_{\alpha\beta\gamma;\delta}$ the *covariant functional derivative of* $B^i{}_{\alpha\beta\gamma}$. *To distinguish this type of functional differentiation from the ordinary functional differentiation we shall use a semi-colon immediately preceding the index added on account of a covariant functional differentiation.*

It is clear now how one can proceed to deduce the infinite sequence of functional tensors

$$B^i_{\alpha\beta\gamma;\,\delta;\,\epsilon}[y]\qquad ,\cdots,\quad B^i_{\alpha\beta\gamma;\,\delta;\,\epsilon;\,\ldots;\,\lambda}[y]\qquad ,\cdots.$$

The general functional operation that yields a functional tensor of one higher covariant order than the functional tensor on which it operates will be called *covariant functional differentiation.* It is always possible to obtain the result of covariant functional differentiation by the processes of ordinary functional differentiation and elimination by means of (5. 1). We shall see in § 8 that covariant functional differentiation is a special case of a more general functional operation that yields functional tensors of higher orders than the functional tensor on which it operates.

For the sake of future reference, we shall write down the *formula giving the covariant functional derivative of a functional tensor* $T^{ij\ldots k}_{ab\ldots c}$

$$\begin{aligned} T^{ij\ldots k}_{ab\ldots c;\,d}[y] = {}'T^{ij\ldots k}_{ab\ldots c,\,d}[y] + T^{\omega j\ldots k}_{ab\ldots c}\ \Gamma^i{}_{\omega\delta} + T^{i\omega\ldots k}_{ab\ldots c}\ \Gamma^j{}_{\omega d} \\ + \cdots + T^{ij\ldots\omega}_{ab\ldots c}\ \Gamma^k{}_{\omega d} - T^{ij\ldots k}_{\omega b\ldots c}\ \Gamma^\omega{}_{ad} - T^{ij\ldots k}_{a\omega\ldots c}\ \Gamma^\omega{}_{bd} \\ - \cdots - T^{ij\ldots k}_{ab\ldots\omega}\,\Gamma^\omega{}_{cd}. \end{aligned} \tag{5. 8}$$

In (5. 8), the functional $\Gamma^i_{\alpha\beta}[y]$ is the affine connection of our affinely connected function space manifold. Of course it is understood that the differential $^{ij\cdots k}_{ab\cdots c}$ is of the Volterra form

$$(5.9) \qquad \delta T^{ij\cdots k}_{ab\cdots c} = {}'T^{ij\cdots k}_{ab\cdots c;\,d}\ \delta y^d$$

and the functional derivative ${}'T^{ij\cdots k}_{ab\cdots c;\,d}$ is continuous both as a functional of y^i and as a function of $ij\cdots kab\cdots cd$.

The functional operation of covariant differentiation yields a tensor when it operates on a tensor. The question arises as to the existence of *functional operations which yield a tensor when the operation is applied on a functional which has a definite law of transformation that is not the tensor law. We shall see later that such functional operations are of great importance in the theory of Riemannian function space manifolds* (see § 10).

To make these statements more definite let us consider a covariant functional tensor $\xi_i[y]$ with a differential

$$(5.10) \qquad \delta\xi_i[y] = \eta_{(i)}[y]\,\delta y^i + {}'\xi_{i,a}[y]\,\delta y^a \quad \text{(the variable } i \text{ is not an integration variable).}$$

In (5. 10), the functional $\eta_{(i)}[y]$ is of necessity a scalar functional, i. e.,

$$(5.11) \qquad \bar{\eta}_{(i)}[\bar{y}] = \eta_{(i)}[y].$$

The law of transformation of ${}'\xi_{\alpha,\beta}[y]$

$$(5.12) \qquad {}'\xi_{\alpha,\beta}[\bar{y}] = {}'\xi_{\alpha,\beta}[y] + {}'\xi_{\lambda,\beta}{}'y_\alpha{}^\lambda[\bar{y}] + {}'\xi_{\alpha,\lambda}{}'y_\beta{}^\lambda + {}'\xi_{\lambda,\mu}{}'y_\alpha{}^\lambda\,{}'y_\beta{}^\mu + \xi_\lambda{}''y^\lambda{}_{\alpha\beta}[\bar{y}] + \eta_{(\alpha)}{}'y_\beta{}^\alpha + \eta_{(\beta)}{}'y_\alpha{}^\beta + \eta_{(\lambda)}{}'y_\alpha{}^\lambda\,{}'y_\beta{}^\lambda$$

(α and β are not integration variables).

Hence by elimination of $''y^\lambda{}_{\alpha\beta}$, we obtain the relation

$$(5.13) \qquad \bar{\xi}_{\alpha;\beta}[\bar{y}] = \xi_{\alpha;\beta}[y] + \xi_{\lambda;\beta}{}'y_\alpha{}^\lambda + \xi_{\alpha;\lambda}{}'y_\beta{}^\lambda + \xi_{\lambda;\mu}{}'y_\alpha{}^\lambda\,{}'y_\beta{}^\mu + \eta_{(\alpha)}[y]\,{}'y_\beta{}^\alpha + \eta_{(\beta)}{}'y_\alpha{}^\beta + \eta_{(\lambda)}{}'y_\alpha{}^\lambda\,{}'y_\beta{}^\lambda$$

(α and β are not integration variables)

where

$$(5.14) \qquad \begin{aligned} \xi_{\alpha;\beta}[y] &= {}'\xi_{\alpha,\beta}[y] - \Gamma^\mu{}_{\alpha\beta}[y]\,\xi_\mu[y] \\ \bar{\xi}_{\alpha;\beta}[\bar{y}] &= {}'\bar{\xi}_{\alpha,\beta}[\bar{y}] - \Gamma^\mu{}_{\alpha\beta}[\bar{y}]\,\bar{\xi}_\mu[\bar{y}]. \end{aligned}$$

The following theorem is immediate.

THEOREM (5. I). *A necessary and sufficient condition that the covariant functional derivative of a vector* $\xi_i[y]$ *with a differential* (5. 10) *be a tensor* *is that the scalar functional* $\eta_i[y]$ *be identically zero.* Similar theorems can be proved for functional tensors of any order.

In case

$$\eta_{(i)}[y] \equiv 1 \tag{5.15}$$

we see from (5.13) and (2.30) that under the group of functional transformations, a sub-group of the transformations $\mathfrak{G}_s$, that induce an orthogonal Fredholm group in the differentials δy^i, and under that alone, does $\xi_{a;\beta}[y]$ transform like a tensor.

Let us assume now that

$$\delta\, '\xi_{i,a}[y] = ''\xi_{i,a\beta}\delta y^\beta \tag{5.16}$$

and that (5.15) holds. It is then a consequence of (5.14) that

$$\delta\, \xi_{a;\beta}[y] = '\xi_{a;\beta,\gamma}[y]\, \delta y^\gamma \tag{5.17}$$

The functional $\xi_{a\beta\gamma}[y]$ defined by

$$\xi_{a\beta\gamma}[y] = '\xi_{a;\beta,\gamma}[y] - \xi_{\lambda;\beta}[y]\, \Gamma^\lambda{}_{a\gamma}[y] - \xi_{a;\lambda}\, \Gamma^\lambda{}_{\beta\gamma} - \Gamma^a{}_{\beta\gamma} - \Gamma^\beta{}_{a\gamma} \tag{5.18}$$

is a covariant functional tensor of order three. This result can be deduced from (5.13) by a method similar to the one used in arriving at (5.13) from the vector law of transformation of $\xi_i[y]$. We have thus derived a functional tensor $\xi_{a\beta\gamma}[y]$ by applying a more general functional operation than covariant functional differentiation to a functional which is not a tensor. The reader will see that a multitude of similar results can be obtained by starting with a general tensor instead of a covariant vector ξ_i.

The functional curvature tensor $B^i{}_{a\beta\gamma}[y]$ and its successive covariant derivatives are fixed functionals of the affine connection $\Gamma^i{}_{a\beta}$ and the functional derivatives of the affine connection. This sequence of tensors may be called affine functional tensor invariants.

DEFINITION. *A functional (not necessarily a tensor)*

$$F^{ij\,\cdots\, l}_{ab\,\cdots\, c}\ [\Gamma, '\Gamma, \cdots, {}^{(r)}\Gamma]$$

of the affine connection $\Gamma^i{}_{a\beta}$ *and its functional derivatives up to and including the rth will be said to be an affine functional invariant of order r and weight w if under the group of transformations* $\mathfrak{G}_s$

(1) *there is a definite law connecting its expression in the variable* $\bar{y}^i$ *in terms of its expression in the variable* y^i

(2) *the form of the functional remains unchanged when the functional arguments are taken to be* $\Gamma^i{}_{a\beta}, '\Gamma^i{}_{a\beta,\gamma_1}, \cdots, {}^{(r)}\Gamma^i{}_{a\beta,\gamma_1\cdots\gamma_r}$

(3) *the wth power of the Fredholm determinant* $D['y_a{}^i]$ *of the functional derivative* $'y_a{}^i[\bar{y}]$ *of a transformation* $\mathfrak{G}_s$ *is a factor of*

$$F^{ij\,\cdots\,l}_{ab\,\cdots\,c}\ [\bar{\Gamma}, '\bar{\Gamma}, \cdots, {}^{(r)}\bar{\Gamma}].$$

Clearly the affine connection $\Gamma^i{}_{\alpha\beta}[y]$ is an absolute (weight $w = 0$) affine functional invariant of order zero and the curvature tensor $B^i{}_{\alpha\beta\gamma}$ is an absolute affine functional invariant of order one.

The definition (4.24) shows that $B^i{}_{\alpha\beta\gamma}[y]$ satisfies the functional identities

$$\begin{aligned} B^i{}_{\alpha\beta\gamma} + B^i{}_{\gamma\alpha\beta} + B^i{}_{\beta\gamma\alpha} &= 0, \\ B^i{}_{\alpha\beta\gamma} &= - B^i{}_{\alpha\gamma\beta}. \end{aligned} \tag{5.19}$$

6. *The fundamental integro-differential equation of affinely connected function space manifolds.* Consider a continuous functional $\Lambda^i{}_{\alpha\beta}[y^\lambda]$ that depends in particular on the variables i, α and β and is a symmetric function of α and β. Let $M^i{}_a[y^\lambda]$ stand for a continuous functional that depends in particular on the variables i and α. Form now the non-linear integro-differential equation

$$\frac{\partial t^2}{\partial^2 y^i(t)} + \Lambda^i{}_{\alpha\beta}[y^\lambda(t)]\ \frac{\partial y^\alpha(y)}{\partial t}\ \frac{\partial y^\beta(t)}{\partial t} + M_\alpha{}^i[y^\lambda(t)]\left(\frac{\partial y^\alpha(t)}{\partial t}\right)^2 = 0. \tag{6.1}$$

In (6.1) we understand that the functional *operations* $\Lambda^i{}_{\alpha\beta}[\ ]$ and $M^i{}_a[\ ]$ are independent of the parameter t. We wish to determine the conditions under which equation (6.1) is left invariant in form by the group of transformations $\mathfrak{G}_s$. Let us make a change of variable

$$y^i = y^i[\bar{y}^\lambda]$$

so that we have

$$y^i(t) = y^i[\bar{y}^\lambda(t)] \tag{6.2}$$

in which $\bar{y}^\lambda(t)$ is continuous in λ and analytic in t. We shall assume that the functional operation $y^i[\ ]$ yields an analytic function of t when it operates on $\bar{y}^\lambda(t)$ and eliminates the variable λ. Obviously

$$\bar{y}^\lambda(t+\epsilon) - \bar{y}^\lambda(t) = \epsilon\ \ [(\partial\bar{y}^\lambda/\partial t) + \sigma^\lambda(t)]$$

where

$$\lim_{\epsilon\to 0}\sigma^\lambda(t) = 0.$$

Hence from the properties of the Fréchét differentials of y^i it follows that

$$\frac{\partial y^i(t)}{\partial t} = \frac{\partial \bar{y}^i(t)}{\partial t} + {}'y_\alpha^{\,i}[\bar{y}^\lambda(t)]\,\frac{\partial \bar{y}^\alpha(t)}{\partial t}$$

(6.3)

$$\frac{\partial^2 y^i(t)}{\partial t^2} = \frac{\partial^2 \bar{y}^i(t)}{\partial t^2} + {}''y^{i}{}_{\alpha\beta}[\bar{y}^\lambda(t)]\,\frac{\partial \bar{y}^\alpha(t)}{\partial t}\,\frac{\partial \bar{y}^\beta(t)}{\partial t} + {}'y_\alpha^{\,i}[\bar{y}^\lambda(t)]\,\frac{\partial^2 \bar{y}^\alpha(t)}{\partial t^2}.$$

A substitution in (6.1) of the expressions for $\partial y^i(t)/\partial t$ and $\partial^2 y^i(t)/\partial t^2$ as given by (6.3) gives

$$\frac{\partial^2 \bar{y}^i(t)}{\partial t^2} + {}'y_\alpha^{\,i}[\bar{y}]\,\frac{\partial^2 \bar{y}^\alpha(t)}{\partial t^2} = \theta^i(t) \tag{6.4}$$

where

$$\theta^i(t) = -\,({}''y^{i}{}_{\alpha\beta}[\bar{y}] + \Lambda^{i}{}_{\alpha\beta} + \Lambda^{i}{}_{\lambda\beta}\,{}'y_\alpha^{\,\lambda} + \Lambda^{i}{}_{\alpha\lambda}\,{}'y_\beta^{\,\lambda} + \Lambda^{i}{}_{\lambda\mu}\,{}'y_\alpha^{\,\lambda}\,{}'y_\beta^{\,\mu}$$
$$+ \mathrm{M}_\alpha^{\,i}\,{}'y_\beta^{\,\alpha} + \mathrm{M}_\beta^{\,i}\,{}'y_\alpha^{\,\beta} + \mathrm{M}_\lambda^{\,i}\,{}'y_\alpha^{\,\lambda}\,{}'y_\beta^{\,\lambda})\,\frac{\partial \bar{y}^\alpha(t)}{\partial t}\,\frac{\partial \bar{y}^\beta(t)}{\partial t} - \mathrm{M}_\alpha^{\,i}\left(\frac{\partial \bar{y}^\alpha(t)}{\partial t}\right)^2.$$

The solution of the Fredholm integral equation (6.4) is

$$\frac{\partial^2 \bar{y}^i(t)}{\partial t^2} + \bar{\Lambda}^{i}{}_{\alpha\beta}[\bar{y}^\lambda(t)]\,\frac{\partial \bar{y}^\alpha(t)}{\partial t}\,\frac{\partial \bar{y}^\beta(t)}{\partial t} + \bar{\mathrm{M}}_\alpha^{\,i}[\bar{y}^\lambda(t)]\left(\frac{\partial \bar{y}^\alpha(t)}{\partial t}\right)^2 = 0 \tag{6.5}$$

where

$$\bar{\Lambda}^{i}{}_{\alpha\beta}[\bar{y}^\lambda(t)] = \left[{i \atop \alpha\beta}\right] + {}'\bar{y}_\sigma^{\,i}[y]\left[{\sigma \atop \alpha\beta}\right], \tag{6.6}$$

$$\left[{i \atop \alpha\beta}\right] = {}''y^{i}{}_{\alpha\beta}[\bar{y}] + \Lambda^{i}{}_{\alpha\beta} + \Lambda^{i}{}_{\lambda\beta}\,{}'y_\alpha^{\,\lambda} + \Lambda^{i}{}_{\alpha\lambda}\,{}'y_\beta^{\,\lambda} + \Lambda^{i}{}_{\lambda\mu}\,{}'y_\alpha^{\,\lambda}\,{}'y_\beta^{\,\mu}$$
$$+ \mathrm{M}_\alpha^{\,i}\,{}'y_\beta^{\,\alpha} + \mathrm{M}_\beta^{\,i}\,{}'y_\alpha^{\,\beta} + \mathrm{M}_\lambda^{\,i}\,{}'y_\alpha^{\,\lambda}\,{}'y_\beta^{\,\lambda}$$

and

$$\bar{\mathrm{M}}_\alpha^{\,i}[\bar{y}^\lambda(t)] = \mathrm{M}_\alpha^{\,i} + {}'\bar{y}_\sigma^{\,i}[y]\,\mathrm{M}_\alpha^{\,\sigma}[y^\lambda(t)]. \tag{6.7}$$

It is to be observed that in the expression for $\left[{i \atop \alpha\beta}\right]$, the terms $\mathrm{M}_\alpha^{\,i}\,{}'y_\beta^{\,\alpha}$ and $\mathrm{M}_\beta^{\,i}\,{}'y_\alpha^{\,\beta}$ are to be understood in the light of an ordinary product—an exception to the integration convention for repeated indices. *Thus the form of equation* (6.1) *persists under a transformation* (6.2) *if and only if the functionals* $\Lambda^{i}{}_{\alpha\beta}$ *and* $\mathrm{M}_\alpha^{\,i}$ *transform according to the rules* (6.6) *and* (6.7) *respectively.*

In this paper we shall direct our attention mainly to equations (6.1) for which

$$\mathrm{M}_\alpha^{\,i} = 0,$$

a condition which persists under a change of variable y^i. The following important theorem is now immediate.

THEOREM (6. I). *A necessary and sufficient condition that the form of the integro-differential equation*

$$\frac{\partial^2 y^i(t)}{\partial t^2} + \Lambda^i_{\alpha\beta}[y^\lambda(t)]\,\frac{\partial y^\alpha(t)}{\partial t}\,\frac{\partial y^\beta(t)}{\partial t} = 0 \tag{6.8}$$

persist under a transformation of the variable y^i is that the functional $\Lambda^i_{\alpha\beta}$ have the law of transformation of an affine connection.

The solution of equation (6. 8), if it exists, defines a system of parameterized curves with a parameterization t in our general function space. Let us consider an affinely connected function space manifold determined by the affine connection $\Gamma^i_{\alpha\beta}[y]$. For different functionals $\Lambda^i_{\alpha\beta}$, equations (6. 8) give rise to various systems of curves in this affinely connected function space manifold. The particular system of parameterized curves which satisfy the equation

$$\frac{\partial^2 y^i(t)}{\partial t} + \Gamma^i_{\alpha\beta}[y^\lambda(t)]\,\frac{\partial y^\alpha(t)}{\partial t}\,\frac{\partial y^\beta(t)}{\partial t} = 0 \tag{6.9}$$

will be called the parameterized *geodesics* or the parameterized *paths* of our affinely connected function space manifold.

To obtain a formal solution of the integro-differential equation (6. 9) we write down the Taylor development

$$y^i(t) = y^i(t_0) + \left(\frac{\partial y^i(t)}{\partial t}\right)_{t=t_0}(t-t_0) + (1/2\,!)\left(\frac{\partial^2 y^i(t)}{\partial t^2}\right)_{t=t_0}(t-t_0)^2 + \cdots + \cdots \tag{6.10}$$

For initial conditions let us take

$$q^i = y^i(t_0) \qquad p^i = \left(\frac{\partial y^i(t)}{\partial t}\right)_{t=t_0} \tag{6.11}$$

where q^i and p^i are arbitrary continuous functions of the real variable i. One observes that (6. 9) gives the derivative $\left(\frac{\partial^2 y^i}{\partial t^2}\right)_{t=t_0}$ as a homogeneous quadratic functional of p^i. Hence the mth derivative of y^i taken for $t = t_0$ will be a homogeneous functional polynomial of degree m in p^i. Thus a formal solution of (6. 9) that takes on the initial conditions (6. 11) is given by the development

$$y^i(t) = q^i + p^i(t-t_0) - (1/2\,!)\,\Gamma^i_{\alpha\beta}[q^\lambda]\,p^\alpha p^\beta(t-t_0)^2 - (1/3\,!)\Gamma^i_{\alpha\beta\gamma}[q^\lambda]\,p^\alpha p^\beta p^\gamma(t-t_0)^3 - \cdots \tag{6.12}$$

where

$$(6.13)\quad \Gamma^{i}{}_{\alpha\beta\gamma}[q^{\lambda}] = (1/3)P('\Gamma^{i}{}_{\alpha\beta,\gamma}[q^{\lambda}] - \Gamma^{i}{}_{\sigma\beta}\Gamma^{\sigma}{}_{\alpha\gamma} - \Gamma^{i}{}_{\alpha\sigma}\Gamma^{\sigma}{}_{\beta\gamma})$$

and in general

$$(6.14)\quad \Gamma^{i}_{\alpha\beta\gamma\cdots\mu\nu}[q^{\lambda}] = (1/N)P\,('\Gamma^{i}_{\alpha\beta\gamma\cdots\mu,\nu}[q^{\lambda}] - \Gamma^{i}_{\sigma\beta\gamma\cdots\mu}[q^{\lambda}]\quad \Gamma^{\sigma}{}_{\alpha\nu}[q^{\lambda}] - \cdots \Gamma^{i}_{\alpha\beta\gamma\cdots\sigma}\quad \Gamma^{\sigma}{}_{\mu\nu}).$$

In (6.14), N denotes the number of variables $\alpha\,\beta\,\gamma\cdots\mu\,\nu$ and the symbol P stands for the operation that adds to the terms already in the parenthesis all terms obtainable from them by a cyclic permutation of the variables $\alpha\,\beta\,\gamma\cdots\mu\,\nu$.

The formal development in (6.12) is a power series in $(t - t_0)$

$$(6.15)\qquad a_0{}^{i} + a_1{}^{i}(t - t_0) + a_2{}^{i}(t - t_0)^2 + \cdots$$

in which the coefficients $a_0{}^{i}, a_1{}^{i}, \cdots,$ are continuous functions of the real variable i $(a \leq i \leq b)$. The development in (6.12) will then be uniformly and absolutely convergent in i and t for

$$a \leq i \leq b, \qquad t_0 \leq t \leq t_1$$

if $|\, t_1 - t_0 \,|$ is sufficiently small and the affine connection with its functional derivatives obeys certain obvious inequalities. Under these restrictions then, the formal development on the right side of (6.12) will represent a function analytic in t and continuous in i for t and i ranging over the respective intervals (t_0, t_1) and (a, b). This function $y^{i}(t)$ will then be the unique solution, analytic in t and continuous in i, of (6.9) under the initial conditions (6.11).

7. *Geodesic and normal functional coördinates.* The analysis in the theory of the affinely connected manifold can often be simplified by the introduction of suitable canonical variables. More definitely, the introduction of a functional coördinate system $\bar{y}^{i}$ such that

$$(7.1)\qquad (\Gamma^{i}{}_{\alpha\beta}[\bar{y}^{\lambda}])_{\bar{y}^{\lambda}=0} = 0$$

is frequently useful. We shall call such a system of functional coördinates $\bar{y}^{i}$ a *geodesic* system. The *origin* of this geodesic system of coördinates is the point P whose coördinates are $\bar{y}^{\lambda} = 0$. To show the existence of a geodesic system for each point P of our affinely connected manifold, let us consider the functional transformation

$$(7.2)\qquad y^{i} = q^{i} + \bar{y}^{i} - \tfrac{1}{2}\,\Gamma^{i}{}_{\alpha\beta}[q^{\lambda}]\,\bar{y}^{\alpha}\bar{y}^{\beta}$$

in which q^i is a given continuous function of i and represents the coördinates of a given point P in the y^i coördinate system. Clearly the set of values

$$y^i = q^i, \qquad \bar{y}^i = 0 \tag{7.3}$$

satisfy the functional equation (7. 2). Assuming that $\Gamma^i{}_{\alpha\beta}[q^\lambda]$ is such that (7. 2) is a transformation $\mathfrak{G}_s$, we see that (7. 2) is a transformation of coördinates such that

$$({}'y_\alpha{}^i\,[\bar{y}])_{\bar{y}\lambda=0} = 0, \qquad ({}''y_\alpha{}^i[\bar{y}])_{\bar{y}\lambda=0} = -\Gamma^i{}_{\alpha\beta}\,[q^\lambda]. \tag{7.4}$$

Hence (7. 1) follows from (5. 1) by means of the relations (7. 3) and (7. 4). Thus (7. 2) defines a geodesic coördinate system $\bar{y}^i$ with $\bar{y}^i = 0$ as origin. Such a system $\bar{y}^i$ is, however, merely one of an infinitude of geodesic systems; for, any transformation $\mathfrak{G}_s$ that satisfies conditions (7. 4) is a geodesic system.

An arbitrary transformation on the general coördinates y^i induces in general a complicated transformation in the geodesic variables $\bar{y}^i$. There is a particular type of geodesic coördinates, however, which suffer merely a Fredholm transformation.

If in equation (6. 12), we put

$$z^i = p^i\,(t - t_0) \tag{7.5}$$

we shall have the equation

$$y^i = q^i + z^i - (1/2\,!)\Gamma^i{}_{\alpha\beta}[q^\lambda]\,z^\alpha z^\beta - (1/3\,!)\Gamma^i{}_{\alpha\beta\gamma}[q^\lambda]\,z^\alpha z^\beta z^\gamma - \cdots - \cdots. \tag{7.6}$$

If the affine connection is suitably restricted, equation (7. 6) will be a transformation $\mathfrak{G}_\infty$ with an inverse

$$\begin{aligned} z^i = y^i - q^i &+ (1/2\,!)a^i{}_{\alpha\beta}(y^\alpha - q^\alpha)(y^\beta - q^\beta) \\ &+ (1/3\,!)a^i{}_{\alpha\beta\gamma}(y^\alpha - q^\alpha)(y^\beta - q^\beta)(y^\gamma - q^\gamma) + \cdots. \end{aligned} \tag{7.7}$$

The coördinates z^i defined by (7. 6) are a particular kind of geodesic coördinates. Such geodesic coördinates will be called *normal coördinates.* Equation (7. 5) for a fixed function p^i gives then the equation of a path (solution of (6. 9)) through the origin $y^i = q^i$ (i. e., $z^i = 0$) of the normal coördinate system z^i defined by (7. 6).

A functional transformation

$$\bar{y}^i = \bar{y}^i[y] \tag{7.8}$$

transforms the equations of the paths

$$y^i = y^i(t) \tag{7.9}$$

into

$$\bar{y}^i = \bar{y}^i(t). \tag{7.10}$$

The transformed equations (7.10) of the paths satisfy the integro-differential equation

$$(7.11) \qquad \frac{\partial^2 \bar{y}^i(t)}{\partial t^2} + \Gamma^i_{\alpha\beta}[\bar{y}^\lambda(t)]\, \frac{\partial \bar{y}^\alpha(t)}{\partial t}\, \frac{\partial \bar{y}^\beta(t)}{\partial t} = 0.$$

The transformation (7.8) changes the initial conditions (6.11) for equation (6.9) into

$$(7.12) \qquad \begin{aligned} \bar{q}^i &= \bar{y}^i(t_0) = \bar{q}^i[q^\lambda] \\ \bar{p}^i &= \left(\frac{\partial \bar{y}^i(t)}{\partial t}\right)_{t=t_0} \end{aligned}$$

for the transformed equation (7.11). Since

$$(7.13) \qquad \left(\frac{\partial \bar{y}^i(t)}{\partial t}\right)_{t=t_0} = \left(\frac{\partial y^i(t)}{\partial t} + {}'\bar{y}_\alpha{}^i\, [y^\lambda(t)]\, \frac{\partial y^\alpha(t)}{\partial t}\right)_{t=t_0}$$

it follows that

$$(7.14) \qquad \bar{p}^i = p^i + {}'\bar{y}_\alpha{}^i[q^\lambda]\, p^\alpha.$$

Multiplying both sides of (7.14) by $(t - t_0)$ we obtain

$$(7.15) \qquad \bar{z}^i = z^i + {}'\bar{y}_\alpha{}^i[q^\lambda]\, z^\alpha.$$

Since q^λ is a fixed function, it follows that (7.15) is a Fredholm transformation. Hence we have the important theorem.

THEOREM (7.I). *An arbitrary transformation $\mathfrak{G}_\infty$ in the coördinates y^i induces a Fredholm transformation in the normal coördinates z^i which are determined by the coördinates y^i and a point P with coördinates $y^i = q^i$.*

Let $N^i_{\alpha\beta}[z]$ be the expression for the affine connection in a normal coördinate system z^i. Then since (7.5) must satisfy

$$(7.16) \qquad \frac{\partial^2 z^i(t)}{\partial t^2} + N^i_{\alpha\beta}[z^\lambda(t)]\, \frac{\partial z^\alpha(t)}{\partial t}\, \frac{\partial z^\beta(t)}{\partial t} = 0$$

we see that

$$(7.17) \qquad N^i_{\alpha\beta}[z^\lambda(t)]\, p^\alpha p^\beta = 0.$$

Consequently we have the result

$$(7.18) \qquad N^i_{\alpha\beta}[o] = 0.$$

8. *A generalization of covariant functional differentiation.* In § 5 we found by means of elimination processes a functional operation, covariant functional differentiation, which produced a functional tensor of one higher order than the tensor on which it operated. Furthermore we saw in the same paragraph that a slightly different functional operation from covariant func-

tional differentiation gave rise to a functional tensor when it operated on a functional with a law of transformation (5. 13) in which $\eta_{(a)} = 1$. It is the purpose of this paragraph to show that functional tensors of any order can be built up by means of functional operations which include those of § 5 as a special case. We shall use the theory of normal coördinates. It should be observed, however, that the final results *do not* depend for their validity on the actual existence or non-existence of normal coördinates—even if the series (7. 6) does not converge we can use the formal theory of normal coördinates as a *heuristic guide* to obtain the final results, the validity of which can be established by the elimination processes of § 5.

Let $f^i{}_{\alpha\beta}[y^\lambda]$ be a continuous functional of y^λ that has a differential of the Volterra form

$$\delta f^i{}_{\alpha\beta}[y^\lambda] = {}'f^i{}_{\alpha\beta,\gamma}[y^\lambda]\, \delta y^\gamma \tag{8. 1}$$

and a definite law of transformation

$$\begin{aligned}\bar{f}^i{}_{\alpha\beta}[\bar{y}] = {} & f^i{}_{\alpha\beta}[y] + f_\sigma{}^i{}_\beta{}'y_\alpha{}^\sigma[\bar{y}] + f^i{}_{\alpha\sigma}{}'y_\beta{}^\sigma + f^i{}_{\sigma\tau}{}'y_\alpha{}^\sigma\, {}'y_\beta{}^\tau + \rho^i{}_{\alpha\beta} \\ & + {}'\bar{y}_\lambda{}^i (f^\lambda{}_{\alpha\beta} + f^\lambda{}_{\sigma\beta}\, {}'y_\alpha{}^\sigma + f^\lambda{}_{\alpha\sigma}{}'y_\beta{}^\sigma + f^\lambda{}_{\sigma\tau}{}'y_\alpha{}^\sigma{}'y_\beta{}^\tau + \rho^\lambda{}_{\alpha\beta}).\end{aligned} \tag{8. 2}$$

The quantity $\rho^i{}_{\alpha\beta}$ in (8. 2) is assumed to be independent of any normal coördinates z^i whenever it is referred to such a system of coördinates. Let $\phi^i{}_{\alpha\beta}[z^\lambda]$ be the expression for $f^i{}_{\alpha\beta}$ when it is referred to a system of normal coördinates z^i which is determined by the arbitrary coördinates y^i and a point P with coördinates q^i. By calculation we obtain

$${}'\bar{\phi}^i{}_{\alpha\beta,\gamma}[z] = \left({}^{\;i}_{\alpha\beta\gamma}\right) + {}'\bar{z}_\lambda{}^i \left({}^{\;\lambda}_{\alpha\beta\gamma}\right) \tag{8. 3}$$

in which

$$\begin{aligned}\left({}^{\;i}_{\alpha\beta\gamma}\right) = {} & {}'\phi^i{}_{\alpha\beta,\gamma}[z] + {}'\phi^i{}_{\sigma\beta,\gamma}{}'z_\alpha{}^\sigma + {}'\phi^i{}_{\alpha\sigma,\gamma}{}'z_\beta{}^\sigma + {}'\phi^i{}_{\alpha\beta,\sigma}{}'z_\gamma{}^\sigma + {}'\phi^i{}_{\alpha\sigma,\tau}{}'z_\beta{}^\sigma\, {}'z_\gamma{}^\tau \\ & + {}'\phi^i{}_{\sigma\beta,\tau}{}'z_\alpha{}^\sigma\, {}'z_\gamma{}^\tau + {}'\phi^i{}_{\sigma\tau,\gamma}{}'z_\alpha{}^\sigma\, {}'z_\beta{}^\tau + {}'\phi^i{}_{\sigma\tau,\nu}{}'z_\alpha{}^\sigma\, {}'z_\beta{}^\tau\, {}'z_\gamma{}^\nu.\end{aligned}$$

From the arbitrariness of the point P with coördinates $y^i = q^i$, from (8. 3) and from (7. 15) we see that the functional $f^i{}_{\alpha\beta;\gamma}[y]$ of y^λ defined by

$$f^i{}_{\alpha\beta;\gamma} = ({}'\phi^i{}_{\alpha\beta,\gamma}[z])_{z^\lambda=0} \tag{8. 4}$$

is a functional tensor covariant of order three and contravariant of order one.

The explicit form of the functional $f^i{}_{\alpha\beta;\gamma}[y]$ in terms of the affine connection $\Gamma^i{}_{\alpha\beta}[y]$ and its functional derivatives of course depends on the quantity $\rho^i{}_{\alpha\beta}$ in (8. 2). We shall treat presently a few important cases. For the case

$$\rho^i{}_{\alpha\beta} = {}''y^i{}_{\alpha\beta}[\bar{y}] \tag{8. 5}$$

we have

$$\phi^i{}_{\alpha\beta}[z] + \phi^\sigma{}_{\alpha\beta}[z]\,{}'y_\sigma{}^i[z] = {}''y^i{}_{\alpha\beta}[z] + f^i{}_{\alpha\beta}[y] + f^i{}_{\sigma\beta}\,{}'y_\alpha{}^\sigma[z] + f^i{}_{\alpha\sigma}\,{}'y_\beta{}^\sigma + f^i{}_{\sigma\tau}\,{}'y_\alpha{}^\sigma\,{}'y_\beta{}^\tau. \tag{8.6}$$

Hence

$$ {}'\phi^i{}_{\alpha\beta,\gamma}[z] = {}'''y^i{}_{\alpha\beta\gamma}[z] + {}'f^i{}_{\alpha\beta,\gamma}[y] + f^i{}_{\sigma\beta}\,{}''y^\sigma{}_{\alpha\gamma}[z] + f^i{}_{\alpha\sigma}\,{}''y^\sigma{}_{\beta\gamma} - \phi^\sigma{}_{\alpha\beta}\,{}''y^i{}_{\sigma\gamma}[z] + * \tag{8.7}$$

where the $*$ stands for terms which vanish with ${}'y_\alpha{}^\sigma[z]$. Formula (7.6) shows that at the origin of normal coördinates z^i we have

$$ {}'y_\alpha{}^\sigma = 0; \quad {}''y^\sigma{}_{\alpha\beta} = -\Gamma^\sigma{}_{\alpha\beta}; \quad {}'''y^\sigma{}_{\alpha\beta\gamma} = -\Gamma^\sigma{}_{\alpha\beta\gamma}. \tag{8.8}$$

With the aid of these formulae as well as (8.6) and (8.7) we finally obtain

$$f^i{}_{\alpha\beta;\gamma}[y] = {}'f^i{}_{\alpha\beta,\gamma}[y] - \Gamma^i{}_{\alpha\beta\gamma} - f^i{}_{\sigma\beta}\Gamma^\sigma{}_{\alpha\gamma} - f^i{}_{\alpha\sigma}\Gamma^\sigma{}_{\beta\gamma} + f^\sigma{}_{\alpha\beta}\Gamma^i{}_{\sigma\gamma} - \Gamma^\sigma{}_{\alpha\beta}\Gamma^i{}_{\sigma\gamma}. \tag{8.9}$$

It is easy to show that the case

$$\rho^i{}_{\alpha\beta} \equiv 0$$

yields the covariant functional derivative $f^i{}_{\alpha\beta;\gamma}[y]$ of the functional tensor $f^i{}_{\alpha\beta}[y]$.

Suppose now that a functional $f^{ij\cdots k}_{ab\cdots c}[y]$ has differentials of the Volterra form

$$ {}^{(\alpha)}f^{ij\cdots k}_{ab\cdots c,\,\lambda_1\lambda_2\cdots\lambda_\alpha}[y]\quad \delta y^{\lambda_1}\,\delta y^{\lambda_2}\cdots\delta^{\lambda_\alpha} \qquad (\alpha = 1, 2, \cdots, p) \tag{8.10}$$

in which the functional derivatives ${}^{(\alpha)}f$ are continuous functionals of y^i and continuous functions of $ij\cdots k\,ab\cdots c\,\lambda_1\cdots\lambda_\alpha$. Let $\phi^{ij\cdots k}_{ab\cdots c}[z]$ represent the expression for $f^{ij\cdots k}_{ab\cdots c}$ in a normal coördinate system z^i. Let us assume that the law of transformation of the functional $f^{ij\cdots k}_{ab\cdots c}[y]$ is such that ${}^{(p)}\phi^{ij\cdots k}_{ab\cdots c,\,\lambda_1\lambda_2\cdots\lambda_p}[y]$ transforms like a functional tensor under the Fredholm group (7.15) of the normal coördinates z^i. *The functional* $f^{ij\cdots k}_{ab\cdots c;\,\lambda_1\lambda_2\cdots\lambda_p}[y]$ *of* y^i *defined by*

$$f^{ij\cdots k}_{ab\cdots c;\,\lambda_1\lambda_2\cdots\lambda_p}[q] = \left({}^{(p)}\phi^{ij\cdots k}_{ab\cdots c,\,\lambda_1\lambda_2\cdots\lambda_p}[z]\right)_P \tag{8.11}$$

is a functional tensor with $ij\cdots k$ *as contravariant indices and* $a\cdots c\lambda_1\cdots\lambda_p$ *as covariant indices.—In* (8.11) *we understand that the functional* ${}^{(p)}\phi$ *is evaluated at the origin* P *of the normal coördinates* z^i. The functional $f^{ij\cdots k}_{ab\cdots c;\,\lambda_1\cdots\lambda_p}$ will be said to be the *pth extension* of the functional $f^{ij\cdots k}_{ab\cdots c}$. When $p = 1$ and $f^{ij\cdots k}_{ab\cdots c}[y]$ is a functional tensor, then $f^{ij\cdots k}_{ab\cdots c}$ is merely the covariant functional derivative of the functional tensor $f^{ij\cdots k}_{ab\cdots c}$. We observe that the pth successive covariant functional derivative $f^{ij\cdots k}_{ab\cdots c;\lambda_1;\lambda_2;\cdots;\lambda_p}$ is not in general the same as the pth extension of the functional $f^{ij\cdots k}_{ab\cdots c}$. The proof that the functional $f^{ij\cdots k}_{ab\cdots c;\lambda_1\cdots\lambda_p}$ defined by (8.11) is a functional tensor is obvious on the basis of our assumptions about the functional $f^{ij\cdots k}_{ab\cdots c}$. It is interesting to observe that if the differentials of the functional $f^{ij\cdots k}_{ab\cdots c}$ are not of the Volterra form [see (8.10)], the functional defined by (8.11) will not be in general a functional tensor. This can be seen most easily from the fact that the functional derivatives ${}^{(p)}\phi^{ij\cdots k}_{ab\cdots c,\lambda_1\cdots\lambda_p}[z]$ will not transform like functional tensors under the Fredholm group (7.15). As an example we may take the case of a covariant functional vector $\xi_a[y]$ with a differential (5.10). The functional $\xi_{a;\beta}[y]$ defined by the process (8.11) will obey the non-tensor law (5.13).

Any functional $f^{ij\cdots k}_{ab\cdots c;\lambda_1\cdots\lambda_p}$ defined by (8.11) can be expressed as functional of $f^{ij\cdots k}_{ab\cdots c}$, $\Gamma^i_{\alpha\beta}$, and their functional derivatives. The procedure is similar to the simple case (8.9) already treated.

9. *Normal functional tensors and replacement theorems.* The curvature tensor $B^i_{\alpha\beta\gamma}$ and its successive covariant functional derivatives are, as we have seen, a sequence of affine functional invariants. There is, however, another important sequence of tensors each of which is an affine functional invariant. These affine functional invariants $A^i_{\alpha\beta\gamma_1\cdots\gamma_p}[y]$ are defined by

$$(9.1)\qquad A^i_{\alpha\beta\gamma_1\cdots\gamma_p} = A^i_{\alpha\beta;\gamma_1\cdots\gamma_p}$$

where

$$(9.2)\qquad A^i_{\alpha\beta;\gamma_1\cdots\gamma_p}[q] = \left({}^{(p)}N_{\alpha\beta,\gamma_1\cdots\gamma_p}[z]\right)_P .$$

In (9.2), we understand that $N^i_{\alpha\beta}[z]$ is the affine connection $\Gamma^i_{\alpha\beta}[y]$ referred to a system of normal coördinates z^i with origin at the point P. It follows from the discussion of the last paragraph that the functionals $A^i_{\alpha\beta\gamma_1\cdots\gamma_p}[y]$ are functional tensors covariant of order $p+2$ and contravariant of order one. We shall call the functional $A^i_{\alpha\beta\gamma_1\cdots\gamma_p}[y]$ the *pth normal affine functional tensor.*

Let us assume now that the affine connection $N^i_{\alpha\beta}[z]$ possesses a uniformly and absolutely convergent Volterra development about the functional neighborhood of the origin P of the normal coördinates z^i. Since $N^i_{\alpha\beta}$ vanishes at the origin of normal coördinates, it follows that

$$(9.3)\qquad N^i_{\alpha\beta}[z] = A^i_{\alpha\beta\gamma}[q]\, z^\gamma + (1/2!)A^i_{\alpha\beta\gamma\delta}[q]\, z^\gamma z^\delta + (1/3!)A^i_{\alpha\beta\gamma\delta\epsilon} z^\gamma z^\delta z^\epsilon + \cdots + \cdots .$$

Hence the sequence of normal functional tensors A constitutes a *complete set of affine functional invariants* in the sense that the affine connection $N^i{}_{\alpha\beta}[z]$ is completely determined by them, which in turn determines the affinely connected manifold.

Equation (7.17) shows that

$$N^i{}_{\alpha\beta}[z] z^\alpha z^\beta = 0. \tag{9.4}$$

Consequently

$$A^i{}_{\alpha\beta\gamma} z^\alpha z^\beta z^\gamma + (1/2!)\, A^i{}_{\alpha\beta\gamma\delta} z^\alpha z^\beta z^\gamma z^\delta + \cdots = 0. \tag{9.5}$$

The functional derivatives of (9.5) must vanish identically in the arbitrary function z^i. In particular they must vanish at the origin $z^i = 0$. Hence

$$A^i{}_{\alpha\beta\gamma}[y] + A^i{}_{\gamma\alpha\beta} + A^i{}_{\beta\gamma\alpha} = 0 \tag{9.6}$$

and in general

$$S(A^i{}_{\alpha\beta\gamma\ldots\omega}) = 0 \tag{9.7}$$

in which S stands for the operation that adds to the functional $A^i{}_{\alpha\beta\gamma\ldots\omega}$ all functionals obtained from it by a permutation of the variables $\alpha\beta\gamma \cdots \omega$. It is obvious from the definition (9.2) that

$$A^i{}_{\alpha\beta\gamma\ldots\omega} = A^i{}_{\beta\alpha\gamma\ldots\omega}, \tag{9.8}$$

$$A^i{}_{\alpha\beta\gamma\ldots\omega} = A^i{}_{\alpha\beta\lambda\ldots\rho}, \tag{9.9}$$

where $\lambda \cdots \rho$ is any permutation of the variables $\gamma \cdots \omega$.

As a special case of formula (8.9) we have

$$A^i{}_{\alpha\beta\gamma}[y] = {}'\Gamma^i{}_{\alpha\beta,\gamma}[y] - \Gamma^i{}_{\alpha\beta\gamma} - \Gamma^i{}_{\sigma\beta}\Gamma^\sigma{}_{\gamma\alpha} - \Gamma^i{}_{\sigma\alpha}\Gamma_{\beta\gamma}{}^\sigma. \tag{9.10}$$

To obtain the corresponding formula for the pth normal tensor one need only employ the same method that led to formula (8.9).

It is possible to express every affine tensor functional invariant as a functional of the normal tensors A alone. In fact let us consider an affine tensor invariant T of order r and weight w. We have

$$\begin{aligned} T^{ij\cdots l}_{ab\cdots c}\big[N^\sigma{}_{\alpha\beta}[z],\ {}'N^\sigma{}_{\alpha\beta,\gamma_1}, \cdots, {}^{(r)}N^\sigma{}_{\alpha\beta,\gamma_1\ldots\gamma_r}\big] \\ = (D[{}'y_\nu{}^\mu])^\omega\,(T[\Gamma[y], \cdots], {}'y). \end{aligned} \tag{9.11}$$

In (9.11), $N^i{}_{\alpha\beta}[z]$ stands for the affine connection $\Gamma^i{}_{\alpha\beta}[y]$ referred to a system or normal coördinates z^i determined by the arbitrary coördinates z^i and an origin $y^i = q^i$. If we evaluate (9.11) at the origin $y^i = q^i$, we obtain the relation

$$\begin{aligned} T^{ij\cdots l}_{ab\cdots c}\big[0,\ A^\sigma{}_{\alpha\beta\gamma_1}[q], \cdots, \mathrm{A}^\sigma{}_{\alpha\beta\gamma_1\ldots\gamma_r}\big] \\ = T^{ij\cdots l}_{ab\cdots c}\big[\Gamma^\sigma{}_{\alpha\beta}[q],\ {}'\Gamma^\sigma{}_{\alpha\beta,\gamma_1}, \cdots, {}^{(r)}\Gamma^\sigma{}_{\alpha\beta,\gamma_1\ldots\gamma_r}\big] \end{aligned}$$

since the Fredholm determinant $D[{}'y_\nu{}^\mu]$ has the value one at the origin of normal coördinates. The following theorem follows immediately.

THEOREM (9. I). *Any affine tensor functional invariant of order r (relative or absolute) can be put into the form*

$$(9.12) \qquad T^{ij\,\cdots\, l}_{ab\,\cdots\, c}[0,\ A^\sigma{}_{\alpha\beta\gamma_1}, \cdots,\ A^\sigma{}_{\alpha\beta\gamma_1\ldots\gamma_r}]$$

by replacing the affine connection $\Gamma^i{}_{\alpha\beta}$ by zero and the functional derivatives of $\Gamma^i{}_{\alpha\beta}$ by the corresponding affine normal functional tensors $A^i{}_{\alpha\beta\gamma\ldots\delta}$.

Corresponding theorems may be proved for affine invariants that are not tensors. For example, if an affine functional invariant $f^i{}_{\alpha\beta}$ of order r and weight w has the law of transformation

$$(9.13) \qquad f^i{}_{\alpha\beta}[\bar\Gamma^\sigma{}_{ij},\ \bar\Gamma^\sigma{}_{ij,k_1}, \cdots,\ {}^{(r)}\bar\Gamma^\sigma{}_{ij,k_1\ldots k_r}] = (D[{}'y_\nu{}^\mu])^w \left[\binom{i}{\alpha\beta} + {}'\bar y_\lambda{}^i \binom{\lambda}{\alpha\beta} \right],$$

where

$$\binom{i}{\alpha\beta} = {}''y^i{}_{\alpha\beta} + f^i{}_{\alpha\beta}[\Gamma^\sigma{}_{ij}, \cdots,\ {}^{(r)}\Gamma^\sigma{}_{ij,k_1\ldots k_r}] + f^i{}_{\rho\beta}{}'y_\alpha{}^\rho + f^i{}_{\alpha\rho}{}'y_\beta{}^\rho + f^i{}_{\rho\phi}{}'y_\alpha{}^\rho\,{}'y_\beta{}^\phi,$$

it can be put in the form

$$(9.14) \qquad f^i{}_{\alpha\beta}[0,\ A^\sigma{}_{ijk_1}, \cdots,\ A^\sigma{}_{ijk_1\ldots k_r}] - \Gamma^i{}_{\alpha\beta}.$$

Furthermore, theorem (9. I) holds also for the case of an affine invariant $f_{\alpha\beta}[\Gamma^i{}_{\gamma\delta},\ {}'\Gamma^i{}_{\gamma\delta,\epsilon_1}, \cdots,\ {}^{(r)}\Gamma^i{}_{\gamma\delta,\epsilon_1\ldots\epsilon_r}]$ which transforms like the functional $\xi_{i;\beta}$ in (5. 13).

Before we turn our attention to the study of a particular type of affinely connected manifold, it is worth while to observe that the problem of the *equivalence* of two affinely connected function space manifolds is a difficult problem in the theory of functional algebraic forms (see § 2) and in the theory of functional equations with functional derivatives.

10. *Quadratic differential functional forms and their functional invariants.* The theory of quadratic differential functional forms is of fundamental importance in a theory of Riemannian function space manifolds.

DEFINITION. *A functional expression*

$$(10.1) \qquad g_{\alpha\beta}[y]\,\delta y^\alpha\,\delta y^\beta + g_\alpha[y]\,(\delta y^\alpha)^2;\quad g_{\alpha\beta}[y] = g_{\beta\alpha}[y]$$

will be said to be an absolute quadratic differential functional form if

$$(10.2)\qquad \bar{g}_{\alpha\beta}[\bar{y}]\,\delta\bar{y}^{\alpha}\,\delta\bar{y}^{\beta}+\bar{g}_{\alpha}[\bar{y}]\,(\delta\bar{y}^{\alpha})^{2}=g_{\alpha\beta}[y]\,\delta y^{\alpha}\,\delta y^{\beta}+g_{\alpha}[y]\,(\delta y^{\alpha})^{2}$$

holds under an arbitrary functional transformation $\mathfrak{G}_s$.

We shall restrict ourselves in this paper to quadratic forms

$$(10.3)\qquad g_{\alpha\beta}[y]\,\delta y^{\alpha}\,\delta y^{\beta}+\int_{a}^{b}(\delta y^{\alpha})^{2}\,d\alpha;\quad g_{\alpha\beta}[y]=g_{\beta\alpha}[y]$$

in which $g_{\alpha\beta}$ is a continuous functional of y^{i} and a continuous function of the variables α and β. A necessary and sufficient condition that

$$(10.4)\qquad \bar{g}_{\alpha\beta}[\bar{y}]\,\delta\bar{y}^{\alpha}\,\delta\bar{y}^{\beta}+\int_{a}^{b}(\delta\bar{y}^{\alpha})^{2}\,d\alpha=g_{\alpha\beta}[y]\,\delta y^{\alpha}\,\delta y^{\beta}+\int_{a}^{b}(\delta y^{\alpha})^{2}\,d\alpha$$

under an arbitrary transformation $\mathfrak{G}_s$ is that

$$(10.5)\qquad \begin{aligned}\bar{g}_{\alpha\beta}[\bar{y}]=g_{\alpha\beta}[y]+g_{\sigma\beta}{}'y_{\alpha}{}^{\sigma}[\bar{y}]+g_{\alpha\sigma}{}'y_{\beta}{}^{\sigma}+g_{\sigma\tau}{}'y_{\alpha}{}^{\sigma}\,{}'y_{\beta}{}^{\tau}&\\ +{}'y_{\beta}{}^{\alpha}+{}'y_{\alpha}{}^{\beta}+\int_{a}^{b}{}'y_{\alpha}{}^{\sigma}\,{}'y_{\beta}{}^{\sigma}\,d\sigma.&\end{aligned}$$

On the basis of theorem (2. II) one can easily prove the following theorem.

THEOREM (10. I). *The Fredholm determinant* $D[g_{\alpha\beta}]$ *of the functional coefficients* $g_{\alpha\beta}[y]$ *of the form* (10. 3) *is a relative scalar invariant of weight two;* i. e.,

$$(10.6)\qquad D[\bar{g}_{\alpha\beta}]=(D[{}'y_{\mu}{}^{\lambda}])^{2}\,D[g_{\alpha\beta}]$$

We shall assume throughout our whole discussion that the Fredholm determinant

$$(10.7)\qquad D[g_{\alpha\beta}]\neq 0.$$

We observe that the Fredholm determinant $D[g_{\alpha\beta}]$ is a continuous functional of y^{i}. Hence if it does not vanish at a point $y^{i}=q^{i}$, it will continue not to vanish in the functional neighborhood of the point q^{i}.

THEOREM (10. II). *The resolvent kernel* $g^{\alpha\beta}[g_{\lambda\mu}]$ *of the functional coefficients of the form* (10. 3) *is an absolute functional invariant of the quadratic differential functional form* (10. 3) *while the first minor* $D^{\alpha\beta}[g_{\lambda\mu}]$ *of the Fredholm determinant* $D[g_{\lambda\mu}]$ *is a relative functional invariant of weight two.* The *law of transformation of* $g^{\alpha\beta}$ *and* $D^{\alpha\beta}$ *is given respectively by*

$$(10.8)\qquad \left\{\begin{aligned}g^{\alpha\beta}[\bar{g}_{\lambda\mu}]=g^{\alpha\beta}[g_{\lambda\mu}]+g^{\sigma\beta}\,{}'\bar{y}_{\sigma}{}^{\alpha}[y]+g^{\alpha\sigma}\,{}'\bar{y}_{\sigma}{}^{\beta}+g^{\sigma\tau}\,{}'\bar{y}_{\sigma}{}^{\alpha}\,{}'\bar{y}_{\tau}{}^{\beta}&\\ +{}'\bar{y}_{\beta}{}^{\alpha}+{}'\bar{y}_{\alpha}{}^{\beta}+\int_{a}^{b}{}'\bar{y}_{\sigma}{}^{\alpha}\,{}'\bar{y}_{\sigma}{}^{\beta}\,d\sigma,&\end{aligned}\right.$$

$$(10.9)\qquad \left\{\begin{aligned}&D^{\alpha\beta}[\bar{g}_{\lambda\mu}]=(D[{}'y_{j}{}^{i}])^{2}\,\Big[D^{\alpha\beta}[g_{\lambda\mu}]+D^{\sigma\beta}\,{}'\bar{y}_{\sigma}{}^{\alpha}+D^{\alpha\sigma}\,{}'\bar{y}_{\sigma}{}^{\beta}\\ &\quad+D^{\sigma\tau}\,{}'\bar{y}_{\sigma}{}^{\alpha}\,{}'\bar{y}_{\tau}{}^{\beta}-D[g_{\alpha\beta}]\,\Big({}'\bar{y}_{\beta}{}^{\alpha}+{}'\bar{y}_{\alpha}{}^{\beta}+\int_{a}^{b}{}'\bar{y}_{\sigma}{}^{\alpha}\,{}'\bar{y}_{\sigma}{}^{\beta}\,d\sigma\Big)\Big].\end{aligned}\right.$$

In order to study the functional differential form (10. 3) effectively, we shall assume that the functional $g_{\alpha\beta}[y]$ has differentials of all orders of the Volterra form. We shall make the further desirable restriction that all the functional derivatives involved in the differentials are continuous functionals of y^i as well as continuous functions of the variables on which they depend in particular.

The analogue of the *Christoffel symbols* of the first kind is the functional $[\alpha\beta, \gamma]$ that is defined by

$$(10.10)\qquad [\alpha\beta, \gamma] = \tfrac{1}{2}({}'g_{\gamma\alpha,\beta} + {}'g_{\beta\gamma,\alpha} - {}'g_{\alpha\beta,\gamma})$$

in which ${}'g_{\alpha\beta,\gamma}[y]$ is the functional derivative of the coefficients $g_{\alpha\beta}[y]$ of the form (10. 3). It is a consequence of (10. 5) that the law of transformation of the functional $[\alpha\beta, \gamma]$ is

$$(10.11)\qquad \begin{aligned} \overline{[\alpha\beta, \gamma]} = {} & [\alpha\beta, \gamma] + [\lambda\beta, \gamma]\, {}'y_\alpha^\lambda[\bar{y}] + [\alpha\lambda, \gamma]\, {}'y_\beta^\lambda + [\alpha\beta, \lambda]\, {}'y_\gamma^\lambda \\ & + [\alpha\lambda, \mu]\, {}'y_\alpha^\lambda\, {}'y_\beta^\mu + [\lambda\beta, \mu]\, {}'y_\alpha^\lambda\, {}'y_\gamma^\mu + [\lambda\mu, \gamma]\, {}'y_\alpha^\lambda\, {}'y_\beta^\mu \\ & + [\lambda\mu, \nu]\, {}'y_\alpha^\lambda\, {}'y_\beta^\mu\, {}'y_\gamma^\nu + g_{\lambda\gamma}\, {}''y^\lambda_{\alpha\beta} + g_{\lambda\mu}\, {}'y_\gamma^\lambda\, {}''y^\mu_{\alpha\beta} \\ & + {}''y^\gamma_{\alpha\beta} + \int_a^b {}'y_\gamma^\lambda\, {}''y^\lambda_{\alpha\beta}\, d\lambda. \end{aligned}$$

In (10. 11) we understand, of course, that

$$\overline{[\alpha\beta, \gamma]} = \tfrac{1}{2}({}'\bar{g}_{\gamma\alpha,\beta}[\bar{y}] + {}'\bar{g}_{\beta\gamma,\alpha}[\bar{y}] - {}'\bar{g}_{\alpha\beta,\gamma}[\bar{y}]).$$

The functional $\Gamma^i_{\alpha\beta}[y]$ defined by

$$(10.12)\qquad \Gamma^i_{\alpha\beta} = [\alpha\beta, i] + g^{i\sigma}[g_{\lambda\mu}]\, [\alpha\beta, \sigma]$$

is an affine connection uniquely determined by the quadratic functional differential form (10. 3). The notation $\Gamma^i_{\alpha\beta}$ need not cause any confusion, since throughout this paragraph and the next we shall be concerned exclusively with affinely connected manifolds whose affine connection is uniquely determined by a quadratic form (10. 3).

Since $g^{\alpha\beta}$ is the resolvent kernel of $g_{\alpha\beta}[y]$, it follows readily from the theory of the Fredholm integral equation that the *tensor* functional identity

$$(10.13)\qquad g_{\alpha\beta} + g^{\alpha\beta} + g_{\alpha\sigma}g^{\sigma\beta} = 0$$

holds for all continuous functions y^i which are in a functional neighborhood of a point for which the Fredholm determinant $D[g_{\alpha\beta}]$ does not vanish. By means of (10. 8), (10. 11), (10. 12), (10. 13), (3. 5) and (3. 6) we obtain by calculation the law of transformation

$$(10.14)\qquad \bar{\Gamma}^i{}_{\alpha\beta} + \bar{\Gamma}^\sigma{}_{\alpha\beta}{}'\bar{y}_\sigma{}^i[\bar{y}] = {}''y^i{}_{\alpha\beta}[\bar{y}] + \Gamma^i{}_{\alpha\beta} + \Gamma^i{}_{\sigma\beta}{}'y_\alpha{}^\sigma + \Gamma^i{}_{\alpha\sigma}{}'y_\beta{}^\sigma + \Gamma^i{}_{\sigma\tau}{}'y_\alpha{}^\sigma{}'y_\beta{}^\tau$$

in which

$$\bar{\Gamma}^i{}_{\alpha\beta} = \overline{[\alpha\beta, i]} + \bar{g}^{i\sigma}[\bar{y}]\,\overline{[\alpha\beta, \sigma]}.$$

This then shows the truth of our assertion about the functional $\Gamma^i{}_{\alpha\beta}$ as defined by (10.12).

The identity (10.13) enables us to re-write the defining relation (10.12) in the form

$$(10.15)\qquad [\alpha\beta, i] = g_{i\lambda}\,\Gamma^\lambda{}_{\alpha\beta} + \Gamma^i{}_{\alpha\beta}.$$

We have thus expressed the Christoffel symbol $[\alpha\beta, i]$ of the first kind in terms of the Christoffel symbol $\Gamma^i{}_{\alpha\beta}$ of the second kind.

It is perhaps worth while to mention explicitly a few properties of the functional $g^{\alpha\beta}$ considered as a functional of y^i. The differential of $g^{\alpha\beta}$ is of the Volterra form

$$(10.16)\qquad \delta g^{\alpha\beta}[y] = {}'g_\gamma{}^{\alpha\beta}\,\delta y^\gamma.$$

This can be seen most easily as follows. By definition we have

$$(10.17)\qquad g^{\alpha\beta} = -\frac{D^{\alpha\beta}[g_{\lambda\mu}]}{D[g_{\lambda\mu}]}$$

in which $D^{\alpha\beta}[g_{\lambda\mu}]$ is the first minor of the Fredholm determinant $D[g_{\lambda\mu}]$. It is known* that the differentials of the functionals $D^{\alpha\beta}$ and D, considered as functionals of $g_{\lambda\mu}$, are linear homogeneous functionals of the differentials $\delta g_{\lambda\mu}$. But $g_{\lambda\mu}$ has a differential of the Volterra form. Hence the differential of $g^{\alpha\beta}$ when we consider $g^{\alpha\beta}$ as a functional of y^i takes the form (10.16).

To develop the functional invariant theory of the quadratic form (10.3) effectively we must impose further restrictions on the functional $g_{\alpha\beta}[y]$. Let us assume that $g_{\alpha\beta}$ has not only differentials of all orders but that it is given by an absolutely and uniformly convergent Volterra development

$$(10.18)\qquad g_{\alpha\beta}[q^i + y^i] = g_{\alpha\beta}[q^i] + {}'g_{\alpha\beta,\gamma}[q^i]\,y^\gamma + (1/2!)''g_{\alpha\beta,\gamma\delta}[q^i]\,y^\gamma y^\delta + (1/3!)'''g_{\alpha\beta,\gamma\delta\epsilon}[q^i]\,y^\gamma y^\delta y^\epsilon + \cdots$$

about an arbitrarily given point q^i in our function space.

Since the functional $\Gamma^i{}_{\alpha\beta}[y]$ possesses functional derivatives and differentials of all orders, we can construct the curvature tensor $B^i{}_{\alpha\beta\gamma}$ and its successive covariant functional derivatives on the basis of the particular affine connection (10.12). The functional $B_{\alpha\beta\gamma\delta}[y]$ defined by

$$(10.19)\qquad B_{\alpha\beta\gamma\delta} = g_{\alpha\sigma}\,B^\sigma{}_{\beta\gamma\delta} + B^\alpha{}_{\beta\gamma\delta}$$

---

* Cf. G. C. Evans, *American Mathematical Monthly*, Vol. 34 (1927), p. 148. See also Fredholm, *loc. cit.*

will be called the covariant form of the curvature tensor $B^i{}_{\alpha\beta\gamma}$. One can show by a direct calculation that $B_{\alpha\beta\gamma\delta}$ is a covariant tensor of order four.

By means of (10.13) we can re-write formula (10.19) in the form

$$B^\alpha{}_{\beta\gamma\delta} = g^{\alpha\sigma} B_{\sigma\beta\gamma\delta} + B_{\alpha\beta\gamma\delta}. \tag{10.20}$$

Relations (10.19) and (10.20) are special cases of a general result. In fact, if $T^{i \cdots j \cdots k}_{a \cdots b \cdots c}$ is a functional tensor covariant of order l and contravariant of order m, then

$$g_{j\sigma} T^{i \cdots \sigma \cdots k}_{a \cdots b \cdots c} + T^{i \cdots j \cdots k}_{a \cdots b \cdots c} \tag{10.21}$$

is a functional tensor covariant of order $l+1$ and contravariant of order $m-1$ while

$$g^{b\sigma} T^{i \cdots j \cdots k}_{a \cdots \sigma \cdots c} + T^{i \cdots j \cdots k}_{a \cdots b \cdots c} \tag{10.22}$$

is a functional tensor covariant of order $l-1$ and contravariant of order $m+1$.

On the basis of the results of the preceding paragraphs we can obviously set up the machinery of covariant functional differentiation for functional tensors as well as for functionals with a law of transformation (10.5) or (10.8). Differentiating (10.5) functionally and eliminating the functional derivatives $''y^i{}_{\alpha\beta}$ by means of (10.14) we obtain the tensor law of transformation of the covariant derivative $g_{\alpha\beta;\gamma}$ of $g_{\alpha\beta}$. The explicit form of $g_{\alpha\beta;\gamma}$ is

$$g_{\alpha;\beta\gamma}[y] = {}'g_{\alpha\beta,\gamma} - g_{\lambda\beta}\Gamma^\lambda{}_{\alpha\gamma} - g_{\alpha\lambda}\Gamma^\lambda{}_{\beta\gamma} - \Gamma^\alpha{}_{\beta\gamma} - \Gamma^\beta{}_{\alpha\gamma}. \tag{10.23}$$

In a similar manner we obtain the covariant functional derivative $g_{;\gamma}{}^{\alpha\beta}$ of $g^{\alpha\beta}$

$$g_{;\gamma}{}^{\alpha\beta} = {}'g_\gamma{}^{\alpha\beta} + g^{\alpha\lambda}\Gamma^\beta{}_{\lambda\gamma} + g^{\lambda\beta}\Gamma^\alpha{}_{\lambda\gamma} + \Gamma^\beta{}_{\alpha\gamma} + \Gamma^\alpha{}_{\beta\gamma}. \tag{10.24}$$

On employing the defining relation (10.12) and the identity (10.13) in (10.23) we obtain the result

$$g_{\alpha\beta;\gamma} = 0. \tag{10.25}$$

We also have

$$g_{;\gamma}{}^{\alpha\beta} = 0. \tag{10.26}$$

To prove this, let us differentiate (10.13) functionally. We evidently obtain

$$'g_{\alpha,\beta\gamma} + {}'g_\gamma{}^{\alpha\beta} + {}'g_{\alpha\sigma,\gamma}g^{\sigma\beta} + g_{\alpha\sigma}{}'g_\gamma{}^{\sigma\beta} = 0,$$

a Fredholm integral equation with $g_{\alpha\sigma}$ as its kernel and $'g_\gamma{}^{\alpha\beta}$ as the unknown. The unique continuous solution of this equation is

$$'g_\gamma{}^{\alpha\beta} = -{}'g_{\alpha\beta,\gamma} - {}'g_{\alpha\sigma,\gamma}g^{\sigma\beta} - {}'g_{\sigma\beta,\gamma}g^{\sigma\alpha} - g^{\sigma\alpha}g^{\tau\beta}\,{}'g_{\sigma\tau,\gamma}. \tag{10.27}$$

Since

$$'g_{\alpha\beta,\gamma}[y] = [\alpha\gamma, \beta] + [\beta\gamma, \alpha], \tag{10.28}$$

we find that

$$'g_{\gamma}{}^{\alpha\beta} = - g^{\alpha\lambda}\Gamma^{\beta}{}_{\lambda\gamma} - g^{\lambda\beta}\,\Gamma^{\alpha}{}_{\lambda\gamma} - \Gamma^{\beta}{}_{\alpha\gamma} - \Gamma^{\alpha}{}_{\beta\gamma}. \tag{10.29}$$

Substituting in (10.24) this expression for $'g_{\gamma}{}^{\alpha\beta}$ we finally obtain (10.26).

It is a consequence of (10.25) that the covariant derivatives $B_{\alpha\beta\gamma\delta;\epsilon}$ of $B_{\alpha\beta\gamma\delta}$ is given by

$$B_{\alpha\beta\gamma\delta;\epsilon} = g_{\alpha\sigma}\,B^{\sigma}{}_{\beta\gamma\delta;\epsilon} + B^{\alpha}{}_{\beta\gamma\delta;\epsilon}. \tag{10.30}$$

More generally we have

$$B_{\alpha\beta\gamma\delta\,;\,\epsilon\,;\,\zeta;\,\ldots\,;\,\kappa} = g_{\alpha\sigma}\,B^{\sigma}_{\beta\gamma\delta\,;\,\epsilon\,;\,\zeta;\,\ldots\,;\,\kappa} + B^{\alpha}_{\beta\gamma\delta\,;\,\epsilon\,;\,\zeta;\,\ldots\,;\,\kappa} \tag{10.31}$$

Let $\eta_{\alpha\beta}[z]$ denote the coefficients $g_{\alpha\beta}[y]$ in a normal coördinate system z^i with origin at the point $y^i = q^i$ and based on the affine connection (10.12). On account of (10.25), the Volterra development for $\eta_{\alpha\beta}[z]$ takes the form

$$\eta_{\alpha\beta}[z] = \eta_{\alpha\beta}[q] + (1/2\,!)\,g_{\alpha\beta;\gamma\delta}[q]\,z^{\gamma}z^{\delta} + (1/3\,!)\,g_{\alpha\beta;\gamma\delta\epsilon}z^{\gamma}z^{\delta}z^{\epsilon} + \cdots \tag{10.32}$$

where the tensor $g_{\alpha\beta;\,\gamma_1\,\ldots\,\gamma_p}[y]$ is the pth extension of the functional $g_{\alpha\beta}[y]$. The importance of this sequence of functional tensors is made evident by the following theorem.

THEOREM (10. III). *Any tensor functional invariant of order r of the quadratic functional form* (10.3)

$$T^{ij\,\ldots\,l}_{ab\,\ldots\,c}\,[g_{\alpha\beta},\ 'g_{\alpha\beta,\gamma_1}, \cdots,\ {}^{(r)}g_{\alpha\beta,\gamma_1\,\ldots\,\gamma_r}] \tag{10.33}$$

can be put into the form

$$T^{ij\,\ldots\,l}_{ab\,\ldots\,c}\,[g_{\alpha\beta},\ 0,\ g_{\alpha\beta,\gamma_1\gamma_2}, \cdots,\ g_{\alpha\beta,\gamma_1\,\ldots\,\gamma_r}] \tag{10.34}$$

by replacing $g_{\alpha\beta}$ by $g_{\alpha\beta}$, $'g_{\alpha\beta,\gamma_1}$ by zero, and the higher functional derivatives of $g_{\alpha\beta}$ by the corresponding extensions of $g_{\alpha\beta}$. The proof of this theorem is entirely similar to that of Theorem (9. I). An obvious modification of Theorem (10. III) enables us to give a replacement theorem for the analogues of the n-dimensional differential parameters.

Let $f[y]$ be a scalar functional of y^i such that

$$\delta f = {}'f_{\alpha}\,\delta y^{\alpha}, \quad \delta' f_{\alpha} = {}''f_{\alpha\beta}\,\delta y^{\beta} \tag{10.35}$$

The functional

$$g^{\alpha\beta}[y]\ 'f_{\alpha;\beta}[y] + \int_a^b {}'f_{\alpha;\alpha}[y]\,d\alpha \quad ('f_{\alpha;\beta} = {}''f_{\alpha\beta} - {}'f_{\sigma}\Gamma^{\sigma}{}_{\alpha\beta} = f_{;\alpha\beta}) \tag{10.36}$$

is a tensor and the functional equation with functional derivatives

$$g^{\alpha\beta}\,'f_{\alpha;\beta} + \int_a^b {}'f_{a;a}[y]\,d\alpha = 0 \tag{10.37}$$

is an analogue of Laplace's equation.

The functional

$$g^{\alpha\beta}\,'f_\alpha\,'f_\beta + \int_a^b (f_a)^2\,d\alpha$$

and

$$g^{\alpha\beta}\,'f_\alpha\,'\phi_\beta + \int_a^b {}'f_a\,'\phi_a\,d\alpha$$

are analogues of the two n-dimensional differential parameters of the first order.

11. *Riemannian function space manifolds.* A quadratic functional differential form evaluated at a fixed point y^i is an algebraic quadratic functional form in the differentials δy^i. With this in mind let us make some further observations on algebraic quadratic functional forms

$$g_{\alpha\beta}y^\alpha y^\beta + \int_a^b (y^a)^2\,d\alpha. \tag{11.1}$$

Suppose that (11.1) is a *positive definite functional form.* Analytically this means that

$$g_{\alpha\beta}y^\alpha y^\beta + \int_a^b (y^a)^2\,d\alpha > 0 \tag{11.2}$$

for all functions y^a which do not vanish identically on the interval (a, b). It is easily shown that the *Fredholm determinant* $D[g_{\alpha\beta}]$ *of a positive definite form* (11.1) *does not vanish* (i. e., unity is not a characteristic value of $D[g_{\alpha\beta}]$). In fact suppose

$$D[g_{\alpha\beta}] = 0.$$

Then the homogeneous Fredholm integral equation

$$y^a + g_{a\beta}y^\beta = 0$$

will possess a continuous solution y^a which does not vanish identically. Obviously this contradicts (11.2).

Clearly the group of Fredholm transformations leaves invariant the positive definite property of a form (11.1).

Let us consider now a positive definite quadratic differential form

$$g_{\alpha\beta}[y]\,\delta y^\alpha\,\delta y^\beta + \int_a^b (\delta y^a)^2\,d\alpha. \tag{11.3}$$

We shall understand that

$$g_{\alpha\beta}\,\delta y^\alpha\,\delta y^\beta + \int_a^b (\delta y^a)^2\,d\alpha > 0 \tag{11.4}$$

for all continuous functions y^i such that

$$| q^i - y^i | < N$$

and for all differentials δy^i which do not vanish identically on the interval (a, b).

A function space manifold possessing a positive definite quadratic form (11.3) will be called a *Riemannian function space manifold.* We shall define the element of length ds by

$$ds^2 = g_{\alpha\beta}[y]\, \delta y^\alpha\, \delta y^\beta + \int_a^b (\delta y^\alpha)^2\, d\alpha = Q[\delta y, \delta y]. \tag{11.5}$$

Let

$$y^i = y^i(u)$$

be the equation of a parameterized curve with a real parameterization u. The *length* s of this curve between the points $y^i(u_0)$ and $y^i(u_1)$ will by definition be

$$s = \int_{u_0}^{u_1} \left[g_{\alpha\beta}[y^i(u)]\, \frac{\partial y^\alpha(u)}{\partial u}\, \frac{\partial y^\beta(u)}{\partial u} + \int_a^b \left(\frac{\partial y^\alpha(u)}{\partial u} \right)^2 d\alpha \right]^{1/2} du \tag{11.6}$$

The functional s so defined is a function of function space curves since its value depends only on the curve and not on any accidental parameterization u or coördinate system y^i.

The metrical ground-form (11.5) may be used to define the angle θ between two direction vectors $\delta_1 y^\alpha$ and $\delta_2 y^\alpha$. By definition

$$\cos\theta = Q[\delta_1 y, \delta_2 y] / \, (Q[\delta_1 y, \delta_1 y])^{1/2}\, (Q[\delta_2 y, \delta_2 y])^{1/2}. \tag{11.7}$$

Since the metrical ground-form $Q[\delta y, \delta y]$ is a positive definite functional form, it follows that

$$(Q[\delta_1 y, \delta_2 y])^2 < Q[\delta_1 y, \delta_1 y]\;\; Q[\delta_2 y, \delta_2 y] \tag{11.8}$$

for all non-proportional functions $\delta_1 y^i$ and $\delta_2 y^i$.

If the parameter u is taken to be the arc length s, we evidently have

$$g_{\alpha\beta}\,[y^i(s)]\, \frac{\partial y^\alpha(s)}{\partial s}\, \frac{\partial y^\beta(s)}{\partial s} + \int_a^b \left(\frac{\partial y^\alpha(s)}{\partial s} \right)^2 d\alpha = 1. \tag{11.9}$$

The problem of finding the *geodesics* of our Riemannian function space manifold is a problem in *minima of functionals.* The functional to be minimized is the distance functional (11.6). The generalized Euler-Lagrange equations for this variational problem are readily found to be

$$\frac{\partial^2 y^\alpha(s)}{\partial s^2} + g_{\alpha\beta}[y^i(s)]\, \frac{\partial^2 y^\beta(s)}{\partial s^2} + [\gamma\delta, \alpha]\, \frac{\partial y^\gamma(s)}{\partial s}\, \frac{\partial y^\delta(s)}{\partial s} = 0 \tag{11.10}$$

h which

$$[\gamma\delta, \alpha] = \tfrac{1}{2}\left({}'g_{\alpha\gamma,\delta}[y^i(s)] + {}'g_{\delta\alpha,\gamma}[y^i(s)] - {}'g_{\gamma\delta,\alpha}[y^i(s)]\right).$$

We may look upon (11. 10) as a Fredholm integral equation to be solved for $\partial^2 y^\alpha(s)/\partial s^2$. The unique solution, continuous as a function of α, is given by

$$\frac{\partial^2 y^\alpha(s)}{\partial s^2} + \Gamma^\alpha{}_{\gamma\delta}[y^i(s)]\frac{\partial y^\gamma(s)}{\partial s}\,\frac{\partial y^\delta(s)}{\partial s} = 0. \tag{11. 11}$$

In (11. 11) the functional $\Gamma^\alpha{}_{\gamma\delta}[y^i(s)]$ is the affine connection (10. 12) evaluated for the functional argument $y^i(s)$. The solution (11. 11) is, of course, valid for the functional neighborhood in which unity is not a characteristic value of the kernel $g_{\alpha\beta}[y^i(s)]$.

The integro-differential equation (11. 11) is a special case of the equation (6. 9). It is to be observed that in choosing the initial conditions

$$\begin{aligned} q^i &= y^i(s_0) \\ p^i &= (\partial y^i(s)/\partial s)_{s=s_0} \end{aligned} \tag{11. 12}$$

for the equation (11. 11), the function p^i must be taken so as to satisfy the condition

$$g_{\alpha\beta}[q^i]\, p^\alpha p^\beta + \int_a^b (p^\alpha)^2\, d\alpha = 1. \tag{11. 13}$$

In case our Riemannian manifold is a Euclidean function space, for which

$$ds = \left[\int_a^b [\partial y^\alpha(u)/\partial u]^2\, d\alpha\right]^{1/2} du, \tag{11. 14}$$

the equations of the geodesics, parameterized with the arc length s, take the simple form

$$\partial^2 y^\alpha(s)/\partial s^2 = 0. \tag{11. 15}$$

The most general solution of (11. 15) obviously is

$$y^\alpha = A^\alpha s + B^\alpha.$$

The methods of this paper can be extended to function spaces consisting of continuous functions in two or more independent variables. One might suspect that Volterra's theory of *permutable functions* and *functions of composition* would be useful in a geometry of functions of two independent variables.

THE OHIO STATE UNIVERSITY.

A Theorem Concerning the Affine Connection.

BY TRACY YERKES THOMAS.

Consider the equations of the paths:

$$d^2x^i + \Gamma^i_{\alpha\beta}dx^\alpha dx^\beta = 0 \tag{1}$$

of an affinely connected space of affine connection $\Gamma^i_{\alpha\beta}$ and the equations

$$d^mx^i + \Gamma^i_{\alpha\beta\ldots\epsilon}dx^\alpha dx^\beta \cdots dx^\epsilon = 0 \qquad (m = 3, 4, \cdots, \quad) \tag{2}$$

which result from them by differentiation.

The affine connection $\Gamma^i_{\alpha\beta}$ transforms into the connection $C^i_{\alpha\beta}$ according to the equations

$$C^\sigma_{ab}\frac{\partial x^i}{\partial y^\sigma} = \frac{\partial^2 x^i}{\partial y^a\,\partial y^b} + \Gamma^i_{\alpha\beta}\frac{\partial x^\alpha}{\partial y^a}\ \frac{\partial x^\beta}{\partial y^b} \tag{3}$$

when the coördinates x^i undergo an arbitrary analytic transformation by which they are transformed into a set of coördinates y^i. In order to see how the generalized connections $\Gamma^i_{\alpha\beta\ldots\epsilon}$ transform under this same transformation of coördinates let us examine how the equations (3) themselves are obtained. Differentiating the equations of transformation of the differentials dx^i, namely

$$dx^i = \frac{\partial x^i}{\partial y^a}\,dy^a \tag{4}$$

we have

$$d^2x^i = \frac{\partial^2 x^i}{\partial y^a\,\partial y^b}\,dy^a\,dy^b + \frac{\partial x^i}{\partial y^a}\,d^2y^a. \tag{5}$$

Substituting (4) and (5) into the equations (1) we find that the condition that (1) should remain invariant in form is that the $\Gamma^i_{\alpha\beta}$ should transform by (3). Suppose now that it is possible to continue in this way and so get the equations of transformation of the first $p-1$ successive connections $\Gamma^i_{\alpha\beta\ldots\mu}$ as the result of the condition that the equations (1) and (2) for $m = 3, \cdots, p-1$ should be invariant in form. In the y^i coördinates these equations then become

$$d^my^i + C^i_{\alpha\beta\ldots\mu}\,dy^\alpha\,dy^\beta \cdots dy^\mu = 0 \qquad (m = 2, \cdots, p-1). \tag{6}$$

To find the equations of transformation of the connection $\Gamma^i_{\alpha\beta\ldots\epsilon}$ in (2) for $m = p$ we differentiate (5) repeatedly so as to obtain

$$(7) \qquad d^p x^i = \frac{\partial^p x^i}{\partial y^a \ldots \partial y^t} dy^a \cdots dy^t + \frac{\partial x^i}{\partial y^\sigma} d^p y^\sigma + ☆$$

where the ☆ indicates terms involving differentials $d^m x^i$ for which $m < p$. These differentials are to be eliminated by means of (6) after which (7) is to be substituted into (2) for $m = p$. The requirement of the invariance of form then gives

$$(8) \qquad C^\sigma{}_{a \ldots t} \frac{\partial x^i}{\partial y^\sigma} = \frac{\partial^p x^i}{\partial y^a \cdots \partial y^t} + \Gamma^i{}_{a \ldots \epsilon} \frac{\partial x^a}{\partial y^a} \cdots \frac{\partial x^\epsilon}{\partial y^t} + ☆$$

where the ☆ denotes a polynominal composed of a sum of terms each of which involves a component of connection $C^i{}_{a \ldots \mu}$ with less than p subscripts. The equations of transformation of an arbitrary connection $\Gamma^i{}_{\alpha\beta \ldots \epsilon}$ are thus given by (8).

Now consider a set of functions $Z^i(x)$ defined by the power series expansion

$$(9) \qquad Z^i(x) = \sum_{s=2}^{\infty} (1/s!) (\Gamma^i{}_{\sigma_1 \ldots \sigma_s})_q (x^{\sigma_1} - q^{\sigma_1}) \cdots (x^{\sigma_s} - q^{\sigma_s})$$

in which the constants q^i are arbitrary and the coefficients are subject only to the condition that the series converges. The following theorem regarding the series (9) in its relation to the affine connection $\Gamma^i{}_{\alpha\beta}(x)$ can be proved.

THEOREM. *Given an arbitrary affine connection and a set of arbitrary functions* $Z^i(x)$ *defined by* (9). *Then there exists a coördinate system* x *such that the components of the affine connection* $\Gamma^i{}_{\alpha\beta}(x)$ *and of the generalized connections* $\Gamma^i{}_{\alpha\beta \ldots \epsilon}(x)$ *at* $x^i = q^i$ *will take on the values of the coefficients of the power series expansion of the function* $Z^i(x)$.

To prove this theorem we transform the components of the given affine connection to a system of normal coördinates y^i. In the normal coördinates y^i let the components of affine connection be represented by the set of functions $C^i{}_{\alpha\beta}(y)$. Now construct the transformation

$$(10) \qquad x^i = q^i + yi - \sum_{s=2}^{\infty} (1/s!) (\Gamma^i{}_{\sigma_1 \ldots \sigma_s})_q y^{\sigma_1} \cdots y^{\sigma_s}$$

which is determined by the coefficients of the expansion of the functions Z^i. All components $C^i{}_{\alpha\beta}(y)$ vanish at $y^i = 0$ since the coördinates y^i are normal coördinates. Hence if we evaluate both members of (8) at the origin of the coördinates y^i we have

$$\Gamma^i{}_{a \ldots \epsilon}(q) = - \left(\frac{\partial^p x^i}{\partial y^a \cdots \partial y^\epsilon} \right)_0$$

which means on account of (10) that at $x^i = q^i$ the components of the connection $\Gamma^i_{\alpha\beta}(x)$ and of the generalized connections $\Gamma^i_{\alpha\beta\ldots\epsilon}(x)$ are equal to the coefficients of the power series (9).

In particular the affine connection of the above theorem may be a Christoffel symbol:

$$\tfrac{1}{2} g^{i\sigma} \left(\frac{\partial g_{\alpha\sigma}}{\partial x^\beta} + \frac{\partial g_{\sigma\beta}}{\partial x^\alpha} - \frac{\partial g_{\alpha\beta}}{\partial x^\sigma} \right)$$

based on a fundamental metric tensor $g_{\alpha\beta}$.

If we take the given connection in the above theorem to be that of a flat space (a remark by Professor Veblen) we have the

SPECIAL THEOREM. *For a flat space (affine or Riemann) there exists a coördinate system x such that the components of the affine connection $\Gamma^i_{\alpha\beta}(x)$ and of the generalized connections $\Gamma^i_{\alpha\beta\ldots\epsilon}(x)$* at $x^i = q^i$ *will take on the values of the coefficients of the power series expansion of the functions $Z^i(x)$.*

In certain investigations it is of importance to consider invariantive systems of partial differential equations which involve the quantities $g_{\alpha\beta}$ or $\Gamma^i_{\alpha\beta}$ as unknown functions. A simple system of equations of this sort is the system of Einstein's gravitational equations for free space which expresses the vanishing of the contracted curvature tensor. The above functions $Z^i(x)$ belong to the set of arbitrary constants and functions which are determinative of the solutions of such differential equations.

PRINCETON UNIVERSITY.

Concerning Subsets of a Continuous Curve which can be Connected through the Complement of the Continuous Curve.[*]

By W. L. Ayres.

---

1. *Introduction.* One of the present tendencies in the theory of sets of points is the replacing of theorems concerning subsets of space by theorems concerning subsets of a continuous curve. This is evidently a more general point of view since the entire space is a particular case of a continuous curve.[†] But in making this change, it is found that certain theorems involving the notion of connectedness no longer hold true. In studying these difficulties we are led to the introduction of a new notion of a subset of a point set being connected through the complement of the point set.[‡] In this paper we make a study of this new property and, among other results, show that a number of theorems remain true on replacing the entire space by a continuous curve if the property of connectedness is replaced by this new property.

It is understood that all the point sets which are considered in this paper lie in a two-dimensional Euclidean space, which we denote by S.

2. *Definitions.* If the point set N is a subset of the point set M and x and y are points of N, we shall say that x and y *can be connected through N and $S-M$* if the set $N+(S-M)$ contains a connected subset containing both x and y. If the set $N+(S-M)$ contains a continuum containing both x and y, we shall say that *x and y can be connected through N and $S-M$ in the strong sense.* If every two points of N can be connected through N and $S-M$, we shall say that N is connected through N and $S-M$.

A plane domain D will be said to be *simply connected* if, for every simple closed curve J lying wholly in D, either the interior or the exterior of J is a subset of D.[*] A connected subset D of a continuous curve M is said to be an *M-domain* if $M - D$ is closed.[†] We shall say that an M-domain D is *simply connected with respect to M* if, for every simple closed curve J lying wholly in D, either D contains every point of M which lies in the interior of J or D contains every point of M which lies in the exterior of J.

---

[*] Presented to the American Mathematical Society September 9 and October 29, 1927.

[†] We are using the term continuous curve in the so-called "generalized sense," i. e. a set which is closed, connected and connected im kleinen is called a continuous curve. This was suggested independently by R. L. Moore and by S. Mazurkiewicz. See Moore, "Concerning Simple Continuous Curves," *Transactions of the American Mathematical Society,* Vol. 21 (1920), footnote on page 347, and Mazurkiewicz, "Sur les lignes de Jordan," *Fundamenta Mathematicae,* Vol. I (1920), page 193.

[‡] Independently of the author and apparently almost simultaneously, R. L. Moore noticed this new property. See his paper, "Some Separation Theorems, *Proceedings of the National Academy of Sciences,* Vol. 13 (1927), pp. 711-716.

Notations. We shall use the term *arc* as meaning *simple continuous arc.* If x and y are points of an arc α, the subarc xy of α is understood to mean the arc whose end-points are x and y and which is a subset of α. If xy denotes an arc with endpoints x and y, the symbols $<xy, xy>$ and $<xy>$ denote the sets $xy - x$, $xy - y$ and $xy - x - y$ respectively. If x and y are points, $d(x, y)$ denotes the distance from x to y. If x is a point and K is a point set, $d(x, K)$ denotes the lower bound of the numbers $d(x, y)$ for all points y of K. The symbol $[X]$ denotes a set each of whose elements may be represented by X. The elements X may be points or sets of points. If $N(P)$ is a point set defined relative to the point P, the symbol

$$\sum_{P \subset K} N(P)$$

denotes the set of all points Q such that Q belongs to some set $N(P)$ for some point P of the point set K.

3. THEOREM 1. *If the point set K is a subset of the point set M and P is a point of K, then the set of all points of K which can be connected to P through K and $S - M$ is closed with respect to K.*[‡]

Proof. Let K_p denote the set of all points of K which can be connected to P through K and $S - M$, and let X be any limit point of K_p which belongs to K. For each point Q of K_p there exists a connected subset C_Q of $K + (S - M)$ containing P and Q. Let

$$Z = \sum_{Q \subset K_p} C_Q.$$

---

[*] This definition differs slightly from that given by R. L. Moore, "Concerning Continuous Curves in the Plane," *Mathematische Zeitschrift*, Vol. 15 (1922), p. 257. According to our definition the exterior of a circle is a simply connected domain, while by Moore's definition it is not simply connected. For bounded domains the two definitions are equivalent.

[†] R. L. Wilder, "Concerning Continuous Curves," *Fundamenta Mathematicae*, Vol. 7 (1925), p. 341.

[‡] A point set G is said to be closed with respect to the point set H if every limit point of G which belongs to H belongs also to G. Cf. Knaster and Kuratowski, "Sur les ensembles connexes," *Fundamenta Mathematicae*, Vol. 2 (1921), p. 207.

The set Z is connected since every set C_Q is connected and contains the point P. Also the set Z contains the set K_p. Since X is a limit point of the set K_p, it is a limit point of the set Z. Then $X + Z$ is a connected subset of $K + (S - M)$ containing X and P. By definition of K_p, the point X belongs to the set K_p. Then every limit point of K_p which belongs to K belongs also to K_p and thus K_p is closed with respect to K.

THEOREM 2. *In order that a bounded M-domain D of a continuous curve M be simply connected with respect to M it is necessary and sufficient that the M-boundary B of D be connected through B and S — M.*

Proof. The condition is necessary. If M is the entire plane then, by a theorem of R. L. Moore,* the M-boundary B is connected and thus is connected through B and $S - M$. If M is not the entire plane, let P be some point of $S - M$ and let T denote an inversion of the plane with respect to some circle with center at P. Let M' denote $T(M)$ or $T(M) + P$ according as M is or is not bounded. Then M' is a bounded continuous curve,† $T(D)$ is an M'-domain which is simply connected with respect to M' and has $T(B)$ as its M'-boundary. Now let us suppose that B is not connected through B and $S - M$. Since B is bounded and closed, $T(B)$ is bounded, closed, does not contain P and is not connected through $T(B)$ and $S - M'$. Then $T(B)$ contains two points x and y which cannot be connected through $T(B)$ and $S - M'$. By a theorem due to R. L. Moore,‡ there exists a simple closed curve J such that J belongs to M', separates x and y in S and has no point in common with $T(B)$. Since $T(B)$ contains points (x and y) in both the interior and exterior of J, the M'-domain $\cdot T(D)$ contains points in both the interior and exterior of J. Then, as $T(D)$ is connected, $T(D)$ contains a point of J. If there is a point of J which does not belong to $T(D)$, then J must contain at least one point of $T(B)$ contrary to the condition that $T(B)$ and J have no common points. Hence J lies wholly in $T(D)$. But there is a point of $M' - T(D)$ in both the interior and exterior of J and thus $T(D)$ is not simply connected with respect to M'. But this is a contradiction and therefore B must be connected through B and $S - M$.

The condition is sufficient. Suppose D is not simply connected with respect to M. Then there exists a simple closed curve J which lies wholly in D and such that $M - D$ contains a point x in the interior of J and a point

---

* "Concerning Continuous Curves in the Plane," *loc. cit.*

† Knaster and Kuratowski, "Sur les continus non-bornés," *Fundamenta Mathematicae*, Vol. 5 (1924), pp. 23-58.

‡ "Some Separation Theorems," *loc. cit.*, Theorem 3.

y in the exterior of J. Let α be an arc of M whose endpoints are x and y. In the order x to y let $\bar{x}$ be the first point of J on α and $\bar{y}$ be the last point. On the subset $x\bar{x}>$ of α there is a point x' of B and on the subset $y\bar{y}>$ of α there is a point y' of B. Since B is connected through B and $S - M$, there is a connected subset K of $B + (S - M)$ which contains the points x' and y'. The connected set K has no point in common with D and contains the points x' and y' in the interior and exterior of J. But as J is a subset of D, this is impossible.

THEOREM 3. *In order that a bounded and connected subset D of a continuous curve M be an M-domain which is simply connected with respect to M it is necessary and sufficient that there exist a simply connected domain R such that $R \cdot M = D$.*

Proof. Obviously the condition is sufficient. We shall show that it is also necessary. By Theorem 2, the M-boundary B of D is connected through B and $S - M$. There exists a continuum K such that $K \cdot M = B$.* For each point P of D there is a circle with center at P whose interior $I(P)$ contains no point of $K + (M - D)$. Let

$$G = \sum_{P \subset D} I(P).$$

The set G is a domain such that $G \cdot M = D$. Let J be any simple closed curve which lies wholly in G. Suppose that both the interior and exterior of J contain a point of $M - D$. Then it is easily seen that both the interior and exterior of J contain a point of B. The continuum K contains a point interior to J and a point exterior to J but contains no point of G. But this is impossible since J is a subset of G. Then either the interior or the exterior of J contains no point of $M - D$, and let $V(J)$ denote that complementary domain of J containing no point of $M - D$. Let R be the domain consisting of the domain G plus all the domains $V(J)$ for all simple closed curves J of G. Evidently $R \cdot M = D$. It remains to show that R is simply connected.

If R is not simply connected, there exists in R a simple closed curve J', such that the interior and exterior of J' contain points x and y of $S - R$. From the definition of R it is seen that there can exist no simple closed curve lying wholly in G and having x in its interior and y in its exterior. We shall obtain a contradiction by proving that such a simple closed curve does exist. If P is any point of J' which belongs to G, let $J_1(P)$ be a circle such that every point of $J_1(P)$ and its interior $L(P)$ belong to G. If P is any point of J' which belongs to $R - G$, by definition of R, there is a domain $L(P)$ belonging to R and containing P and whose boundary is a simple closed curve $J_1(P)$ belonging to G. Then J' is covered by the set of domains $[L(P)]$. By the Borel-Lebesgue theorem, there is a finite subset, $L(P_1), L(P_2), \cdots, L(P_n)$, of $[L(P)]$ which covers J'. Neither the point x nor the point y belongs to the set $\sum_{i=1}^{n}(L(P_i) + J_1(P_i))$ and it is not difficult to see that the the boundary of the complementary domain of $\sum_{i=1}^{n}(L(P_i) + J_1(P_i))$ containing x is a simple closed curve J'' which belongs to $\sum_{i=1}^{n} J_1(P_i)$ and thus to G. The interior of J'' is a subset of the interior of J' and thus y lies in the exterior of J''. We have the desired contradiction. Hence R must be simply connected.

---

* R. L. Moore, "Some Separation Theorems," *loc. cit.*, Theorem 4. Moore proves that if H is a closed subset of a continuous curve M and x and y are points of H which can be connected through H and $S - M$ then x and y can be connected through H and $S - M$ in the strong sense, but his proof is sufficient to show that if G is any subset of H which is connected through H and $S - M$ then G is connected through H and $S - M$ in the strong sense.

THEOREM 4. *If M is a continuous curve and K is a bounded cutting*[*] *of M such that if H is any maximal connected subset of K the set K-H is not a cutting of M, then K is connected through K and S-M.*

Proof. Let A and B be points of $M - K$ which lie in no connected subset of $M - K$. By a theorem of G. T. Whyburn,[†] the set K contains a set G which is an irreducible cutting of M between A and B. Evidently every maximal connected subset of K contains a point of G. Let

$$M - G = M_1 + M_2,$$

where M_1 and M_2 are mutually separated sets containing A and B respectively. If the set G is not connected through G and S—M, there exists a simple closed curve J lying in M, separating two points x and y of G in S and having no point in common with G. Then J lies wholly in M_1 or wholly in M_2. Suppose it lies in M_1. By a theorem due to G. T. Whyburn,[‡] the set $M_2 + G$

---

[*] A subset K of a continuum M is said to be a *cutting* of M if $M - K$ is not connected. The set K is said to be an *irreducible cutting of M between the points A and B* if $M - K$ is the sum of two mutually separated sets M_A and M_B containing A and B respectively and no subset of K has this property. Cf. G. T. Whyburn, "Concerning Irreducible Cuttings of a Continuum," *Bulletin of the American Mathematical Society*, Vol. 33 (1927), p. 408 (abstract).

[†] *Loc. cit.*

[‡] *Loc. cit.*

is a continuum. But this is impossible since J and $M_2 + G$ have no common points and G contains a point in both the interior and exterior of J. Hence $S - M$ contains a set L such that $G + L$ is connected. Since every maximal connected subset of K contains a point of G, the set $K + L$ is connected.

THEOREM 5. *In order that a continuum M be a continuous curve it is necessary and sufficient that if A is an arc, K is a closed subset of $M + A$ and x and y are points of K which can be connected through K and $S - (M + A)$, then x and y can be connected through K and $S - (M + A)$ in the strong sense.*

R. L. Moore [*] has shown that the condition is necessary and the condition may be shown to be sufficient by a modification of the proof of another theorem of his.[†]

THEOREM 6. *Let M be a continuous curve and N be a closed subset of M. For each point P of N let $C(P)$ be the set of all points of S whose distance from P is less than or equal to a number $r(P)$. Let $K(P)$ be the maximal connected subset of $M \cdot C(P)$ which contains P. Suppose* (1) *the function $r(P)$ is a bounded non-negative upper semi-continuous function*[‡] *of the point P over the set N,* (2) *the set of points $[P]$, for which $r(P) = 0$ contains no continuum. Under these conditions every maximal connected subset of the set*

$$\sum_{P \subset N} K(P)$$

is a continuous curve.

Proof. Let H be a maximal connected subset of $\sum_{P \subset N} K(P)$. Let A be any limit point of H. Let $P_1, P_2, P_3, \cdots$, be a sequence of points of H having A as a sequential limit point. Let P'_i denote the center of the circle $C(P)$ which contains P_i. Since the function $r(P)$ is bounded, there exists a sequence of integers $n_1, n_2, n_3, \cdots$, such that the sequence $P'_{n_1}, P'_{n_2}, P'_{n_3}, \cdots$, has a sequential limit point B. The set N being closed, the point B belongs to N. For every value of i, $r(P'_{n_i}) \geqq d(P_{n_i}, P'_{n_i})$. Then as $r(P)$ is upper semi-continuous, $r(B) \geqq d(A, B)$. The sets $K(P'_{n_1}), K(P'_{n_2}), K(P'_{n_3}), \cdots$, have a closed and connected limiting set containing both A and B.[*] Moreover this limiting set belongs to M and to $C(B)$, and thus it belongs to $K(B)$. Then A belongs to $K(B)$ and thus to H. Therefore H is a closed set.

---

[*] "Some Separation Theorems," *loc. cit.*, Theorem 4. Moore proves only the bounded case and makes use of the fact that only a finite number of the complementary domains of a continuous curve are of diameter greater than a given positive number. While this is not true for unbounded continuous curves, it is true, and this would be sufficient for Moore's proof, that only a finite number of the complementary domains of a continuous curve (bounded or not) that contain points within a given circle are of diameter greater than a given positive number.

[†] *Ibid.*, Theorem 5.

[‡] E. W. Hobson, *Theory of Functions of a Real Variable*, Cambridge University Press, 2nd Edition 1921, Vol. I, p. 290.

If H is not a continuous curve, there exist two concentric circles C_1 and C_2 and an infinite sequence of subcontinua of H, $H_\omega, H_1, H_2, H_3, \cdots$, satisfying the conditions of the Moore-Wilder Lemma.[†] Let r_1 and r_2 be the radii of C_1 and C_2 and let $r = r_2 - r_1 > 0$. Let C_3 and C_4 be circles concentric with C_1 and with radii $r_1 + r/3$ and $r_2 - r/3$ respectively. Let $\bar{H}_\alpha$ $(\alpha = \omega, 1, 2, 3, \cdots)$ be the set of all points of H_α which lie on C_3 or on C_4 or between C_3 and C_4. For each point x of H let x' denote the center of the corresponding circle $C(P)$ containing x and used in defining the set $\sum_{P \subset N} K(P)$.

In case the maximum value of $r(x')$ for all points x of $\bar{H}_n$ approaches zero as n increases indefinitely, every point of $\bar{H}_\omega$ is a limit point of N and thus N contains $\bar{H}_\omega$. By condition (2) of the hypothesis of the theorem, the set $\bar{H}_\omega$ must contain a point Q such that $r(Q) > 0$. Let η be the smaller of the numbers $r(Q)$ and $r/3$. There exists a positive number δ_η such that every point of M whose distance from Q is less than δ_η can be joined to Q by an arc of M whose diameter is less than η. There exists an integer $\bar{n}$ such that $H_{\bar{n}}$ contains a point whose distance from Q is less than δ_η. Then $K(Q)$ contains a point of $H_{\bar{n}}$. But this is contrary to the condition that every set H_n is a maximal connected subset of $H \cdot G$, where G denotes the set consisting of C_1, C_2 and all those points of S which lie between C_1 and C_2.

If the above condition does not hold, there exists a positive number ϵ and an infinite sequence of integers $n_1, n_2, n_3, \cdots$, such that $\bar{H}_{n_i}$ contains a point P_i so that (a) $r(P'_i) > \epsilon$, (b) $[P_i]$ has a sequential limit point U on $\bar{H}_\omega$, (c) $[P'_i]$ has a sequential limit point V. Let

$$\overline{\lim_{i \to \infty}}\, r(P'_i) = m \geqq \epsilon.$$

---

[*] Z. Janiszewski, "Sur les continus irréductibles entre deux points," *Journal de l'Ecole Polytechnique*, Ser. 2, Vol. 16 (1912), p. 109.

[†] For a full statement of the lemma, see R. L. Wilder, "Concerning Continuous Curves," *loc. cit.*, p. 371. See also the earlier papers of R. L. Moore, "A Characterization of Jordan Regions by Properties that have no Reference to Their Boundaries," *Proceedings of the National Academy of Sciences*, Vol. 4 (1918), pp. 364-370, and "Continuous Sets that have no Continuous Sets of Condensation," *Bulletin of the American Mathematical Society*, Vol. 25 (1919), pp. 174-176.

Since $r(P)$ is an upper semi-continuous function, $r(V) \geqq m$. The limiting set of $K(P'_1)$, $K(P'_2)$, $K(P'_3)$, $\cdots$, is closed, connected and contains U and V. If t denotes the smaller of the numbers $\frac{1}{2}\epsilon$ and $r/8$, then all but a finite number of the sets $K(P'_i)$ contain a point x_i whose distance from P_i is less than $r/8$ and lying interior to the circle C_5 with center at V and radius $r(V) - t/2$. The points $[x_i]$ have a limit point x which must belong to H_ω since $d(\bar{x}, U) \leqq r/8$, and $\bar{x}$ must lie interior to or on the circle C_5 and belong to $K(V)$. Since M is a continuous curve there exists a number δ_t such that every point of M whose distance from $\bar{x}$ is less than δ_t can be joined to $\bar{x}$ by an arc of M of diameter less than t. There is a point x_j such that $d(\bar{x}, x_j) < \delta_t$, and let β be an arc of M with end-points $\bar{x}$ and x_j and of diameter less than t. The arc β belongs to G and $C(V)$. Since $\bar{x}$ belongs to $K(V)$, the entire arc β belongs to $K(V)$ and hence to H. But this is contrary to the condition that H_{n_j} is a maximal connected subset of $H \cdot G$, since $\bar{x}$ belongs to H_ω.

Then the assumption that H is not a continuous curve has led to a contradiction, which proves our theorem.

THEOREM 7.* *Let K be a bounded continuum which is a subset of a continuous curve M, T be a closed totally disconnected subset of K such that (1) $K - T$ is the sum of two sets K_1 and K_2, (2) if D_i denotes the set of all points $[x]$ such that x belongs to a complementary domain of M that has points of K_i on its boundary then $K_1 + D_1$ and $K_2 + D_2$ are mutually separated sets. If x_1 and x_2 are points of K_1 and K_2 respectively then there exists a simple closed curve J which is a subset of M, separates x_1 and x_2 in S, and such that every point of $J \cdot K$ is a point of T.*

Proof. The set K_1 is closed except for a subset T_1 of T. Let B_i be the point set consisting of the boundaries of the domains of D_i. No point of K_2 is on the boundary of any domain of D_1 and, since only a finite number of the complementary domains of M that belong to D_1 are of diameter greater than any given positive number, every limit point of B_1 which does not belong to B_1 belongs to $K_1 + T_1$. Hence $K_1 + B_1 + T_1$ is closed, and $K_1 + B_1 - T \cdot B_1$ and K_2 are mutually separated. For each point y of $K_1 + T_1 + B_1$, let $r(y) = \frac{1}{2}d(y, K_2)$ and $C(y)$ be the set of all points whose distance from y is less than or equal to $r(y)$. Let H_y be the maximal connected subset of $M \cdot C(y)$ containing y. It is easily seen that $r(y)$ is continuous on $K_1 + B_1$

---

* In the case where M is the entire plane, this theorem is due to R. G. Lubben. "The Separation of Mutually Separated Subsets of a Continuum by Curves," *Bulletin of the American Mathematical Society*, Vol. 32 (1926), p. 114 (abstract).

$+ T_1$ and is zero only on $T_1 + T \cdot B_1$, a totally disconnected point set. Then all the conditions of Theorem 6 are satisfied and thus every maximal connected subset of

$$\sum_{y \subset K_1+B_1+T_1} H_y$$

is a continuous curve. Let M_1 be the maximal connected subset of $\sum H_y$ containing x_1. It is evident that no point of K_2 belongs to M_1. Let D' be the complementary domain of M_1 containing x_2 and let B' be its boundary. Suppose B' contains a point z of K_1. There exists a circle $\bar{C}(z)$ with center at z such that every point of M in the interior of $\bar{C}(z)$ lies with z in a connected subset of M which lies within $C(z)$. Since z is approachable from D', there is an arc α one of whose end-points is z and such that $\alpha - z$ is a subset of D'. There is a point w of α such that every point of the subarc zw of α lies within $\bar{C}(z)$. If $< zw$ contains any point of M, this point belongs to M_1 since it belongs to H_z. But this is impossible since $< zw$ belongs to D'. Hence $< zw$ belongs to a complementary domain of M and this domain must be one of the set D_1 since there is a point of K_1 on its boundary. Every point of its boundary belongs to M_1 and hence D' is this domain. But x_2 belongs to no domain of the set D_1. Hence B' contains no point of K_1.

Let J be the boundary of D' with respect to the point x_1.[*] Then J is the required simple closed curve.

4. In this section we propose to study the properties of a continuous curve M having the property that every two points x and y of the sum of the boundaries of the complementary domains of M may be connected through $x + y$ and $S - M$. This is equivalent to the condition that every two points of the sum of the boundaries of the complementary domains lie together on the boundary of some complementary domain. For the remainder of this paper let W denote the sum of the boundaries of the complementary domains of a continuous curve M.

---

[*] If D is a domain, B is its boundary and x is a point of $S - (D + B)$, the *boundary of D with respect to x* is the boundary of the maximal domain of $S - (D + B)$ containing the point x. If B is a continuous curve, the boundary of D with respect to x is either a simple closed curve or an open curve. Cf. R. L. Moore, "Concerning the Separation of Point Sets by Curves," *Proceedings of the National Academy of Sciences*, Vol. 11 (1925), footnote 5 on page 475. Cf. also my paper, "Concerning the Boundaries of Domains of a Continuous Curve," *Bulletin of the American Mathematical Society*, Vol. 33 (1927), pp. 565-571.

THEOREM 8. *If every two points x and y of W may be connected through $x + y$ and $S - M$, then W is a continuous curve.*

Proof. Obviously W is connected. Let P be a limit point of W and let $P_1, P_2, P_3, \cdots$, be a sequence of points of W having P as a sequential limit point and such that $P_1 \neq P$. For every $i > 1$, there is a complementary domain D_i of M having P_1 and P_i on its boundary. Since only a finite number of the complementary domains of a continuous curve whose boundaries contain a given point are of diameter greater than a given positive number, there are only a finite number of distinct domains in the set $[D_i]$. Then there is one domain D_j which has infinitely many of the points P_i on its boundary. Hence P belongs to the boundary of D_j and thus to W. Therefore W is closed.

If W is not a continuous curve, there exist two concentric circles C_1 and C_2 and an infinite sequence of continua, $W_\infty, W_1, W_2, W_3, \cdots$, of W satisfying the conditions of the Moore-Wilder Lemma and the further condition that if P_α and Q_α $(\alpha = \infty, 1, 2, 3, \cdots)$ are points of $C_1 \cdot W_\alpha$ and $C_2 \cdot W_\alpha$ respectively then C_1 contains an arc β_1, with end-points X_1 and P_∞ which contains every point P_α and in the order $X_1P_1P_2P_3 \cdots P_\infty$, and C_2 contains an arc β_2 with end-points X_2 and Q_∞ which contains every point Q_α and in the order $X_2Q_1Q_2Q_3 \cdots Q_\infty$. Let the radii of C_1 and C_2 be r_1 and r_2 and let $r_2 - r_1 = r > 0$. Let x, y and z be points of W_∞ whose distances from the center of the circle C_1 are $r_1 + r/6$, $r_2 - r/6$ and $r_1 + \frac{1}{2}r$ respectively. There exists a positive number δ such that any point of M whose distance from x is less than δ can be joined to x by an arc of diameter less than $r/6$, and similar for the point y. There exists an integer k such that W_k contains points x_k and y_k such that $d(x, x_k) < \delta$ and $d(y, y_k) < \delta$, and let α_1 and α_2 be arcs of M with end-points x_k and x and y_k and y respectively and each of diameter less than $r/6$. Let z_{k+1} be a point of W_{k+1} whose distance from the center of C_1 is $r_1 + \frac{1}{2}r$. Then z_{k+1} and z cannot be connected through $z_{k+1} + z$ and $S - M$ since z_{k+1} and z do not lie in the same complementary domain of $W_k + W_{k+2} + \alpha_1 + \alpha_2$. Therefore W is a continuous curve.

THEOREM 9. *If every two points x and y of W may be connected through $x + y$ and $S - M$, then either* (1) *M is the sum of three arcs which have the same end-points and no two of which have any other point in common, or* (2) *every maximal cyclic curve** *of W is a simple closed curve.*

---

* G. T. Whyburn, "Cyclicly connected continuous curves," *Proceedings of the National Academy of Sciences*, Vol. 13 (1927), pp. 31-38.

Proof. Let C be any maximal cyclic curve of W. Then C contains a simple closed curve J. If $C = J$, our theorem is proved. If $C - J$ is non-vacuous, let T be a maximal connected subset of $C - J$. The set T has at least two limit points on J. If w is a point of T there exist arcs wx and wy which lie in T except for the points x and y which are limit points of T on J. The set $wx + wy$ contains an arc xzy with end-points x and y and such that $< xzy >$ lies in T. We shall show that $M = J + xzy$. Suppose that M contains a point which does not belong to $J + xzy$. Let D denote the complementary domain of $J + xzy$ containing this point and α_1 and α_2 denote the arcs of $J + xzy$ from x to y which form the boundary of D and let α_3 be the other arc of $J + xzy$ from x to y. If every point of D belongs to M, a point u of $<\alpha_1>$ cannot be connected to a point v of $<\alpha_2>$ through $u + v$ and $S - M$. Hence D contains a point which does not belong to M. Since the connected set D contains a point of M and a point of $S - M$, it contains a point p of W. Let q be a point of $<\alpha_3>$. But p and q cannot be connected through $p + q$ and $S - M$. Therefore $M = J + xzy$, and our theorem is proved.

THEOREM 10. *If every two points x and y of W may be connected through $x + y$ and $S - M$, then* (1) *if J is any simple closed curve of W then either the interior or the exterior of J contains no point of W,* (2) *if L is an open curve of W then one of the complementary domains of $S - L$ contains no point of W.*

THEOREM 11.[*] *If W is bounded and non-vacuous and every two points x and y of W may be connected through $x + y$ and $S - M$, then either* (1) *there is one complementary domain of M whose boundary is W, or* (2) *M consists of three arcs with the same end-points and no two of which have any other point in common.*

Proof. By Theorem 8, the set W is a continuous curve. If W contains no simple closed curve our theorem is proved since $M = W$ and the complementary domain whose boundary is W is $S - M$. Now suppose W contains a simple closed curve J_1. If $W = J_1$, then the exterior of J_1 can contain no point of M and this is the desired complementary domain of M whose boundary is W. If a maximal connected subset of $W - J_1$ has more than one limit point on J_1 then, by Theorem 9, condition (2) of our theorem holds. Also, by Theorem 10, either $W - J_1$ lies wholly in the exterior or wholly in the interior of J_1.

---

[*] A result closely related to this has recently been published by G. T. Whyburn. See Theorem 27 of his paper, "Concerning the Structure of a Continuous Curve," *American Journal of Mathematics*, Vol. 50 (1928), pp. 167-194.

Case I. Suppose $W - J_1$ is non-vacuous, the interior of J_1 contains $W - J_1$, and no maximal connected subset of $W - J_1$ has more than one limit point on J_1. Let T be a maximal connected subset of $W - J_1$, let P be its limit point on J_1 and let Q be any other point of J_1. Let $P_1, P_2, P_3, \cdots$ be a sequence of distinct points of T having P as a sequential limit point. For every i there is a complementary domain D_i of M having Q and P_i on its boundary. Since there are only a finite number of complementary domains of M of diameter greater than any given positive number that contain points interior to the simple closed curve J_1, there are only a finite number of the domains $[D_i]$ that are distinct. Then there is one domain D_k that has Q and infinitely many of the points P_i on its boundary. Hence P also belongs to the boundary of D_k. Let J_2 be the boundary of D_k with respect to some point in the exterior of J_1. The simple closed curve J_2 contains P and Q and is a subset of the set consisting of J_1 plus the interior of J_1. If any point of J_2 is in the interior of J_1, then some maximal connected subset of $W - J_1$ has two limit points on J_1. Hence $J_2 = J_1$. Then there is a complementary domain D_k of M which lies in the interior of J_1 and whose boundary contains every point of J_1. Now suppose there is a point x of W which is not a boundary point of D_k. Let J_3 be the boundary of D_k with respect to the point x. The set J_3 is a simple closed curve enclosing x, and $J_3 \neq J_1$. Then J_3 has a point x of W in its interior and a point of J_1 (and thus of W) in its exterior, contrary to Theorem 10. Therefore W is the boundary of D_k.

Case II. Suppose $W - J_1$ is non-vacuous, the exterior of J_1 contains $W - J_1$ and no maximal connected subset of $W - J_1$ has more than one limit point on J_1. In a manner very similar to that used in Case I, we may show that there is a complementary domain of M which lies in the exterior of J_1 and has W as its boundary.

THEOREM 12. *If G is a subset of W which is dense in W and every two points x and y of G may be connected through $x + y$ and $S - M$, then every two points p and q of W can be connected through $p + q$ and $S - M$.*

Proof. Let P and Q be points of W. Let $P_1, P_2, P_3, \cdots$ and $Q_1, Q_2, Q_3, \cdots$ be two sequences of points of G which have P and Q as sequential limit points. For every i there is a complementary domain D_i of M whose boundary contains P_i and Q_i. There can only be a finite number of the domains D_i which are distinct, and thus there is one domain D_j which has on its boundary infinitely many of the points $[P_i]$ and infinitely many of the set $[Q_i]$. Then both P and Q are boundary points of the domain D_j, and P and Q can be joined through $P+Q$ and $S-M$.

THEOREM 13. *If W is unbounded and every two points x and y of W can be connected through $x+y$ and $S-M$, then either* (1) *there is one complementary domain of M whose boundary is W, or* (2) *M consists of three rays with the same vertex*[*] *and no two of which have any other point in common, or* (3) *M consists of an open curve plus an arc which has only its end-points in common with the open curve.*

Proof. Let P be a point of a complementary domain of M and let T be an inversion of the plane with respect to some circle whose center is P. Let $W'=T(W)+P$ and $M'=T(M)+P$. The set W' is the sum of the boundaries of the complementary domains of the continuous curve M'. Since every two points x and y of $T(W)$ may be connected through $x+y$ and $S-M'$, by Theorem 12, every two points of W' have this property. The hypothesis of Theorem 11 is satisfied and hence either (a) W' is the boundary of one of the complementary domains of the continuous curve M', or (b) M' consists of three arcs with the same end-points and no two of which have any other point in common. If situation (a) holds, then there is a complementary domain of M whose boundary is W. If situation (b) holds, there are two possibilities according as P is one of the common end-points or is an interior point[†] of one of the arcs. Under the first possibility, we get situation (2) of our theorem; under the second, we have situation (3).

THEOREM 14. *If W is non-vacuous and every three points of W lie together on the boundary of some complementary domain of M, then there is a complementary domain of M whose boundary is the set W.*

Proof. The hypothesis of either Theorem 11 or Theorem 13 is satisfied according as W is bounded or unbounded. The curve M cannot consist of three arcs with the same end-points but no two of which have any other point in common, for if x, y and z are interior points of the three arcs and no two belong to the same arc, then there is no complementary domain of M whose boundary contains x, y and z. Similarly situations (2) and (3) of Theorem 13 cannot exist. Therefore W is the boundary of some complementary domain of M.

---

[*] An *open curve* is a closed set which is in a continuous (1, 1) correspondence with a straight line. If P is a point of an open curve C, the set consisting of P plus either of the maximal connected subsets of $C-P$ is called a *ray*. The point P is called the *vertex* of the ray.

[†] A point of an arc which is not an end-point of the arc is called an *interior point* of the arc.

THEOREM 15. *If every two points x and y of a continuous curve M can be connected through $x + y$ and $S - M$, then $M = W$.*

Proof. Suppose $M - W$ contains a point P. By Theorem 8, the point P is not a limit point of W. Then there exists a circle C with center at P such that every point of C and its interior belongs to $M - W$. Let Q be any point of C. Clearly P and Q cannot be connected through $P + Q$ and $S - M$. But this is contrary to the hypothesis of the theorem.

NATIONAL RESEARCH FELLOW IN MATHEMATICS,
THE UNIVERSITY OF TEXAS.

Admissible Numbers in the Theory of Probability.*

BY ARTHUR H. COPELAND.

I. *The fundamental assumptions of the theory of probability.*

We shall show that the fundamental assumptions of the theory of probability are equivalent to the assumption that there exists a number, x, satisfying a denumerably infinite set of independent equations. Thus, although these assumptions may seem exceedingly plausible on the basis of physical considerations, their consistency requires investigation.

The first assumption to which we refer is:

Assumption (a): If an event has a probability, p, then the ratio of the number of successes, r, to the number of trials, n, approaches the limit, p, as n becomes infinite.†

For such an event it is proved that the probability of obtaining r successes in n trials is ${}_nC_r p^r(1-p)^{n-r}$. More precisely, it is proved that if $m \cdot n$ trials are made of the given event and these trials are grouped so that the first n trials are in the first group, the second n trials in the second group, etc., and if in s of the m groups there are r successes and $n-r$ failures, then as m becomes infinite the ratio s/m approaches ${}_nC_r p^r(1-p)^{n-r}$. In the derivation of this formula the following assumption is made:

Assumption (b): the probability of success is a constant and does not vary from one trial to the next.

The precise meaning of assumption (*b*) is not obvious. That is, it is not obvious what restrictions this assumption is supposed to place upon the behavior of the given event. However, from the applications which are made of assumption (*b*), we are justified in concluding that it at least implies that the probability of success of the rth term in any group of n trials is p. In other words, if t be the number of successes of the rth members of the m groups, then t/m approaches p as m becomes infinite.

It is impossible to devise a physical experiment to test the validity of these assumptions, since we are necessarily restricted to a finite number of trials of the event. We can, however, invent a law of behavior for some imaginary event and see whether it is possible for the behavior of such an event to satisfy assumptions (*a*) and (*b*). Thus our problem is to construct a model universe of events to test the validity of the fundamental assumptions of the theory of probability.

---

* Presented to the American Mathematical Society, Oct. 29, 1927.

† Cf. Guido Castelnuovo, *Calcolo delle probabilita*, prefazione, p. xvi; J. L. Coolidge, *An Introduction to Mathematical Probability*, p. 4.

We shall represent the successes and failures of an event by a sequence of ones and zeroes. These ones and zeroes can be taken as the digits of some number, x. That is, this representation of an event history is also the binary representation of some number, x, lying between zero and one.

Let x represent a given event history, where

$$x = \cdot\, x^1 x^2 x^3 x^4 \cdots x^k \cdots$$

and where x^k is equal to zero or one. Also let

$$p_n(x) = \sum_{k=1}^{n} x^k/n.$$

Then the probability associated with the event history, x, is

$$p(x) = \lim_{n\to\infty} p_n(x)$$

whenever this limit exists.

Next let

$$(r/n)x = \cdot\, x^r x^{r+n} x^{r+2n} \cdots x^{r+kn} \cdots$$

where r, n, and k are integers.[*] We can now express assumptions (*a*) and (*b*) in a very simple form. If an event has a probability, p, then the successes and failures of this event must be represented by a number, x, for which

(1) $$p[(r/n)x] \quad \text{is defined}$$

(2) $$p[(r/n)x] = p$$

for every pair of integers r, n such that $0 < r \leqq n$.

Equations (2) constitute a denumerably infinite set. The equations of this set are not all independent but we shall prove there exists an infinite sub-set of independent equations.[†]

One other assumption is made in the derivation of the formula ${}_nC_r p^r (1 - p)^{n-r}$, namely

(3) $(1/n)x$, $(2/n)x$, $(3/n)x$, $\cdots$, $(n/n)x$ are independent numbers,

---

[*] $(r/n)x$ is read r over n of x. It will be observed that $(1/1)x = x$.

[†] Cf. Section V.

it being understood that a set of numbers is an independent set when and only when the events which these numbers represent are independent.* This assumption is also implied by assumption (b). That is, the probability of success at any trial is p and does not depend upon the success or failure of any other trial.

A number, x, is said to belong to the set $A(p)$, where $0 < p < 1$, if and only if it satisfies conditions (1), (2), and (3). A number is admissible if and only if it belongs to some set, $A(p)$.† If an event has a history which is represented by an admissible number, then the behavior of the event is admissible in the sense that it satisfies the assumptions underlying the theory of probability. The problem of establishing the consistency of these assumptions is identical with that of proving the existence of admissible numbers.

There is one difficulty in the interpretation of our model. We shall see that if we have given an admissible number, we can change arbitrarily a finite number of its digits and obtain in this way a new admissible number. Thus, since any physical experiment involves only a finite number of trials, the assumption that the history of the probable event must be represented by an admissible number, apparently gives no restriction whatsoever on the behavior of physical events. Before attempting to answer this objection let us point out that our model has not introduced a new difficulty but thrown new light on an old one. The above objection has no more justification than the statement that since in a given number of trials of a probable event, all combinations of successes and failures are possible, the assumption that this event has a probability places no restriction on the behavior of the event.

We can imagine that an experimenter makes a series of trials of a given event, then a second experimenter makes another series of trials of the same event, etc. If the object on which the experiments are made, wears out, a similar one is constructed. The work continues indefinitely. We assume that the history thus obtained can be represented by an admissible number. This assumption does restrict the behavior of the event—it restricts what might be described as its asymptotic behavior.

---

* Precise definitions of independence are given on pp. 540 and 541.

† It will be observed that an admissible number cannot be rational since the digits of a rational number repeat periodically and such a number is excluded by condition (2).

II. *The Algebra of Event Histories.*

We shall let $x \cdot y$ represent the compound event which consists in the conjunction of the two events, x and y (where x and y are the events whose histories are represented by the numbers x and y). Then

$$x \cdot y = \cdot u^1 u^2 u^3 \cdots u^i \cdots \quad \text{where} \quad u^i = x^i y^i.$$

This product obeys the two laws:

A. $$x \cdot y = y \cdot x$$

B. $$(x \cdot y) \cdot z = x \cdot (y \cdot z).$$

If the events, x and y, are mutually exclusive, then $x \cdot y = 0$. When this is the case, we can represent the history of the event "either x or y" by

$$x + y = \cdot v^1 v^2 v^3 \cdots v^i \cdots \quad \text{where} \quad v^i = x^i + y^i.$$

The expression, $x + y$, is the algebraic sum of the numbers, x and y.

This algebra of event histories also obeys the laws:

C. $$x + y = y + x \qquad \text{if } x \cdot y = 0$$

D. $$(x + y) + z = x + (y + z) \quad \text{if } x \cdot y = y \cdot z = z \cdot x = 0$$

E. $$(x + y) \cdot z = x \cdot z + y \cdot z \qquad \text{if } x \cdot y = 0.$$

In the case of two events the theorem of total probability can be stated in the form:

THEOREM 1. *If* $x \cdot y = 0$, *then* $p(x + y) = p(x) + p(y)$.

The extension to the case of n events is immediate.

Let us consider the history

$$1 - x = \cdot w^1 w^2 w^3 \cdots w^i \cdots \quad \text{where} \quad w^i = 1 - x^i.$$

We have the following identity

$$x \cdot (1 - x) = 0.$$

The event $1 - x$ is the complement of the event x. That is, $1 - x$ occurs whenever x fails, and fails whenever x occurs.

The following experiment serves to illustrate the use of this algebra: A penny was flipped ninety times and the heads and tails recorded. An event, x, consisted in the occurrence of a head. The successes and failures of x were arranged in groups of three. An event, X, consisted in the occurrence of two heads in a group of three flips. Hence

$$X = (1/3)x \cdot (2/3)x \cdot [1 - (3/3)x] + (2/3)x \cdot (3/3)x \cdot [1 - (1/3)x] + (3/3)x \cdot (1/3)x \cdot [1 - (2/3)x].$$

It was found that $p_{90}(x) = .466$ and $p_{30}(X) = .400$. These results agree fairly well with the theoretical values, $p(x) = .500$, $p(X) = .375$.

As a consequence of theorem 1, we have the equation

$$(4) \qquad p(x_1 \cdot x_3 \cdot x_4 \cdots x_n) + p(x_2 \cdot x_3 \cdot x_4 \cdots x_n) = p[(x_1 + x_2) \cdot x_3 \cdot x_4 \cdots x_n]$$

if $x_1 \cdot x_2 = 0$.

As special cases of this equation we have

$$(5) \quad p(x_1 \cdot x_2 \cdot x_3 \cdots x_n) + p[(1 - x_1) \cdot x_2 \cdot x_3 \cdots x_n] = p(x_2 \cdot x_3 \cdots x_n).$$

$$(6) \quad p(1 - x) = 1 - p(x).$$

The following equations are easily verified:

$$(7) \qquad (r/n) \prod_{k=1}^{N} x_k = \prod_{k=1}^{N} (r/n) x_k$$

$$(8) \qquad (r/n) \sum_{k=1}^{N} x_k = \sum_{k=1}^{N} (r/n) x_k \text{ if } x_i \cdot x_j = 0 \text{ when } i \neq j$$

$$(9) \qquad (r/n)(1 - x) = 1 - (r/n)x$$

$$(10) \qquad (r_2/n_2)[(r_1/n_1)x] = (R/N)x$$

where $N = n_1 n_2$ and $R = r_1 + (r_2 - 1)n_1$.

III. *Independent Numbers.*

It is a fundamental theorem that if a compound event consist in the conjunction of n independent events, then the probability of the compound event is the product of the probabilities of the individual events. This theorem is true in the case of two events and can be made true in the general case by formulating an accurate definition of independence.

Let us consider first the case of two events, x and y. The probability that y will occur if x occurs is

$$\lim_{n \to \infty} \frac{\sum_{i=1}^{n} x^i \cdot y^i}{\sum_{i=1}^{n} x^i} = \frac{p(x \cdot y)}{p(x)}$$

whenever $p(x)$ and $p(x \cdot y)$ exist and $p(x) \neq 0$. The probability that y will occur if x fails is $p[(1 - x) \cdot y]/p(1 - x)$. We can now state the following definition:*

---

* For some purposes it might be convenient to formulate a more restrictive definition of independence, but in the discussion of admissible numbers the above definition is sufficient.

Two events x and y are independent if and only if

$$\frac{p[(1-x)\cdot y]}{p[1-x]}=\frac{p(x\cdot y)}{p(x)} \quad\text{and}\quad \frac{p[(1-y)\cdot x]}{p[1-y]}=\frac{p(x\cdot y)}{p(y)}.$$

It is understood that the probability function is defined for all of the arguments involved and that $p(x)$ and $p(y)$ are different from one or zero.

The theorem of compound probability together with its converse can be stated as follows:

THEOREM 2. *A necessary and sufficient condition that two numbers, x and y, be independent is that*

$$p(x\cdot y)=p(x)\cdot p(y) \quad\text{where}\quad 0<p(x), p(y)<1.$$

This theorem is easily proved by means of equations (5) and (6).

It is natural to extend the definition of independence by saying that n events are independent provided each is independent of each of the other $n-1$. Can we, on the basis of this definition, extend theorem 2 to the case of n events? The following example throws considerable light on this question. The histories of three events x_1, x_2, and x_3, are as follows:

$$\begin{aligned} x_1&=\cdot 1100, 1100, 1100, 1100, \cdots\\ x_2&=\cdot 1010, 1010, 1010, 1010, \cdots \qquad (11)\\ x_3&=\cdot 1001, 1001, 1001, 1001, \cdots . \end{aligned}$$

We have the equations, $p(x_1)=p(x_2)=p(x_3)=\frac{1}{2}$, and $p(x_1\cdot x_2)=p(x_2\cdot x_3)=p(x_3\cdot x_1)=\frac{1}{4}$, but $p(x_1\cdot x_2\cdot x_3)=\frac{1}{4}\neq p(x_1)p(x_2)p(x_3)$. That is, each of the three events is independent of the other two, but the probability of the compound event consisting in the conjunction of x_1, x_2, and x_3 is not the product of the individual probabilities.

A second example is also instructive. An urn, U_1, contains n balls each of which is half red and half green. It contain also n white balls. An urn, U_2, contains n red balls and n green balls. U_3 contains n blue balls and n yellow balls. U_4 contains n blue and yellow balls and n white balls. Two men, A and B, draw balls one at a time from the urns, replacing the balls after each draw. A draws alternately from U_1 and U_2 and B draws from U_3 whenever A draws from U_1, and from U_4 whenever A draws from U_2. Events x_1, x_2, x_3, x_4 consist in the drawing of balls having respectively the colors red, green, blue, yellow. Each event has the probability, a half, and is independent of each of the others, but the compound event which is the conjunction of the four events, has the probability, zero.

These facts can be more easily seen if we examine the event histories.

Let X_1 be the history of the drawing of a red and green ball from U_1, X_2 that of drawing a red ball from U_2, X_3, that of drawing a blue ball from U_3, and X_4, that of drawing a blue and yellow ball from U_4. Then the equations

$$(12)\qquad \begin{array}{ll} (1/2)x_1 = X_1 & (2/2)x_1 = X_2 \\ (1/2)x_2 = X_1 & (2/2)x_2 = 1 - X_2 \\ (1/2)x_3 = X_3 & (2/2)x_3 = X_4 \\ (1/2)x_4 = 1 - X_3 & (2/2)x_4 = X_4 \end{array}$$

define x_1, x_2, x_3, x_4 in terms of X_1, X_2, X_3, X_4. From these equations we get

$$(13)\qquad \begin{array}{ll} (1/2)(x_1 \cdot x_2) = X_1 & (2/2)(x_1 \cdot x_2) = 0 \\ (1/2)(x_3 \cdot x_4) = 0 & (2/2)(x_3 \cdot x_4) = X_4 \\ (1/2)(x_1 \cdot x_3) = X_1 \cdot X_3 & (2/2)(x_1 \cdot x_3) = X_2 \cdot X_4 \\ (1/2)(x_2 \cdot x_4) = X_1 \cdot (1 - X_3) & (2/2)(x_2 \cdot x_4) = (1 - X_2) \cdot X_4 \\ (1/2)(x_1 \cdot x_4) = X_1 \cdot (1 - X_3) & (2/2)(x_1 \cdot x_4) = X_2 \cdot X_4 \\ (1/2)(x_2 \cdot x_3) = X_1 \cdot X_3 & (2/2)(x_2 \cdot x_3) = (1 - X_2) \cdot X_4. \end{array}$$

Hence we get

$$(14)\qquad x_1 \cdot x_2 \cdot x_3 \cdot x_4 = 0.$$

From these equations it follows that $p(x_i \cdot x_j) = p(x_i)p(x_j)$ if $i \neq j$ and $i, j = 1, 2, 3, 4$, but $p(x_1 \cdot x_2 \cdot x_3 \cdot x_4) = 0$.

From these two examples it is clear that in order to insure the independence of N numbers, $x_1, x_2, \cdots, x_N$, it is not sufficient to assume that every x_i is independent of every x_j (where $i \neq j$). A definition which is sufficient is the following:

The numbers $x_1, x_2, x_3, \cdots, x_N$, are independent if and only if every number, $x_{\mu_1} \cdot x_{\mu_2} \cdot x_{\mu_3} \cdots x_{\mu_m}$. is independent of every number, $x_{\nu_1} \cdot x_{\nu_2} \cdots x_{\nu_n}$, where the numbers, $x_{\mu_1}, x_{\mu_2}, \cdots, x_{\nu_1}, \cdots, x_{\nu_n}$, are all different and all belong to the set $x_1, x_2, \cdots, x_N$.[*]

We can now state the general theorem of compound probability.

THEOREM 3. *A necessary and sufficient condition that the distinct numbers, $x_1, x_2, x_3, \cdots, x_N$, be independent is that every subset, $x_{\nu_1}, x_{\nu_2}, x_{\nu_3}, \cdots, x_{\nu_n}$, no two numbers of which are equal, satisfy the condition*

$$(15)\qquad 0 < p(x_{\nu_1} \cdot x_{\nu_2} \cdot x_{\nu_3} \cdots x_{\nu_n}) = p(x_{\nu_1})p(x_{\nu_2})p(x_{\nu_3}) \cdots p(x_{\nu_n}) < 1.$$

Is it not obvious that equations (15) are compatible, and if these equations are incompatible for every N, then there exist no independent events. The following theorems are concerned with the existence of independent events:

---

[*] N is understood to be an unrestricted integer.

THEOREM 4. *There exist n independent numbers,* $X_1, X_2, \cdots, X_k, \cdots, X_n$, *such that* $p(X_k) = \frac{1}{2}$.

We shall show the existence of n independent numbers whose digits repeat periodically, the period being 2^n. Let us write the first 2^n digits of these numbers in tabular form as follows.

$$\begin{array}{l} X_1^1\, X_1^2\, X_1^3 \cdots X_1^{2^n} \\ X_2^1\, X_2^2\, X_2^3 \cdots X_2^{2^n} \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ X_n^1\, X_n^2\, X_n^3 \cdots X_n^{2^n}. \end{array}$$

We shall arrange the columns in $n + 1$ groups, $G_0, G_1, G_2, \cdots, G_n$. Group G_k will contain all possible different columns having k zeroes and $n - k$ ones. Hence G_k contains ${}_nC_k$ columns, and the total number of columns in the period is 2^n.

Let us compute the value of $p(X_{\nu_1} \cdot X_{\nu_2} \cdot X_{\nu_3} \cdots X_{\nu_r})$ where the numbers, $X_{\nu_1}, X_{\nu_2}, \cdots, X_{\nu_r}$, are all different and all belong to the set, $X_1, X_2, \cdots, X_n$. Since the digits of $X_{\nu_1} \cdot X_{\nu_2} \cdots X_{\nu_r}$ repeat periodically, we need only to count the number of ones in a period. The kth group contains ${}_{n-r}C_k$ ones, and hence the total number of ones contributed by all the groups is $\sum_{k=0}^{n-r} {}_{n-r}C_k = 2^{n-r}$. Thus

$$p(X_{\nu_1} \cdot X_{\nu_2} \cdot X_{\nu_3} \cdots X_{\nu_r}) = 2^{n-r}/2^n = 2^{-r}.$$

In particular $p(X_{\nu_k}) = \frac{1}{2}$, and therefore, by theorem 3, the numbers X_1, X_2, X_n are independent.

Before we consider the general problem of the independence of numbers associated with an arbitrary probability, we shall need two more theorems.

THEOREM 5. *If the numbers,* $x_1, x_2, x_3, \cdots, x_N$, *are independent, then the numbers,* $1 - x_1, x_2, x_3, \cdots, x_N$, *are independent.*

We have the equation

$$p[(1 - x_1) \cdot x_{\nu_1} \cdot x_{\nu_2} \cdots x_{\nu_n}] = p(x_{\nu_1} \cdot x_{\nu_2} \cdots x_{\nu_n}) - p(x_1 \cdot x_{\nu_1} \cdot x_{\nu_2} \cdots x_{\nu_n}).$$

By hypothesis the numbers, $x_1, x_2, \cdots, x_N$, are independent and hence

$$\begin{aligned} p(x_{\nu_1} \cdot x_{\nu_2} \cdots x_{\nu_n}) &- p(x_1 \cdot x_{\nu_1} \cdots x_{\nu_n}) \\ &= p(x_{\nu_1}) p(x_{\nu_2}) \cdots p(x_{\nu_n}) - p(x_1) p(x_{\nu_1}) \cdots p(x_{\nu_n}). \end{aligned}$$

Therefore

$$\begin{aligned} p[(1 - x) \cdot x_{\nu_1} \cdot x_{\nu_2} \cdot x_{\nu_n}] &= [1 - p(x_1)] p(x_{\nu_1}) p(x_{\nu_2}) \cdots p(x_{\nu_n}) \\ &= p(1 - x_1) p(x_{\nu_1}) p(x_{\nu_2}) \cdots p(x_{\nu_n}). \end{aligned}$$

It follows that the numbers, $1 - x_1, x_2, x_3, \cdots, x_N$, are independent.

THEOREM 6. *If the numbers $x_1, x_3, x_4, \cdots, x_N$, are independent, and the numbers $x_2, x_3, x_4, \cdots, x_N$, are independent, and if $x_1 \cdot x_2 = 0$, then the numbers $(x_1 + x_2), x_3, x_4, \cdots, x_N$, are independent.*

By hypothesis

$$p(x_1 \cdot x_{\nu_1} \cdot x_{\nu_2} \cdots x_{\nu_n}) = p(x_1) p(x_{\nu_1}) p(x_{\nu_2}) \cdots p(x_{\nu_n})$$

and

$$p(x_2 \cdot x_{\nu_1} \cdot x_{\nu_2} \cdots x_{\nu_n}) = p(x_2) p(x_{\nu_1}) p(x_{\nu_2}) \cdots p(x_{\nu_n}).$$

Adding these equations we get

$$p[(x_1 + x_2) \cdot x_{\nu_1} \cdot x_{\nu_2} \cdots x_{\nu_n}] = p(x_1 + x_2) p(x_{\nu_1}) p(x_{\nu_2}) \cdots p(x_{\nu_n}).$$

Hence the numbers $(x_1 + x_2), x_{\nu_1}, x_{\nu_2}, \cdots, x_{\nu_n}$, are independent.

We can now prove the following theorem:

THEOREM 7. *There exist N independent numbers, $X_1, X_2, \cdots, X_i, \cdots, X_N$, such that*

$$p(X_i) = \cdot p^1 p^2 p^3 \cdots p^j \cdots p^s$$

where $p^j = 0$ or 1, and $p^s = 1$.

We shall prove the theorem by induction. Let us assume that there exist N numbers, $Y_1, Y_2, \cdots, Y_N$, such that $p(Y_i) = p$, and K numbers, $X_1, X_2, \cdots, X_K$, such that $p(X_i) = \frac{1}{2}$. Let us assume that the numbers, $Y_1, Y_2, \cdots, Y_N, X_1, X_2, \cdots, X_K$, are all independent and that $K > \nu N$. We shall show that the numbers, $V_1, V_2, \cdots, V_N, X_{\nu N+1}, \cdots, X_K$, are independent, where $V_i = U_i \cdot Y_i + (1 - X_{\nu_i})$ and where $U_i = X_{(i-1)\nu+1} \cdot X_{(i-1)\nu+2} \cdots X_{i\nu}$.

By theorem 3 the numbers, $U_1 \cdot Y_1, U_2 \cdot Y_2, \cdots, U_N \cdot Y_N, X_{\nu N+1} \cdots X_K$, are independent. By theorem 5 the numbers, $1 - X_\nu, U_2 \cdot Y_2, \cdots, U_N \cdot Y_N, X_{\nu N+1} \cdots X_K$, are independent. Hence by theorem 6 the numbers, $V_1, U_2 \cdot Y_2, U_3 \cdot Y_3, \cdots, U_N \cdot Y_N, X_{\nu N+1}, \cdots, X_K$, are independent. Similarly $V_1, 1 - X_{2\nu}, U_3 \cdot Y_3, \cdots, U_N \cdot Y_N, \cdots, X_K$, are independent and therefore $V_1, V_2, U_3 \cdot Y_3, \cdots, U_N \cdot Y_N, \cdots, X_K$ are independent. Similarly it can be proved that $V_1, V_2, \cdots, V_N, X_{\nu N+1}, \cdots, X_K$ are independent. Furthermore $p(V_i) = \frac{1}{2} + p2^{-\nu}$ and $p(U_i \cdot Y_i) = p2^{-\nu}$. We have seen that it is possible to satisfy our hypotheses for the case $p = \frac{1}{2}$. It follows that we can satisfy these hypotheses for the case $p = \frac{1}{2} + 2^{-\nu-1}$ or $2^{-\nu-1}$ and hence, by induction, for $p = \cdot p^1 p^2 \cdots p^i \cdots p^s$.

THEOREM 8. *There exist N independent events, $X_1, X_2, \cdots, X_N$, such that $p(X_i) = p$, where p is any number such that $0 < p < 1$.*

Let $p = \cdot p^1 p^2 p^3 \cdots p^s \cdots$. For any s we can choose N independent numbers, $X_1{}^s, X_2{}^s, \cdots, X_N{}^s$, such that $p(X_i{}^s) = \cdot p^1 p^2 p^3 \cdots p^s$. Hence we can find a sequence of numbers

$$\epsilon_1 > \epsilon_2 > \epsilon_3 > \cdots > \epsilon_s > \cdots$$

such that $\lim\limits_{s \to \infty} \epsilon_s = 0$ and such that

$$| \, p(X_{\nu_1}{}^s \cdot X_{\nu_2}{}^s \cdots X_{\nu_k}{}^s) - p^k \, | \quad < \quad \epsilon_s/3$$
$$| \, p(X_{\nu_1}{}^{s+1} \cdot X_{\nu_2}{}^{s+1} \cdots X_{\iota_k}{}^{s+1}) - p^k \, | \quad < \quad \epsilon_s/3$$

for every set of numbers, $X_{\nu_1}{}^s, X_{\nu_2}{}^s, \cdots, X_{\nu_k}{}^s$ contained in the set, $X_1{}^s, \cdots, X_N{}^s$, and for every set, $X_{\nu_1}{}^{s+1}, \cdots, X_{\nu_k}{}^{s+1}$, contained in $X_1{}^{s+1}, \cdots, X_N{}^{s+1}$. We can choose two sets of integers $m_1, m_2, m_3, \cdots, m_s, \cdots$, and $n_1, n_2, \cdots, n_s, \cdots$, such that

$$\text{(a)} \qquad | \, p_n(X_{\nu_1}{}^{s+1} \cdot X_{\nu_2}{}^{s+1} \cdots X_{\nu_k}{}^{s+1}) - p^k \, | < \epsilon_s/3 \text{ if } n \geqq m_s$$

$$\text{(b)} \qquad | \, p_n(X_{\nu_1}{}^{s} \cdot X_{\nu_2}{}^{s} \cdots X_{\nu_k}{}^{s}) - p^k \, | + (N_s + m_s)/n_s < \epsilon_s/3 \text{ if } n \geqq n_s$$

where $N_s = n_1 + n_2 + n_3 \cdots n_{s-1}$.

Let X_i be such that the digits $N_s + 1$ to N_{s+1} of X_i are the same as the first n_s digits of $X_i{}^s$. Then

$$\text{(c)} \qquad | \, p_n(X_{\nu_1} \cdot X_{\nu_2} \cdots X_{\nu_k}) - (n_s/n) p_{n_s} (X_{\nu_1}{}^s \cdot X_{\nu_2}{}^s \cdots X_{\nu_k}{}^s) \, | < (N_s + m_s)/n < \epsilon_s/3$$

if $N_{s+1} \leqq n \leqq N_{s+1} + m_s$.

Combining (b) and (c) we get

$$\text{(d)} \qquad | \, p_n(X_{\nu_1} \cdot X_{\nu_2} \cdots X_{\nu_k}) - (n_s/n) p^k \, | < (\epsilon_s/3) \, [1 + (n_s/n)] < (2/3)\epsilon_s$$

if $N_{s+1} \leqq n \leqq N_{s+1} + m_s$.

But $| \, (n_s/n) p^k - p^k \, | = (n - n_s/n) p^k < \epsilon_s/3$. Therefore

$$\text{(e)} \qquad | \, p_n(X_{\nu_1} \cdot X_{\nu_2} \cdots X_{\nu_k}) - p^k \, | < \epsilon_s$$

if $N_{s+1} \leqq n \leqq N_{s+1} + m_s$. We have also the inequality

$$\text{(f)} \qquad | \, p_n(X_{\nu_1} \cdots X_{\nu_k}) - (n_s/n) p_{n_s} \, (X_{\nu_1}{}^s \cdots X_{\nu_k}{}^s) - (n - N_{s+1}/n) p_{n-N_{s+1}} (X_{\nu_1}{}^{s+1} \cdots X_{\nu_k}{}^{s+1}) \, | \leqq (N_s/n) < \epsilon_s/3$$

if $N_{s+1} + m_s \leqq n \leqq N_{s+2}$.

Combining (a), (b) and (f) we get

$$\text{(g)} \qquad | \, p_n(X_{\nu_1} \cdots X_{\nu_k}) - (n_s + n - N_{s+1}/n) p^k \, | < (\epsilon_s/3) \, [1 + (n_s + n - N_{s+1}/n)] < \tfrac{2}{3}\epsilon_s.$$

But

$$\text{(h)} \qquad | \, p^k - p^k(n_s + n - N_{s+1})/n \, | = p^k(N_s/n) < \epsilon_s/3$$

if $N_{s+1} + m_s \leqq n \leqq N_{s+2}$. Therefore

(i) $| p_n(X_{\nu_1} \cdot X_{\nu_2} \cdots X_{\nu_k}) - p^k | < \epsilon_s$

if $N_{s+1} + m_s \leqq n \leqq N_{s+2}$.

Therefore $p(X_{\nu_1} \cdot X_{\nu_2} \cdot X_{\nu_3} \cdots X_{\nu_k}) = p^k$. Hence the numbers, X_1, X_2, $\cdots$, X_N, are independent and $p(X_i) = p$.

IV. *The Existence of Admissible Numbers.*

By means of theorem 3 we can express the condition for admissibility in a more compact form as follows:

THEOREM 9. *A necessary and sufficient condition that a number, x, be admissible is that*

$$0 < p[(r_1/n)x \cdot (r_2/n)x \cdots (r_k/n)x] = p^k < 1$$

for every positive integer, n and for every set of distinct integers, r_1, r_2, $\cdots$, r_i, $\cdots$, r_k, such that $0 < r_i \leqq n$.

In order to investigate the existence of admissible numbers we need also the following theorem:

THEOREM 10. *If $p[(r/n)x] = p_r$ for $r = 1, 2, 3, \cdots, n$, then*

$$p(x) = \sum_{r=1}^{n} (p_r/n).$$

We shall now prove the existence of admissible numbers.

THEOREM 11. *Each set, $A(p)$, contains at least one point, x.*

Given any integer, s, we can select $s!$ independent numbers, $X_1{}^s$, $X_2{}^s$, $\cdots$, $X_r{}^s$, $\cdots$, $X_{s!}{}^s$, such that $p(X_r{}^s) = p$. Let U_s be defined by the equations

$$(r/s!)U_s = X_r{}^s. \tag{a}$$

Let n be any integer less than or equal to s, and let $m = s!/n$. Then m is an integer. It follows from equations (7) and (10) that

$$\begin{aligned} p\{(q/m)[(r_1/n)U_s \cdot (r_2/n)U_s \cdots (r_k/n)U_s]\} \\ = p(X_{R_1}{}^s \cdot X_{R_2}{}^s \cdots X_{R_k}{}^s) = p^k \end{aligned} \tag{b}$$

where $0 < r_i \leqq n$ and $r_i \neq r_j$ if $i \neq j$, and where $R_i = r_i + (q-1)n$. Equation (b) holds for $q = 1, 2, 3, \cdots, m$. Hence by theorem 10,

$$p[(r_1/n)U_s \cdot (r_2/n)U_s \cdots (r_k/n)U_s] = p^k.$$

Let $\epsilon_1 > \epsilon_2 > \epsilon_3 > \cdots > \epsilon_s > \cdots$ be a sequence such that $\lim_{s \to \infty} \epsilon_s = 0$. We can select a set of integers, $M_1, M_2, \cdots, M_s, \cdots$, such that

$$\text{(c)} \qquad | p_N[r_1/n) U_{s+1} \cdot (r_2/n) U_{s+1} \cdots (r_k/n) U_{s+1}] - p^k | < \epsilon_s/3$$

whenever $N \geqq M_s/n$ and $n \leqq s$. We can choose a second set of integers, $N_1, N_2, \cdots, N_s, \cdots$, such that

$$\text{(d)} \qquad | p_N[(r_1/n) U_s \cdot (r_2/n) U_s \cdots (r_k/n) U_s] - p^k | + (\nu_s + M_s)/N_s < \epsilon_s/3$$

when $N \geqq N_s/n$ and $n \leqq s$, where $\nu_s = N_1 + N_2 \cdots + N_{s-1}$. At the same time the numbers, N_s, can be chosen so that ν_s/n is an integer when $n \leqq s$. Then N_s/n is an integer.

It is understood that conditions (c) and (d) hold for every n such that $n \leqq s$, and for every set of numbers, $r_1, r_2, \cdots, r_k$, such that $r_i \leqq n$ and $r_i \neq r_j$ if $i \neq j$.

Let x be such that the digits $\nu_s + 1$ to ν_{s+1} of x are the same as the digits 1 to N_s of U_s. Then the digits $(\nu_s/n) + 1$ to ν_{s+1}/n of $(r_1/n)x \cdot (r_2/n)x \cdots (r_k/n)x$ are the same as digits 1 to N_s/n of $(r_1/n)U \cdot (r_2/n)U \cdots (r_k/n)U$. Hence

$$\begin{aligned} \text{(e)} \qquad & | p_N\{(r_1/n)x \cdot (r_2/n)x \cdots (r_k/n)x\} \\ & \qquad - (N_s/nN) p_{(N_s/n)}\{r_1/n) U_s \cdots (r_k/n) U_s\} | \\ & \qquad\qquad \leqq (Nn - N_s)/nN \leqq (\nu_s + M_s)/N_s < \epsilon_s/3 \end{aligned}$$

if $\nu_{s+1}/n \leqq N \leqq (\nu_{s+1} + M_s)/n$.

Combining (d) and (e) we get

$$\begin{aligned} \text{(f)} \qquad & | p_N\{(r_1/n)x \cdot (r_2/n)x \cdots (r_k/n)x\} \\ & \qquad - (N_s/nN) p^k | < (\epsilon_s/3)[1 + (N_s/nN)] < \tfrac{2}{3}\epsilon_s. \end{aligned}$$

But

$$\text{(g)} \qquad | p^k - (N_s/nN) p^k | = (Nn - N_s)/nN \cdot p^k < (\nu_s + M_s)/N_s < \epsilon_s/3$$

if $\nu_{s+1}/n \leqq N \leqq (\nu_{s+1} + M_s)/n$. Therefore

$$\text{(h)} \qquad | p_N\{(r_1/n)x \cdot (r_2/n)x \cdots (r_k/n)x\} - p^k | < \epsilon_s$$

if $\nu_{s+1}/n \leqq N \leqq (\nu_s + M_s)/n$.

We have also the inequality

$$\begin{aligned} \text{(i)} \qquad & | p_N\{(r_1/n)x \cdots (r_k/n)x\} - (N/nN) p_{(N_s/n)}\{(r_1/n) U_s \cdots (r_k/n) U_s\} \\ & \quad - [N - (\nu_{s+1}/n)/N] \, p_{[N-(\nu_{s+1}/n)]}\{(r_1/n) U_{s+1} \cdots \} | \leqq \nu_s/nN \leqq \epsilon_s/3 \end{aligned}$$

if $(\nu_{s+1} + M_s)/n \leqq N \leqq \nu_{s+2}/n$.

Combining (c), (d) and (i) we get

(j) $| p_N\{(r_1/n)x \cdots (r_k/n)x\} - [N\ /nN + (nN - \nu_{s+1})/nN]\, p^k | < \frac{2}{3}\epsilon_s.$

But

(k) $| p^k - [N_s/nN + (nN - \nu_{s+1})/nN]\, p^k | < \epsilon_s/3$

if $(\nu_{s+1} + M_s)/n \leqq N \leqq \nu_{s+2}/n$. Therefore

(l) $| p_N\{(r_1/n)x \cdot (r_2/n)x \cdots (r_k/n)x\} - p^k | < \epsilon_s$

if $(\nu_{s+1} + M_s)/n \leqq N \leqq \nu_{s+2}/n$.

It follows from (h) and (l) that

$$p\{(r_1/n)x \cdot (r_2/n)x \cdots (r_k/n)\} = p^k.$$

Hence x is a point of the set $A(p)$.

THEOREM 12. *Each set, $A(p)$, has the power of the continuum.*

Given any s, we can find $2(s!)$ independent numbers, $X_1{}^s, X_2\ , \cdots, X_{s!}{}^s$, $Y_1{}^s, \cdots, Y_{s!}{}^s$, such that $p(X_r{}^s) = p(Y_r{}^s) = p$. Let U_s and V_s be defined by the equations

$$(r/s!)\,U_s = X_r{}^s \qquad\qquad (r/s_s)\,V_s = Y_r{}^s$$

where $r = 1, 2, 3, \cdots, s!$. We can define two sets of integers, M_s and N_s, such that conditions (c) and (d) of theorem 11, hold for both sets of numbers, U_s and V_s.

Let a be any number such that $0 \leqq a \leqq 1$. Then $a = \cdot a^1a^2a^3 \cdots a^s \cdots$ where $a^s = 0$ or 1. Let us define a number, x, as follows. The digits $\nu_s + 1$ to ν_{s+1} of x are the same as the digits 1 to N_s of U_s or V_s according as a^s is one or zero. Then x belongs to $A(p)$. Moreover, there is a one to one correspondence between the numbers, x, and the numbers, a. Hence $A(p)$ has the power of the continuum.

Another property of the sets, $A(p)$, is the following: If x belongs to the set, $A(p)$ and if $y = \sum_{k=1}^{n} a^k 2^{-k} + x \cdot 2^{-n}$ where $a^k = 0$ or 1, then y belongs to the set, $A(p)$. Conversely, if y belongs to the set, $A(p)$, then x does also. That is, if the interval $(0, 1)$, be divided into 2^n equal parts, then the points of $A(p)$ in each of the sub-intervals, form an image of the whole set, $A(p)$. The numbers, x and y, are related by the equation $(n + 1/1)y = x$. Thus we have to prove the following theorem:

THEOREM 13. *$(N/1)x$ belongs to $A(p)$ if any only if x belongs to $A(p)$.*

Let us assume that s belongs to $A(p)$ and prove that $(N/1)x$ belongs to $A(p)$. We have the equation

$$p\{r_1/n)[(N/1)x]\cdot(r_2/n)[(N/1)x]\cdots(r_k/n)[(N/1)x]\}$$
$$=p[(R_1/n)x\cdot(R_2/n)x\cdots(R_k/n)x]$$

where $R_i=N+r_i-1$, $r_i\leqq n$ and $r_i\neq r_j$ if $i\neq j$.

Case I. If $(N-1)/n$ is an integer, then we can let $(N-1)=(\rho-1)n$ where ρ is an integer. Hence

$$p[(R_1/n)x\cdot(R_2/n)x\cdots(R_k/n)x]=p[(\rho/1)\{(r_1/n)x\cdots(r_k/n)x\}]$$
$$p[(r_1/n)x\cdot(r_2/n)x\cdots(r_k/n)x]=p^k.$$

Case II. If $N-1\leqq n$, then $R_i\leqq 2n$ and $R_i\neq R_j$ if $i\neq j$. Let q and m be two integers such that $0<q<m$. Then

$$p\{q/m)[(R_1/n)x\cdots(R_k/n)x]\}=p[(Q_1/M)x\cdots(Q_k/M)x]=p^k$$

where $M=n\cdot m$ and $Q_i=R_i+(q-1)n\leqq M$. Given any positive number, ϵ, we can first choose m so that $1/m<\epsilon/2$ and then choose a number, μ_0, so that

$$p_k-(\epsilon/2)<p_\mu\{(Q_1/M)x\cdots(Q_k/M)x\}<p^k+(\epsilon/2)$$

whenever $\mu\geqq\mu_0$, where $q=1,2,3,\cdots,m-1$. Then

$$p^k-\epsilon<[(m-1)/m]p^k-(\epsilon/2)<p_\nu\{(R_1/n)x\cdots(R_k/n)x\}$$
$$<p^k+(1/m)+(\epsilon/2)<p^k+\epsilon$$

whenever $\nu\geqq m\mu_0$. That is

$$p\{(R_1/n)x\cdots(R_k/n)x\}=p^k.$$

Case III. If $N-1>n$ and $(N-1)/n$ is not an integer, we can find two numbers, N' and N'' such that $(N'-1)/n$ is an integer $N''-1\leqq n$ and $N=N'+N''-1$. Then $(N/1)x=(N''/1)\{(N'/1)x\}$ and

$$p^k=\{(r_1/n)x\cdots(r_k/n)x\}$$
$$p[(r_1/n)\{(N'/1)x\}\cdots(r_k/n)\{(N'/1)x\}]$$
$$p[(r_1/n)\{(N/1)x\}\cdots(r_k/n)\{(N/1)x\}].$$

Therefore $(N/1)x$ belongs to the set, $A(p)$.

Next let us start with the assumption that $(N/1)x$ belongs to $A(p)$. Then $(N'/1)\{(N/1)x\}=[(N+N'-1)/1]x$ belongs to $A(p)$. Let us choose N' so that $N+N'-2=(\rho-1)n$ where ρ is an integer. Then

$$p^k=p[(r_1/n)\{[(N+N'-1)/1)x]\}\cdots(r_k/n)\{[(N+N'-1)/1]x\}]$$
$$=p[(\rho/1)\{(r_1/n)x\cdot(r_2/n)x\cdots(r_k/n)x\}]$$
$$=p[(r_1/n)x\cdots(r_k/n)x].$$

Therefore x belongs to $A(p)$.

COROLLARY. *If x belongs to $A(p)$ we can change arbitrarily a finite number of the digits of x and obtain in this way a new number, y, which belongs to $A(p)$.*

THEOREM 14. *Every point of the interval $(0,1)$ is a point of condensation of each of the sets, $A(p)$.*

This theorem is an immediate consequence of theorems 12 and 13.

From theorem 14 it follows that between any two distinct points of the interval $(0,1)$, each of the non-denumerable infinity of non-overlapping sets, $A(p)$, has a non-denumerable infinity of points.

V. *Independence of the conditions for admissibility.*

Equations (2) are not all independent. In fact, the equations $p\{(r/n)x\} = p$, where $r = 1, 2, 3, \cdots, n$, imply the equation, $p(x) = p$. Moreover, if n is the product of two integers, n_1, n_2, then since

$$(r_2/n_2)\{(r_1/n_1)x\} = [(r_1 + (r_2 - 1)n_1)/n_1n_2]x$$

and

$$(r_1/n_1)\{(r_2/n_2)x\} = [(r_2 + (r_1 - 1)n_2)/n_1n_2]x,$$

the equations $p\{(r/n)x\} = p$ imply the equations, $p\{(r_1/n_1)x\} = p$ and $p\{(r_2/n_2)x\} = p$, where $r_1 = 1, 2, \cdots, n_1$ and $r_2 = 1, 2, \cdots, n_2$.

However, the equations of the set, $p\{(r/n)x\} = p$ (where n is any prime number and $0 < r < n$), are independent. These equations constitute an infinite sub-set of equations (2). To establish their independence, it will be sufficient to prove the following theorem:

THEOREM 15. *Given an arbitrary prime number, N, and an integer, Q, such that $0 < Q < N$, there exists a number, x, such that*

$$p\{(Q/N)x\} \neq p$$

but $\quad p\{(R/N)x\} = p$ *if* $0 < R < N$ and $R \neq Q$

and $\quad p\{(r/n)x\} = p$ *if* $0 < r < n$

where n is an arbitrary prime number distinct from N.

Let X_R belong to the set, $A(p)$, if $0 < R < N$ and $R \neq Q$. Let X_Q and X_N belong respectively to $A(p_1)$ and $A(p_2)$, where $(p_1 + p_2)/2 = p$ and $p_1, p_2 \neq p$. Let x be defined by the equations

$$(R/N)x = X_R \text{ where } R = 1, 2, \cdots, N.$$

Then

$$p\{(Q/N)x\} = p_1 \neq p \quad \text{and} \quad p\{(R/N)x\} = p$$

if $0 < R < N$ and $R \neq Q$.

We have to show that $p\{(r/n)x\} = p$ if $0 < r < n$ and n is any prime number such that $n \neq N$. Given the numbers, n, N, r, R, there exist two numbers, r' and R', such that

$$(R/N)\{(r/n)x\} = (r'/n)\{(R'/N)x\}$$

The pair of numbers, r' and R', is a solution of the equation,

(a) $$r + (R-1)n = R' + (r'-1)N \qquad \text{where}$$

(b) $$0 < r' < n \quad \text{and} \quad 0 < R' < N.$$

The solution of (a) and (b) is unique, and since n and N are prime and $n \neq N$, corresponding to two distinct values of R there are two distinct values of R'. Hence as R takes on the values, 1, 2, 3, $\cdots$, N, R' also takes on the values, 1, 2, $\cdots$, N (but in different order). Thus, since

$$p[(R/N)\{(r/n)x\}] = p[(r'/n)X_{R'}] = p(X_{R'})$$

it follows from theorem 10 that

$$p\{(r/n)x\} = [((N-2)p + p_1 + p_2)/N] = p.$$

We have now proved that every admissible number must satisfy an infinite set of independent equations.

VI. *Sufficiency of the assumptions.*

In selecting our assumptions we have been using as a guide the formula, ${}_nC_r p^r(1-p)^{n-r}$. It is easily seen that our assumptions are such that this formula is verified. That is, if an event, x, is such that the number, x, belongs to $A(p)$, and if the trials of x are grouped in groups of n (in the manner previously described), then the event, X, which consists in the occurrence of r successes in a group of n, is such that $p(X) = P = {}_nC_r p^r(1-p)^{n-r}$.

However, we usually mean more than this when we say that an event, X, has the probability, P; we mean that X belongs to the set, $A(P)$. The fact that X does belong to the set, $A(P)$, is a consequence of the following theorem:

THEOREM 16. *If x belongs to the set, $A(p)$ and if $X = \sum_{j=1}^{M} Y_j$ and*

$$Y_j = \prod_{i=1}^{r_j} (r_{ij}/n)x \cdot \prod_{i=r_j+1}^{n} \{1 - (r_{ij}/n)x\}.$$

where $0 < r_{ij} \leqq n$ *and* $r_{i'j} \neq r_{ij}$ *if* $i' \neq i$ *and where* $Y_{j'} \neq Y_j$ *if* $j' \neq j$; *then* X *belongs to* $A(P)$, *where* $P = \sum_{j=1}^{M} p^{r_j}(1 - p)^{n-r_j}$.

By hypothesis $Y_{j'} \cdot Y_j = 0$ if $j' \neq j$, so that X is defined. We have to prove that

$$p\{\prod_{k=1}^{L} (\rho_k/\nu)X \cdot\} = P^L \text{ where } 0 < \rho_k \leqq \nu.$$

Let $Y_j^k = (\rho_k/\nu)Y_j)$ then

$$Y_j^k = \prod_{i=1}^{r_j} (R_{ij}^k/N)x \cdot \prod_{i=r_j+1}^{n} \{1 - (R_{ij}^k/N)x\}.$$

where $N = n\nu$ and $R_{ij} = r_{ij} + (\rho_k - 1)n$. Hence $Y_j^k \cdot Y_{j'}^k = 0$ if $j' \neq j$. Therefore

$$\prod_{k=1}^{L} [(\rho_k/\nu)X] = \prod_{k=1}^{L} [\sum_{j=1}^{M} Y_j^k] = \sum [\prod_{k=1}^{L} Y_{j_k}^k]$$

where the expression on the right is the formal algebraic expansion of the expression in the middle.

Since the numbers of each set, $Y_{j_1}^1, Y_{j_2}^2, \cdots, Y_{j_M}^M$, are independent, we have

$$p[\prod_{k=1}^{L} \{(\rho_k/\nu)X\}] = \sum \{\prod_{k=1}^{L} p(Y_{j_k}^k)\} = \prod_{k=1}^{L} [\sum_{j=1}^{M} p(Y_j^k)]$$

But

$$\sum_{j=1}^{M} p(Y_j^k) = p\{(\rho_k/\nu)X\} = p(X) = P.$$

Therefore

$$p[\prod_{k=1}^{L} \{(\rho_k/\nu)X\} = P^L,$$

and hence X belongs to $A(P)$.[*]

If the summation, $\sum_{j=1}^{M} Y_j$, includes all the possible numbers, Y, for which $z_1 \leqq (r_j/n) \leqq z_2$, then the event, X, consists in the occurrence of a success ratio lying between z_1 and z_2, in n trials of the event, x. If the summation, $\sum_{j=1}^{L} Y_j$ includes all numbers, Y_j, for which $r_j = r$, then $M = {}_nC_r$ and X consists in the occurrence of r successes in n trials of x. In either case the assumptions about the event, x, are such that the related event, X, is a probable event.

Even though the assumptions (a) and (b) demand more, or at least as much as (1), (2), (3), the former seem natural and the latter, somewhat artificial. The method of grouping defined in assumptions (1), (2), (3) is a simple one which would very likely be chosen if, for example, one wished to test experimentally the formula, ${}_nC_rp^r(1-p)^{n-r}$. On the other hand there is no physical reason why other methods of grouping should not work equally well.

---

[*] It follows from Theorem 16 that every combination of ones and zeroes must be displayed infinitely many times as successive digits of every admissible number.

Suppose we choose a set of positive integers, $\lambda_1 < \lambda_2 < \lambda_3 \cdots \lambda_n \cdots$ and make λ_n trials of an event, x, with probability, p. If we record the trials numbered λ_1, λ_2, etc., and if n is large, we should expect approximately pn successes of the recorded trials. We should also expect the success ratio to approach p as n becomes infinite. That is, if y is such that $y^i = x^{\lambda_i}$ then we should expect to have the equation, $p(y) = p$. But this is impossible for every set of numbers, $\lambda_1, \lambda_2, \lambda_3, \cdots$, since in particular a set might be chosen (even in advance) so that $y = 0$. It seems obvious that such a choice is not probable. On the other hand it might be probable that y should not be admissible. Let us consider the following theorem.

THEOREM 17. *If* $\lambda_1 < \lambda_2 < \lambda_3 < \cdots < \lambda_n \cdots$ *is any set of positive integers, and if* y *is defined by the equations*

$$(R/N)y = (\lambda_R/\lambda_N)x$$

where x *belongs to the set,* $A(p)$*; then* y *belongs to* $A(p)$.

We have to prove that $p\{(r_1/n)y \cdot (r_2/n)y \cdots (r_k/n)y\} = p^k$. We have the equation, $(R/N)\{r_i/n)y\} = (r_i'/n)\{(R_i'/N)y\}$ where r_i' and R_i' are two integers satisfying the equation, $r_i + (R-1)n = R_i' + (r_i'-1)N$ and such that $0 < r' < n$ and $0 < R' < N$. Hence

$$(R/N)\{r_i/n)y\} = (\rho_i/\nu)x \quad \text{where} \quad \nu = n\lambda_N \quad \text{and}$$

$\rho_i = \lambda_{R_i'} + (r_i'-1)\lambda_N$ and where $\rho_i \neq \rho_j$ if $i \neq j$. Therefore

$$p\{(r_1/n)y \cdot (r_2/n)y \cdots (r_k/n)y\} = p\{(\rho_1/\nu)x \cdot (\rho_2/\nu)x \cdots (\rho_k/\nu)x\} = p^k.$$

It follows from Bernoulli's theorem and from theorem 17 that if N is large, then it is very probable that $\sum_{i=1}^{N} (x^{\lambda_i}/N)$ will differ from $p(x)$ by a small amount.* Thus if an event has an admissible history, its behavior is in accordance with what should be expected of a probable event.

THE RICE INSTITUTE,
HOUSTON, TEXAS.

---

* Jacobus Bernoulli, *Ars Conjectandi*, Basle (1713), p. 236.

The Differential Invariants of Inversive Geometry.

By Boyd C. Patterson.

---

1. *The inversive curvature.* The idea of an intrinsic equation in geometries other than ordinary geometry seems to have originated with Pick about 1906.[*] In 1914 Nohel [†] published a paper in which he calculated the differential and integral invariants for all geometries under the plane transformation groups, but he did not consider explicitly what is now known as inversive geometry—that is, the geometry defined by the six parameter group of transformations

$$Z = (az + b)/(cz + d)$$

together with

$$w = \bar{Z}$$

The bilinear transformation constitutes, strictly speaking, the homographic group and only when we add an inversion in a circle, or in particular a reflexion in the axis of reals, do we have the complete inversive group as given above. The homography contains an even number of inversions and is a subgroup of the inversive group.

G. W. Mullins in a Columbia dissertation of 1917 gives the differential invariant of lowest order under the inversive group expressed (1) in rectangular coördinates and (2) in terms of the ordinary invariants r, the radius of curvature, and s, the arc-length.

Liebmann,[‡] Kubota,[§] and Morley [¶] later calculated, by different methods, this invariant which we shall call the inversive curvature in analogy with the corresponding differential invariant of ordinary geometry. We shall now develop a simple expression for the same—an expression by means of which the inversive curvature may easily be calculated for any curve given parametrically. Later sections will be devoted to applications of this formula.

---

[*] Georg Pick, "Natürliche Geometrie ebener Transformationsgruppen," *Sitzungsberichte der Akademie der Wissenschaften*, Wien, Bd. 115 (1906), S. 139.

[†] Emil Nohel, "Zur natürlichen Geometrie ebener Transformationsgruppen," *Ibid.*, Bd. 123 (1914), S. 2085.

[‡] Liebmann, "Beitrage zur Inversionsgeometrie der Kurven," *Sitzungsberichte der Bayerischen Akademie der Wissenschaften zu München*, Heft I (1923), S. 79.

[§] Kubota, "Beitrage zur Inversionsgeometrie," *Tohoku Imperial University, Science Reports* 13, (1924-25), p. 243.

[¶] Morley, "On Differential Inversive Geometry," *American Journal of Mathematics*, Vol. 48 (1926), p. 144.

Under the group of homographies,

$$Z = (az + b)/(cz + d),$$

a subgroup of the inversive group, we have as a seminvariant of a curve expressed as a complex function of a real parameter μ, $z = z(\mu)$, the Schwarzian derivative of z with respect to μ:

$$\{z, \mu\} = \frac{d}{d\mu}\left(\frac{d^2z/d\mu^2}{dz/d\mu}\right) - \frac{1}{2}\left(\frac{d^2z/d\mu^2}{dz/d\mu}\right)^2.$$

By a seminvariant we mean an expression invariant under bilinear transformations of the dependent variable but not invariant under changes of the parameter. The real and imaginary parts of $\{z, \mu\}$ give us in fact two real seminvariants.

We shall use one of these seminvariants to fix a parameter λ (which we call the inversive arc-length) and the other seminvariant when expressed in terms of λ will become a true differential invariant.[*] We shall, in fact, so fix the parameter λ that $\{z, \lambda\}$ will have its imaginary coefficient equal to 1. The real part of $\{z, \lambda\}$ which is now a true differential invariant we call I_5 for reasons which will appear later.

We have here fixed λ by making the imaginary coefficient of $\{z, \lambda\}$ equal to 1. Might we not have fixed λ so that the real part of $\{z, \lambda\}$ is 1 and the imaginary coefficient the differential invariant? It will be seen in the following paragraph that this cannot be done since λ would then be the solution of a third order differential equation and the number of arbitrary constants would be too great; whereas λ so chosen that

$$\{z, \lambda\} = I_5 + i$$

involves only one arbitrary constant which can easily be disposed of by a suitable selection of the origin of arc-lengths.

Under change of parameter from μ to λ, λ being a real function of μ, we have the formula, due to Cayley, for change of the Schwarzian:

$$\{z, \mu\} = \lambda'^2\{z, \lambda\} + \{\lambda, \mu\},$$

---

[*] Thus in ordinary geometry if a curve is expressed in terms of the ordinary arc-length, s, then the seminvariant $(dx/ds)^2 + (dy/ds)^2$ has the value 1.

primes denoting differentiation with respect to μ. Putting now

$$\{z, \mu\} = A + iB$$

and

$$\{z, \lambda\} = I_5 \pm i$$

we have

$$A + iB = \lambda'^2(I_5 \pm i) + \{\lambda, \mu\},$$

whence λ, the integral invariant, is defined by

$$\lambda = \int (\pm B)^{1/2}\, d\mu, \tag{1.1}$$

the sign being chosen so that λ is real, and I_5, the differential invariant, by

$$I_5 = (A - \{\lambda, \mu\})/B.$$

Since now

$$\pm B = \lambda'^2$$

we can easily calculate $\{\lambda, \mu\}$ in terms of B and its derivatives with respect to μ. It is found to be

$$\{\lambda, \mu\} = (4BB'' - 5B'^2)/8B^2$$

whence

$$I_5 = (8AB^2 - 4BB'' + 5B'^2)/8B^3. \tag{1.2}$$

Given now any curve expressed as a complex function of a real parameter, μ, we find the real and imaginary parts of the Schwarzian $\{z, \mu\}$, and the inversive curvature can be calculated by means of (1.2).

To reduce formula (1.2) to an expression involving the ordinary differential and integral invariants, r and s, it is necessary to make use of the relation

$$(d^2z/ds^2)/(dz/ds) = i/r.$$

This may be easily verified by writing $z = x + iy$ and recalling that $(dx/ds)^2 + (dy/ds)^2 = 1$.

If now we assume a curve to be given as a complex function of the real parameter s, the real and imaginary parts of $\{z, s\}$ are found to be $1/2r^2$ and $-(dr/ds)/r^2$ respectively. Making these substitutions for A and B in (1.2) we have I_5 in terms of the ordinary invariant r and its derivatives with respect to s—viz.

$$I_5 = [4r^2(dr/ds)(d^3r/ds^3) - 5r^2(d^2r/ds^2)^2 - 4r(dr/ds)^2(d^2r/ds^2) + 4(dr/ds)^4 - 4(dr/ds)^2]/8(dr/ds)^3. \tag{1.3}$$

Since r is of the second order in rectangular coördinates, I_5 is of the fifth order. Hence the notation I_5.

Another convenient form of the inversive curvature is to have it expressed in terms of the homologic (equiform) invariants ρ $(=dr/ds)$ and ϕ $(=\int ds/r)$. Making use of (1.3) and the relations

$$dr/ds=\rho\,;\ d^2r/ds^2=d\rho/rd\phi\,;\ d^3r/ds^3=(d^2\rho/d\phi^2-\rho d\rho/d\phi)/r^2$$

and indicating differentiation with respect to the homologic integral invariant, ϕ, by primes we obtain

$$I_5=(4\rho\rho''-5\rho'^2-8\rho'\rho^2+4\rho^4-4\rho^2)/8\rho^3. \tag{1.4}$$

We have previously defined λ as a function of any real parameter, μ, in terms of which a curve may be given. If now the parameter of the curve is the ordinary arc-length, s, since B, the imaginary coefficient of the Schwarzian, is in this case $-(dr/ds)/r^2$ we have

$$d\lambda^2=\pm(dr/ds)ds^2/r^2=\pm drds/r^2=\pm dkds$$

where $k=1/r$ is the ordinary curvature. Similarly the inversive arc-length may be defined in terms of ρ and ϕ:

$$d\lambda^2=\pm\,\rho d\phi^2.$$

We have now the fundamental invariants of differential inversive geometry—the differential invariant I_5 and the integral invariant λ. To obtain differential invariants of higher order we differentiate I_5 with respect to λ. That is,

$$I_n=dI_{n-1}/d\lambda.$$

In the foregoing we have been concerned exclusively with invariants under the group of homographies, speaking of them however as inversive invariants. And this is perfectly proper, for we notice that since expressions invariant under all inversions must be invariant under homographies and also the transformation

$$w=\bar{z}$$

(which is simply a reflexion in the real axis) and since I_5 and λ are both real they are therefore invariant under the latter transformation as well as under the bilinear transformation.

At points where the differential of arc becomes zero, B, the imaginary coefficient of $\{z,\mu\}$, may change sign. Thus the inversive arc-length associated with a portion of a given curve between two points M and N may be a complex number if there is between M and N a circular point, i. e. a point where $k = 1/r$ is constant or $dk = 0$. For if we call such a point P, the number associated with the arc-length MP is real, but on passing through P the number associated with the arc PN becomes pure imaginary. The sum then of the inversive arc-lengths MP and PN is complex.

To avoid this difficulty we shall adopt the convention previously mentioned of defining λ as in (1. 1) and choosing the sign under the radical so that λ is *always* real.

A relation between the differential invariant I_5 and the integral invariant λ of a curve will be spoken of as an inversive intrinsic equation of the curve. All curves having the same inversive intrinsic equation are inversively equivalent—that is, they are the same or may be made the same by an inversive transformation.

The next question which arises is: What are the curves of constant inversive curvature? These curves are known to be the so-called "double logarithmic spirals. If we were concerned here with the theory of groups we might easily find them as path curves of one-parameter subgroups of the homographic group. As it is, however, we will content ourselves with an examination of the differential equation obtained by equating (1. 4) to a constant, which shows $\rho = \text{const.}$ to be one solution. Hence the homologic intrinsic equation of the curves corresponding to this solution is

$$\rho = a$$

and since $\rho = dr/ds$ the ordinary intrinsic equation is that of a logarithmic spiral, viz.

$$r = as + b.$$

Making $b = 0$ by shifting the origin of arc-lengths we have

$$r = as. \tag{1. 5}$$

We find now from (1. 3)

$$I_5 = (a^2 - 1)/2a,$$

the inversive intrinsic equation of the logarithmic spiral. In this discussion we prefer to speak of loxodromes rather than logarithmic spirals. By definition a loxodrome is a curve cutting arcs of circles from a point to another under a constant angle.

By an inversive transformation such that zero and infinity are sent into two finite points of the plane, m and n, the loxodrome whose ordinary intrinsic equation is (1. 5) becomes a "double logarithmic spiral" or as we prefer to call it, simply a loxodrome. Inversively there is no difference between these two loxodromes; each has two poles, in the one they are at zero and infinity and in the other at two finite points, m and n.

The loxodrome given by the ordinary intrinsic equation (1. 5) has as its equation in polar coördinates

$$R = ce^{a\theta}$$

where $a = \cot \alpha$, α being the constant angle of the loxodrome. This gives us a simple expression for I_5, the inversive curvature:

$$I_5 = \cot 2\alpha.$$

We see that for a loxodrome of angle α all points have the same inversive properties for I_5 has the same value at all points of the curve.

The geometrical interpretation of the differential invariant I_5 is that

$$I_5 - \cot 2\alpha = 0 \tag{1.6}$$

is the differential equation of the ∞^5 family of loxodromes with fixed angle α.* Given a curve Γ, it may happen that there are points, P, on it at which Γ is very much like the loxodrome; which means that there the loxodrome has fifth order contact. The condition for this is that at P the value of the differential invariant I_5 of Γ must be equal to $\cot 2\alpha$ where α is the angle of the loxodrome. Furthermore, since for all loxodromes $I_6 = 0$, the condition that there be points of Γ at which some loxodrome hyperosculates is that at certain points of Γ, $I_6 = 0$. The angle α of the hyperosculating loxodrome in this case is determined by the equation

$$\alpha = \tfrac{1}{2} \operatorname{arccot} I_5]_P.$$

2. *The differential invariants of the biquadratic.* We shall now investigate the differential inversive invariants of the biquadratic curve, taking up first the case of a biquadratic with a double-point, then a biquadratic with a cusp, and finally the general biquadratic.

If a biquadratic should have a double-point that double-point may be sent by an inversive transformation to infinity and we will then have what is customarily called an hyperbola; a rectangular hyperbola is the inverse of a biquadratic with a double-point in which branches cut each other at right angles.

---

* G. W. Mullins, *Dissertation on Differential Inversive Invariants*, Columbia, 1917.

Expressed as a complex function of a real parameter the hyperbola is given by

$$z = \mu + t/\mu$$

where μ is the real parameter, and t, a complex number of absolute value unity, is constant. For $t = i$ we have the parametric representation of a rectangular hyperbola and we shall take this as our example of a nodal biquadratic. The values of μ corresponding to the double-point are $\mu = 0, \infty$.

Upon calculating the Schwarzian $\{z, \mu\}$ and substituting its real and imaginary parts in formula (1.2) we find

$$I_5 = -[5\mu^2/3(\mu^4 - 1)^3]\,[3(\mu^4 + 1)^2 - 8\mu^4].$$

Differentiating I_5 with respect to the integral invariant

$$\lambda = 6^{\frac{1}{2}} \int \frac{(\mu^4 - 1)^{\frac{1}{2}}}{\mu^4 + 1}\, d\mu \text{ we obtain}$$

$$I_6 = \frac{10\mu}{6^{\frac{1}{2}}(\mu^4 - 1)^{9/2}}\,(\mu^4 + 1)^4,$$

and in the same manner

$$I_7 = \frac{5(\mu^4 + 1)^4}{3(\mu^4 - 1)^6}\,[(\mu^4 + 1)^2 + 32\mu^4].$$

An examination of these invariants shows that

$$I_5 = 0 \text{ when } \mu = 0, \infty.$$
$$I_6 = 0 \text{ when } \mu = 0, \infty.$$
$$I_7 \text{ never vanishes for real } \mu.$$

Now for $\mu = 0, \infty$; $z = \infty$. And at this point alone (the double-point) $I_6 = 0$. From the results of Section 1 we know that this is the condition that some loxodrome hyperosculates the curve at $z = \infty$; and it is precisely that loxodrome for which $I_5 = 0$, i. e. a 45° loxodrome. At the double-point then of a biquadratic which cuts itself orthogonally we find the curve behaving very much like a 45° loxodrome.

A cuspidal biquadratic may be inverted with the center of inversion at the cusp. This inversion will put the cusp at infinity and the curve thus inverted becomes a parabola. The parametric equation of the curve is and

$$z = 2\mu + i\mu^2,$$

and we find its invariants to be as follows:

$$I_5 = (5/24\mu^3)\ \ [3(\mu^2-1)^2-4]$$
$$I_6 = (5/8\cdot 3^{1/2}\cdot \mu^{9/2})\ \ (1+\mu^2)^3$$
$$I_7 = (-5/16\mu^6)\ \ (\mu^2+1)^3\ (\mu^2-3)$$

and $I_5 = 0$ when $\mu = \pm(1+2/3^{1/2})^{1/2}$

I_6 never vanishes for real μ

$I_7 = 0$ when $\mu = \pm\ 3^{1/2}$.

The map equation of the general biquadratic is given by the Weierstrassian p-function,

$$z = p(u), \tag{2.1}$$

with a rectangular network. For such a network the coefficients of the p-function are real and hence

$$\bar{z} = \bar{p}(\bar{u}) = p(\bar{u}).$$

It is shown in Harkness and Morley *Theory of Functions,* p. 336 that as u varies along a fixed horizontal line the map in the z-plane is a Cartesian curve consisting of two ovals. When it is necessary for us to bring into prominence the fact that the variable u moves along a horizontal line we shall write u in the form $\alpha + i\beta_0$, thus indicating that the imaginary part of the variable u is fixed. Limiting u in this manner, the complete Cartesian mapped by (2.1) includes the representation of four horizontal lines of the rectangle of periods; the lines $i\beta_0$, $2\omega_2 - i\beta_0$ correspond to the outer oval and the lines $\omega_2 \pm i\beta_0$ to the inner oval, $2\omega_1$ and $2\omega_2$ being the real and imaginary periods of the network.

A Cartesian is a biquadratic one of whose foci is at infinity. Since foci invert into foci any general biquadratic may be sent into a Cartesian. The three real foci of the curve (2. 1) are e_1, e_2, e_3, where $e_i = p(\omega_i)$.

When u varies along a quarter line of the rectangle, i. e. when $i\beta_0 = \omega_2/2$, $3\omega_2/2$, the two ovals of the Cartesian come together into a circle with center at e_2 and having e_1 and e_3 as inverse points. And when u traces an eighth line, i. e. $i\beta_0 = \omega_2/4$, $3\omega_2/4$, $5\omega_2/4$, $7\omega_2/4$, the map in the z-plane is a Cartesian which is the inverse of a Cassinian.

What has been said concerning the mapping of horizontal lines holds equally well for the mapping of vertical lines. The vertical lines of course map on the z-plane a system of confocal Cartesians orthogonal to those mapped by horizontal lines.

The Cartesian mapped by (2.1) is therefore expressed as a complex function of a real parameter α, since $u = \alpha + i\beta_0$. We can calculate its invariants by the method of Section 1.

Let us recall the fundamental relation between the p and p' functions, viz.

$$p'^2(u) = 4p^3(u) - g_2 p(u) - g_3,\text{*} \tag{2.2}$$

from which it follows that

$$p''(u) = 6p^2(u) - g_2/2$$
$$p'''(u) = 12p(u)p'(u).$$

Following the customary notation in the theory of elliptic functions we shall denote by primes (′) differentiation with respect to the variable indicated. For example

$$p'u = d(pu)/du; \quad p'2u = d(p2u)/d(2u) = d(p2u)/2du.$$

We now have

$$\{z, u\} = z'''/z' - 3z''^2/2z'^2 = 12pu - 3p''^2u/2p'^2u$$

and by the addition formula for the p-function,

$$(p'u - p'v)^2/4(pu - pv)^2 = p(u + v) + pu + pv, \tag{2.3}$$

when $u = v$ we have the further fact that

$$p''^2u/4p'^2u = p2u + 2pu.$$

Making these simplifications the Schwarzian reduces to the compact form:

$$\{z, u\} = -6p2u.$$

Separating the Schwarzian into its real and imaginary parts each of which is a real function of u we have

$$A(u) = -3(p2u + p2\bar{u})$$
$$B(u) = 3i(p2u - p2\bar{u})$$
$$B'(u) = 6i(p'2u - p'2\bar{u})$$
$$B''(u) = 12i(p''2u - p''2\bar{u})$$

and

$$d\lambda = B^{1/2}du = B^{1/2}d(2u)/2$$

or

$$d(2u)/d\lambda = 2/B^{1/2}.$$

---

[*] The coefficients g_2 and g_3 are invariants of the p-function; in terms of the e_i's they have the values: $g_2 = -4(e_2e_3 + e_3e_1 + e_1e_2)$; $g_3 = 4e_1e_2e_3$.

Making use of formula (1.2)

$$I_5 = \frac{i}{8\cdot 27(p2u - p2\bar{u})}\left[8\cdot 27(p2u + p2\bar{u}) + 144\,\frac{p''2u - p''2\bar{u}}{p2u - p2\bar{u}} - 180\left(\frac{p'2u - p'2\bar{u}}{p2u - p2\bar{u}}\right)^2\right]$$

But from (2.2)

$$(p''2u - p''2\bar{u})/(p2u - p2\bar{u}) = 6(p2u + p2\bar{u})$$

and from (2.3)

$$(p'2u - p'2\bar{u})/(p2u - p2\bar{u}) = 4[p4\alpha + p2u + p2\bar{u}]$$

since $u + \bar{u} = 2\alpha$.

Therefore

$$I_5 = \frac{5i[p2u + p2\bar{u} - 2p4\alpha]}{3(p2u - p2\bar{u})}\ .$$

To obtain I_6 we must differentiate I_5 with respect to $2u$ and multiply by $2/B^{1/2}$. This gives

$$I_6 = \frac{k_1}{(p2u - p2\bar{u})^{3/2}}\left[p'2u + p'2\bar{u} - 4p'4\alpha + [2p4\alpha - p2u - p2\bar{u}]\,\frac{p'2u - p'2\bar{u}}{p2u - p2\bar{u}}\right]$$

$$I_6 = \frac{2k_1}{(p2u - p2\bar{u})^{3/2}}\left[p4\alpha\,\frac{p'2u - p'2\bar{u}}{p2u - p2\bar{u}} - 2p'4\alpha + \frac{p2u\cdot p'2\bar{u} - p2\bar{u}\cdot p'2u}{p2u - p2\bar{u}}\right]$$

which can be arranged as follows:

$$I_6 = \frac{2k_1}{(p2u - p2\bar{u})^{3/2}}\left[(p4\alpha - p2\bar{u})\,\frac{p'2u - p'2\bar{u}}{p2u - p2\bar{u}} - 2p'4\alpha + p'2\bar{u}\right]\cdot$$

As a consequence of the addition formula (2.3) if $u_1 + u_2 + u_3 = 0$ we have the following relation:

$$\frac{p'u_2 - p'u_3}{pu_2 - pu_3} = \frac{p'u_3 - p'u_1}{pu_3 - pu_1} = \frac{p'u_1 - p'u_2}{pu_1 - pu_2}$$

which gives for $u_1 = 2u$, $u_2 = 2\bar{u}$, $u_3 = -4\alpha$ when we recall that p' is an odd function and p an even function:

$$\frac{p'2u - p'2\bar{u}}{p2u - p2\bar{u}} = \frac{p'2\bar{u} + p'4\alpha}{p2\bar{u} - p4\alpha}\ .$$

With this further substitution we now obtain:

$$I_6 = Kp'4\alpha/(p2u - p2\bar{u})^{3/2}.^*$$

We see that $I_5 = \infty$ when $p2u = p2\bar{u}$. Let us write $u = \alpha + i\beta_0$ and then we will have $I_5 = \infty$

when $$p(2\alpha + 2i\beta_0) = p(2\alpha - 2i\beta_0)$$

or, since p is an even function,

when $$p(2\alpha + 2i\beta_0) = p(2i\beta_0 - 2\alpha).$$

This equality is satisfied when

$$2\alpha = n\omega_1, \qquad n, \text{ an integer.}$$

Hence $\alpha = n\omega_1/2$ and we need consider only those values of α which lie in the fundamental rectangle, i. e.

$$\alpha = 0, \ \omega_1/2, \ \omega_1, \ 3\omega_1/2.$$

For α equal to any of these values, the denominator of I_5 vanishes and the numerator becomes infinite,—hence $I_5 = \infty$.

Let us see what points on the Cartesian correspond to these values of α in the rectangle. We have previously mentiond the fact that the system of confocal curves orthogonal to those given by (2.1) with $u = \alpha + i\beta_0$ are those mapped by the equation

$$z = p(\alpha_0 + i\beta).$$

For $\alpha_0 = 0$, ω_1, the map in the z-plane consists of the real axis and for $\alpha_0 = \omega_1/2$, $3\omega_1/2$ the map in the z-plane is the circle with center e_1 and about e_3 and e_2. The eight points cut out of the Cartesian by this circle and real axis are therefore those points for which $I_5 = \infty$.

These eight points are special points of the Cartesian. To show this we proceed as follows. The parametric equation

$$z = p(\alpha + i\beta_0) \tag{2.4}$$

for real values of α maps a Cartesian in the z-plane as mentioned before. The conjugate point $\bar{z} = p(\alpha - i\beta_0)$ is also a point on the curve. For real α equation (2.4) gives points z of the curve.

Let us now allow α to take complex values. The map of

$$z = p(\alpha + i\beta_0)$$

---

* K and k_1, are constant factors and $K = -6k_1$.

in the z-plane is no longer a point of the curve but it is a point which has associated with it an image point, y, in the Cartesian, y being given by the equation

$$\bar{y} = p(\alpha - i\beta_0).$$

Hence (2.4) and its conjugate give for real values of α points on the curve; for complex values of α, a point z and its image point y in the Cartesian.

A circle

$$z\bar{y} - c\bar{y} - \bar{c}z + c\bar{c} - r^2 = 0$$

and the biquadratic (2.4) have either common points of intersection or common inverse points. For the case in hand we suppose the circle and the biquadratic to intersect and we seek their common points in that sense. These points are points whose elliptic parameters, α, are zeros of

$$p(\alpha + i\beta_0) \cdot p(\alpha - i\beta_0) - cp(\alpha - i\beta_0) - \bar{c}p(\alpha + i\beta_0) + c\bar{c} - r^2.$$

This is an elliptic function of α having a double pole at $\alpha = i\beta_0$ and $\alpha = -i\beta_0$—hence it has four poles whose sum is zero and by Abel's Theorem which states that the sum of the zeros of an elliptic function must be congruent to the sum of the poles, it must have also four zeros, α_1, α_2, α_3, α_4; furthermore

$$\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 \equiv 0 \mod 2\omega_1.$$

It is evident that at apses where a circle has four point contact with the biquadratic

$$4\alpha \equiv 0 \mod 2\omega_1$$

or

$$\alpha = n\omega_1/2 \qquad n, \text{ an integer.}$$

But these are the points of the biquadratic where $I_5 = \infty$. Therefore at apses on a biquadratic $I_5 = \infty$.

Returning to the hyperbola and parabola we see that the former has two apses at $\mu = \pm 1$ (the vertices) and the latter one apse at $\mu = 0$ (the vertex).

Let us now examine those points of the biquadratic where $I_6 = 0$. We shall find that they too are special points of the curve. At such points

$$p'4\alpha = 0,$$

that is

$$4\alpha \equiv \omega_1 \mod 2\omega_1$$

or

$$\alpha = \omega_1/4,\ 3\omega_1/4,\ 5\omega_1/4,\ 7\omega_1/4.$$

And, analogous to the preceding case, these points are cut from the biquadratic (2.4) by that curve of the confocal orthogonal system for which α_0 takes values congruent to $\omega_1/4$ mod $\omega_1/2$. This curve is mapped by the vertical lines which divide the period rectangle in eighths and as mentioned before it is the inverse of a Cassinian orthogonal to and having the same foci as the original biquadratic.

Proceeding as before suppose a biquadratic

$$a_{00}z^2\bar{z}^2 + 2a_{01}z^2\bar{z} + 2a_{10}z\bar{z}^2 + a_{02}z^2 + \cdots = 0$$

should intersect the given biquadratic (2.4). The elliptic parameters of these points of intersection are zeros of the elliptic function

$$a_{00}p^2(\alpha + i\beta_0) \cdot p^2(\alpha - i\beta_0) + 2a_{01}p^2(\alpha + i\beta_0) \cdot p(\alpha - i\beta_0) + \cdots.$$

Since this function has a four-fold pole at $\alpha = i\beta_0$ and $\alpha = -i\beta_0$ it must therefore have eight zeros

$$\alpha_1 + \alpha_2 + \cdots + \alpha_8 \equiv 0 \mod 2\omega_1.$$

In general only one biquadratic can be put through eight points; but if the eight points are the points of intersection of two biquadratics, through them a single infinity of biquadratics may be passed.

If now we have two biquadratics U_1, U_2 each of which has sixth order contact with a given curve C at P, then each member of the pencil $U_1 + kU_2$ will have seven points in common at P and an eighth point at, say, P'. Among this single infinity of biquadratics $U_1 + kU_2$, there will be one, U_0, having a double-point at P and one branch in fifth order contact with C and each member of the pencil; and furthermore U_0 will have necessarily the eighth point of intersection at P'. If now U_0 should have one branch in sixth order contact with any one of the pencil, P', the eighth point of intersection must coincide with P. Conversely, if the eighth point of intersection should coincide with P, then U_0 will have one branch in sixth order contact with each one of the pencil $U_1 + kU_2$.

Let us now take a biquadratic U given by

$$z = p(\alpha + i\beta_0).$$

Its points of intersection with a second biquadratic V will, in general, be eight in number. Let furthermore, the elliptic parameters of these eight points of intersection be α_i which must, we know, satisfy the condition

$$a_1 + a_2 + \cdots + a_8 \equiv 0 \mod 2\omega_1.$$

If now the biquadratic V is to have eight points in common with U at a point P, it must have a double-point at P and one of its branches must have sixth order contact with U at P; and if the elliptic parameter of the point P is α, then

$$8\alpha \equiv 0 \mod 2\omega_1.$$

We have seen that for $\alpha = \omega_1/4,\ 3\omega_1/4,\ 5\omega_1/4,\ 7\omega_1/4$ the differential invariant I_6 vanishes and at these points of U, V may have eight points in common with U, since for all of them $8\alpha \equiv 0 \mod 2\omega_1$. They are points cut out of U by the confocal inverted Cassinians mapped by the vertical lines of the rectangle.

These special points of the Cartesian we call Cassinian points or "apses of the second order." This latter terminology is suggested by the analogy with those points cut out by the map of the quarter lines of the rectangle where circles may have four point contact.

At these apses of the second order some loxodrome hyperosculates the biquadratic for there $I_6 = 0$. Furthermore all loxodromes consist entirely of points which are "apses of the second order."

3. *The Cornu Spiral.* As a final example of the differential inversive invariants of a curve we call attention to those curves whose ordinary intrinsic equation is of the type

$$r = as^m.$$

In this case the simplest method of finding I_5 is by making use of the relation

$$(d^2z/ds^2)/(dz/ds) = i/r = i/as^m.$$

Hence
$$\{z, s\} = d[(d^2z/ds^2)/(dz/ds)]/ds - (d^2z/ds^2)^2/2(dz/ds)^2$$
$$= -mi/as^{m+1} + 1/2a^2s^{2m},$$

giving
$$A = 1/2a^2s^{2m}; \quad B = -m/as^{m+1},$$

and now by formula (1.2)

$$I_5 = -1/2mas^{m-1} - a(m+1)(m-3)s^{m-1}/8m$$

and since

$$\lambda = (\pm m/a)^{1/2} \int s^{-(1+m)/2} ds = 2(\pm m)^{1/2} s^{(1-m)/2}/(1-m)a^{1/2} + \text{const.}$$

$$\begin{aligned} I_6 &= (m-1)s^{(1-m)/2}/2m(\pm ma)^{1/2} \\ &\qquad - a^{3/2}(m^2-1)(m-3)s^{3(m-1)/2}/8(\pm m)^{3/2} \\ I_7 &= -(m-1)^2/4m^2 - 3a^2(m-1)^2(m+1)(m-3)s^{2(m-1)}/16m^2 \\ I_8 &= -3a^{5/2}(m-1)^3(m+1)(m-3)s^{5(m-1)/2}/8m^2(\pm m)^{1/2}. \end{aligned} \tag{3.1}$$

For $m = 1$ we have the loxodrome previously considered in Section 1. For $m = -1$ we have the ordinary intrinsic equation of the Cornu Spiral[*] or clothoid:

$$r = a/s.$$

The differential invariants of this curve are given by (3.1) for $m = -1$. They are

$$\begin{aligned} I_5 &= s^2/2a = \lambda^2/2, \qquad \text{since } \lambda = \int ds/a^{1/2} = s/a^{1/2} \\ I_6 &= s/a^{1/2} = \lambda \\ I_7 &= 1 \\ I_8 &= 0. \end{aligned}$$

We have here an example of a curve whose invariant I_7 is constant. The problem of finding all curves for which $I_7 = \text{const.}$ remains at yet unsolved. We have however another special case of curves of this type—namely those curves whose ordinary intrinsic equation is

$$r = As^3.$$

When $A = -1/a$ this is precisely the evolute of the Cornu Spiral, $r = a/s$. Referring again to (3.1) we see that for $m = 3$ the differential invariants of this evolute are

$$\begin{aligned} I_5 &= -1/6As^2 = -\lambda^2/18 \\ I_6 &= 1/3^{3/2} A^{1/2} s = -\lambda/9 \\ I_7 &= -1/9 \\ I_8 &= 0. \end{aligned}$$

---

[*] Loria, "Spezielle Algebraische und Transcendente Ebene Kurven," Art. 194.

We have given two particular examples of curves whose inversive intrinsic equation is

$$I_5 = a\lambda^2 + b\lambda + c$$

Professor Morley, in the reference given on p. 2, discusses the case of curves whose inversive intrinsic equation is

$$I_5 = a\lambda + b$$

and, as he points out in that paper, the general problem of finding those curves having the inversive intrinsic equation $I_5 = f(\lambda)$ involves the solution of the differential equation

$$\{z, \lambda\} = f(\lambda) \pm i.$$

Expansions of the Neumann Type in Terms of Products of Bessel Functions.

By Guy Stevenson.

I. Introduction.

1. *Statement of the Problem.* The object of this paper is the generalization of certain expansions of functions in terms of Bessel functions, and related functions, which expansions have been investigated by Neumann, Gegenbauer and others. In his *Theorie der Bessel'schen Funktionen* Neumann has obtained the expansion [*]

$$f(z) = \sum_{n=0}^{\infty} a_n J_n(z),$$

where $f(z)$ is single-valued and analytic in the neighborhood of the origin, and the coefficients a_n are constants whose values are reproduced in Art. 13. This expansion has been generalized by Gegenbauer [†] to one of the form [‡]

$$z^{\nu} f(z) = \sum_{n=0}^{\infty} b_n J_{\nu+n}(z),$$

where ν is any complex quantity, except a negative integer, and the coefficients b_n are constants whose values are reproduced in Art. 13. Also, to Gegenbauer Watson attributes the expansion [§]

$$z^{\mu+\nu} f(z) = \sum_{n=0}^{\infty} c_n J_{\mu+n/2}(z) J_{\nu+n/2}(z),$$

where μ and ν are any complex quantities, except the halves of negative integers, and the coefficients c_n are constants whose values are also given in Art. 13.

In this paper we extend the expansion theories of Neumann and Gegenbauer in four general directions. In the first place each term in the generalized expansions contains, as factors, any number n of Bessel functions. In the second place the order of each of the Bessel functions in the expansions has the more general form $\mu_i + \theta_i m$ where the μ_i are constants, real or complex, the θ_i are certain positive constants and m is a non-negative integer. In the third place the argument of the Bessel functions is extended in an obvious manner, from z to $z - a$, where a is any real or complex quantity. And, finally, the extension is made to the expansion of functions of several variables.

---

[*] See also Watson, *Theory of Bessel Functions* (London, 1922), p. 523.

[†] Wiener Sitzungsberichte, 74 (2), (1877), pp. 124-130; Watson, *loc cit.*, p. 524.

[‡] By an expression of the form a^r we shall uniformly mean $e^{r \log a}$ where the principal determination of log a is taken.

[§] Wiener Sitzungsberichte, 75, (2), (1877), pp. 218-222; Watson, *loc. cit.*, p. 525.

We first obtain the expansion

$$(z/2)^{\alpha+k} = \sum_{m=0}^{\infty} c_{k,m}\, \mathfrak{J}_{k+2m}(z),$$

in which $\alpha = \mu_1 + \mu_2 + \cdots + \mu_n$, k is a non-negative integer, the coefficients $c_{k,m}$ are constants which are defined in Art. 2, and

$$\mathfrak{J}_m(z) = J_{\mu_1+\theta_1 m}(z) J_{\mu_2+\theta_2 m}(z) \cdots J_{\mu_n+\theta_n m}(z),$$

always subject to the following restrictions: none of the quantities $\mu_i + \theta_i m$ shall be a negative integer, for $i = 1, 2, \cdots, n$, and for $m = 0, 1, 2, \cdots$, and $\theta_1 + \theta_2 + \cdots + \theta_n$ shall be 1 or 2. Since the case presenting the most interest is that for which the sum of the θ's is 1 we make this assumption unless otherwise specified. We next develop an expansion of the function $f(z)$ which is analytic and single-valued within and upon the circle C defined by the relation $| z - a | = R$ in the form

$$f(z) = \sum_{m=0}^{\infty} a_n z^{-\alpha}\, \mathfrak{J}_m(z-a)$$

where

$$a_m = (1/2\pi i) \int_C B_m(t-a)\, f(t)\, dt,$$

and $B_m(t-a)$ is a polynomial in $1/(t-a)$ whose exact form is given in Art. 6. We then obtain expansions for functions in terms of the polynomials $B_m(z-a)$.

In the remaining sections a relation between these expansions and those of Taylor and Laurent is obtained, certain orthogonality properties are derived, some special cases are considered, expansions for the case for which the sum of the θ's is 2 are obtained, and, finally, the extension is made to the expansion of functions of several variables.

II. EXPANSION OF $(z/2)^{a+k}$ IN TERMS OF THE FUNCTIONS $\mathfrak{F}_{k+2m}(z)$.

2. *Expansion of $\mathfrak{F}_m(z)$ in powers of $(z/2)$.* If the absolutely convergent series which represent the Bessel functions $J_{\mu_i+\theta_i m}(z)$, namely,

$$(1) \qquad J_{\mu_i+\theta_i m}(z) = \sum_{r=0}^{\infty} (-1)^r (z/2)^{\mu_i+\theta_i m+2r} / r!\,\Gamma(\mu_i + \theta_i m + r + 1), \qquad (i = 1, 2, \cdots, n),$$

be multiplied together and the product arranged according to the powers of $z/2$ the coefficient of $(z/2)^{a+m+2s}$ is $(-1)^s a_{m,s}$, where

$$a_{m,s} = \sum_{i_1, i_2, \ldots, i_n=0}^{s} 1/ i_1!\, i_2! \cdots i_n!\, \Gamma(\mu_1 + \theta_1 m + i_1 + 1) \Gamma(\mu_2 + \theta_2 m + i_2 + 1) \cdots \Gamma(\mu_n + \theta_n m + i_n + 1),$$

the summation being taken over all non-negative integral solutions of the equation $i_1 + i_2 + \cdots + i_n = s$.

Since the series (1), which define the Bessel functions, all converge absolutely and uniformly in any closed region of the z-plane (the origin being excepted if the real part of $\mu_i + \theta_i m$ is negative) it follows that the expansion

$$(2) \qquad \mathfrak{F}_m(z) = \sum_{s=0}^{\infty} (-1)^s a_{m,s} (z/2)^{a+m+2s}$$

is also absolutely and uniformly convergent in the same region.

3. *Formal expansion of $(z/2)^{a+k}$ in terms of the functions $\mathfrak{F}_{k+2m}(z)$.* In order to determine the coefficients $c_{k,m}$ in the expansion

$$(3) \qquad (z/2)^{a+k} = \sum_{m=0}^{\infty} c_{k,m} \mathfrak{F}_{k+2m}(z), \qquad (k = 0, 1, 2, 3, \cdots),$$

which expansion is momentarily assumed, we substitute for each of the functions $\mathfrak{F}_{k+2m}(z)$ the appropriate power series expansion obtained from (2) and equate coefficients of like powers of $(z/2)$. This gives the recurrence relations

$$(4) \qquad \begin{cases} a_{k,0}c_{k,0} = 1, \\ a_{k,1}c_{k,0} - a_{k+2,0}c_{k,1} = 0, \\ a_{k,2}c_{k,0} - a_{k+2,1}c_{k,1} + a_{k+4,0}c_{k,2} = 0, \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ a_{k,m}c_{k,0} - a_{k+2,m-1}c_{k,1} + \cdots + (-1)^m a_{k+2m,0}c_{k,m} = 0, \end{cases}$$

whence we have

$$(5)\qquad c_{k,m} = \frac{\begin{vmatrix} a_{k,1} & a_{k+2,0} & 0 & 0 & \cdots & 0 \\ a_{k,2} & a_{k+2,1} & a_{k+4,0} & 0 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{k,m-1} & a_{k+2,m-2} & a_{k+4,m-3} & & \cdots & a_{k+2m-2,0} \\ a_{k,m} & a_{k+2,m-1} & a_{k+4,m-2} & & \cdots & a_{k+2m-2,1} \end{vmatrix}}{a_{k,0}a_{k+2,0}\cdots a_{k+2m,0}}$$

$$= \frac{\begin{vmatrix} a'_{k,1} & 1 & 0 & 0 & \cdots & 0 \\ a'_{k,2} & a'_{k+2,1} & 1 & 0 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a'_{k,m-1} & a'_{k+2,m-2} & a'_{k+4,m-3} & & \cdots & 1 \\ a'_{k,m} & a'_{k+2,m-1} & a'_{k+4,m-2} & & \cdots & a'_{k+2m-2,1} \end{vmatrix}}{a_{k+2m,0}}$$

where

$$a'_{r,s} = a_{r,s}/a_{r,0} = \sum_{i_1,\, i_2,\, \ldots,\, i_n=0}^{s} 1/\, i_1!\, i_2! \cdots i_n!\, (\mu_1 + \theta_1 r + i_1)_{i_1} (\mu_2 + \theta_2 r + i_2)_{i_1} \cdots (\mu_n + \theta_n r + i_n)_{i_n}$$

and $$(a)_s = a(a-1)(a-2)\cdots(a-s+1).$$

If we denote by $\Delta_{k,m}(a')$ the determinant in the last member of (5) we have

$$(6)\qquad c_{k,m} = \Delta_{k,m}(a')/a_{k+2m,0}.$$

In the next sections we show that the series

$$(7)\qquad \sum_{m=0}^{\infty} c_{k,m}\, (z/2)^{-a}\, \mathfrak{F}_{k+2m}(z)$$

is absolutely and uniformly convergent in any closed region and that the series in (3) represents the function $(z/2)^{a+k}$.

4. *Expressions dominating* $a_{m,s}$, $c_{k,m}$ *and* $\mathfrak{F}_m(z)$. We first obtain simple dominants for the quantities $a_{m,s}$, $c_{k,m}$ and $\mathfrak{F}_m(z)$, which will be used later in the convergence proofs. From the definition of $a_{m,s}$ (Art. 2) we see that

$$(8)\qquad |\, a_{m,s} \,| = \Big|\sum_{i_1,\, i_2,\, \ldots,\, i_n=0}^{s} 1/i_1!\, i_2! \cdots i_n!\, \Gamma(\mu_1+\theta_1 m+i_1+1) \times \Gamma(\mu_2+\theta_2 m+i_2+1)\cdots\Gamma(\mu_n+\theta_n m+i_n+1)\Big|$$

$$\leq \sum_{i_1,\, i_2,\, \ldots,\, i_n=0}^{s} \frac{s!/(i_1!\, i_2! \cdots i_n!\ |\ (\mu_1+\theta_1 m+s)_{s-i_1} (\mu_2+\theta_2 m+s)_{s-i_2} \cdots (\mu_n+\theta_n m+s)_{s-i_n}\,|}{s!\,|\,\Gamma(\mu_1+\theta_1 m+s+1)\, \Gamma(\mu_2+\theta_2 m+s+1)\cdots\Gamma(\mu_n+\theta_n m+s+1)\,|}$$

where the summation is taken over all non-negative integral solutions of the equation $i_1 + i_2 + \cdots + i_n = s$. If now we allow the summation to be taken over all non-negative integral values consistent with the relation $i_1 + i_2 + \cdots + i_n = s(n-1)$ then the last member of (8) will not be decreased.[*] That is, we have

---

[*] The truth of this statement is established in the following manner:

By a comparison of the terms entering into the sum in the two cases for which $i_1 + i_2 + \cdots + i_n$ is s and $s(n-1)$ respectively, we observe that to each term in the first case there corresponds a term in the second case which differs from it only in the coefficients involving the factorials. Consequently, it is sufficient to show that, for each set of the i's consistent with the relation $i_1 + i_2 + \cdots + i_n = s$ we have

$$s!/i_1!\,i_2! \cdots i_n! \leq [s(n-1)]!/(s-i_1)!\,(s-i_2)! \cdots (s-i_n)!.$$

This is true if

$$s!/[s(n-1)]! \leq i_1!\,i_2! \cdots i_n!/(s-i_1)!\,(s-i_2)! \cdots (s-i_n)!.$$

Denote by $f(s, n;\ i_1, i_2, \cdots, i_n)$ the second member of the last inequality. The last inequality is obviously true for $n = 1$ or 2 if s is any non-negative integer. It is also clearly true when $s = 0, 1, 2$ for all values of n greater than 2. Therefore, in what follows, we may assume that s and n are each greater than 2. Let $p_1, p_2, p_3, \cdots, p_n$ denote a set of values which the i's may assume that will make the product $i_1!\,i_2! \cdots i_n!$ a minimum. Then we have

$$\begin{aligned} s!/i_1!\,i_2! \cdots i_n! &\leq s!/p_1!\,p_2! \cdots p_n! \\ &\leq [s(n-1)]!/(s-p_1)!\,(s-p_2)! \cdots (s-p_n)! \end{aligned}$$

for every possible set of the i's. Hence

$$\begin{aligned} s!/[s(n-1)]! &\leq p_1!\,p_2! \cdots p_n!/(s-p_1)!\,(s-p_2)! \cdots (s-p_n)! \\ &= f(s, n;\ p_1, p_2, \cdots, p_n). \end{aligned}$$

Now the proposition is established if it is shown that $f(s, n;\ p_1, p_2, \cdots, p_n)$ is no greater than $f(s, n;\ i_1, i_2, \cdots, i_n)$ for any set of values which the i's may take. If s and n are each greater than 2, and r is the integer satisfying the relation $r = s/n + \theta$ where $0 \leq \theta < 1$, then $f(s, n;\ i_1, i_2, \cdots, i_n)$ is seen to be simply related to $f(s, n;\ p_1, p_2, \cdots, p_n)$. In fact, after certain obvious reductions have been made, we have

$$\begin{aligned} f(s, n;\ i_1, i_2, \cdots, i_n) = f(s, n;\ p_1, p_2, \cdots, p_n)[(r+a_1)(s-r-a_1+1)] \cdots \\ [(r+a_m)(s-r-a_m+1)]/[(r-b_1)(s-r+b_1+1)] \cdots \\ [(r-b_m)(s-r+b_m+1)], \end{aligned}$$

where a_i and b_i are suitably determined non-negative integers. Since no factor in brackets in the numerator of the fraction in the last relation can be less than $k(s-k+1)$ and no factor in brackets in the denominator can be greater than $k(s-k+1)$, it is clear that $f(s, n;\ i_1, i_2, \cdots, i_n) \geq f(s, n;\ p_1, p_2, \cdots, p_n)$ for every possible set of i's. Hence the proposition is established.

$$(9)\quad |a_{m,s}| \leq \sum_{i_1,\, i_2,\, \ldots,\, i_n=0}^{s(n-1)} \frac{[s(n-1)]!/i_1!\,i_2!\cdots i_n!\,|\,(\mu_1+\theta_1 m+s)_{i_1}(\mu_2+\theta_2 m+s)_{i_2}\cdots(\mu_n+\theta_n m+s)_{i_n}\,|}{s!\,|\,\Gamma(\mu_1+\theta_1 m+s+1)\,\Gamma(\mu_2+\theta_2 m+s+1)\cdots\Gamma(\mu_n+\theta_n m+s+1)\,|}$$

$$= |\,\alpha+m+ns\,|_{s(n-1)}\,/s!\,|\,\Gamma(\mu_1+\theta_1 m+s+1)\times \Gamma(\mu_2+\theta_2 m+s+1)\cdots\Gamma(\mu_n+\theta_n m+s+1)\,|$$

$$\leq \Gamma(|\,\alpha\,|+m+ns+1)\,\Gamma(|\,\alpha\,|+m+s+1)\,|\,\Gamma(\mu_1+\theta_1 m+s+1)\times \Gamma(\mu_2+\theta_2 m+s+1)\cdots\Gamma(\mu_n+\theta_n m+s+1)\,|.$$

The summation indicated in the second member of (9) has been effected by means of the identity

$$(a_1+a_2+\cdots+a_n)_s = \sum_{i_1,\, i_2,\, \ldots,\, i_n=0}^{s} [s!/i_1!\,i_2!\cdots i_n!]\,(a_1)_{i_1}(a_2)_{i_2}\cdots(a_n)_{i_n}$$

an identity which is readily established by mathematical induction. The special case for which $n=2$ was established by Vandermonde.* For convenience we shall denote by $\alpha_{m,s}$ the last member of (9).

From the recurrence relations obtained in Art. 2 we have the relation

$$(10)\qquad |\,c_{k,m}\,| \leq [\,|\,a_{k,m}\,|\cdot|\,c_{k,0}\,| + |\,a_{k+2,m-1}\,|\cdot|\,c_{k,1}\,| + \cdots + |\,a_{k+2m-2,1}\,|\cdot|\,c_{k,m-1}\,|\,]/|\,a_{k+2m,0}\,|$$

$$\leq [\alpha_{k,m}\,|\,c_{k,0}\,| + \alpha_{k+2,m-1}\,|\,c_{k,1}\,| + \cdots + \alpha_{k+2m-2,1}\,|\,c_{k,m-1}\,|\,]/|\,a_{k+2m,0}\,|.$$

We shall denote the last member of (10) by $B_{k,m}$.

If we expand the determinant $\Delta_{k,m}(a')$ by minors with respect to the elements of the last column, and then expand in the same manner the minor of the unit element and so on, we have

$$(11)\qquad \Delta_{k,m}(a') = a'_{k+2m-2,1}\Delta_{k,m-1}(a') - a'_{k+2m-4,2}\Delta_{k,m-2}(a') + \cdots + (-1)^m a'_{k+2,m-1}\Delta_{k,1}(a') + (-1)^{m+1}a'_{k,m}.$$

Now let $P_{k,m}(a')$ denote the sum of the absolute values of the terms in the expansion of $\Delta_{k,m}(a')$ as a polynomial in the symbols a'. Then

$$(12)\qquad P_{k,m}(a') = |\,a'_{k,m}\,| + |\,a'_{k+2,m-1}\,|\,P_{k,1}(a') + |\,a'_{k+4,m-2}\,|\,P_{k,2}(a') + \cdots + |\,a'_{k+2m-2,1}\,|\,P_{k,m-1}(a').$$

If in (12) we replace $|\,a'_{r,s}\,|$ by the dominating quantity $\alpha'_{r,s}$, where

$$\alpha'_{r,s} = \Gamma(-|\,\alpha\,|+r+ns+1)/\,\Gamma(-|\,\alpha\,|+r+s+1)\,(-|\,\mu_1\,|+\theta_1 r+s)_s \times (-|\,\mu_2\,|+\theta_2 r+s)_s\cdots(-|\,\mu_n\,|+\theta_n r+s)_s,$$

---

* See Chrystal's *Algebra*, Vol. 2, p. 9.

we have [*]

$$(13)\quad P_{k,m}(\alpha') = \alpha'_{k,m} + \alpha'_{k+2,m-1}P_{k,1}(\alpha') + \alpha'_{k+4,m-2}P_{k,2}(\alpha') + \cdots + \alpha'_{k+2m-2,1}P_{k,m-1}(\alpha').$$

Moreover, we have

$$(14)\quad |\, c_{k,m}\,| \leq P_{k,m}(a')/|\, a_{k+2m,0}\,| \leq P_{k,m}(\alpha')/|\, a_{k+2m,0}\,|.$$

Making use of the inequality employed by Nielsen,[†]

$$|\, J_\nu(z)\,| \leq |\, (z/2)^\nu\,|\, e^{|z|^2/4|\nu_0+1|}/\Gamma(\nu+1),$$

where $|\,\nu_0+1\,|$ is the least of the quantities $|\,\nu+1\,|, |\,\nu+2\,|, \cdots$, we have the following inequality:

$$(15)\quad |\, \mathfrak{F}_m(z)\,| \leq |\, (z/2)^{\alpha+m}\,| \cdot |\, a_{m,0}\,|\, e^{|z/2|^2 E_m},$$

where $E_m = \sum_{i=1}^{n} 1/\nu_i(m)$, and $\nu_i(m)$ for each i is the smallest of the quantities $|\,\mu_i+\theta_i m + r\,|$ for $r = 1, 2, 3, \cdots$.

5. *Convergence of the series in* (3). In view of the inequalities (10) and (15) it is clear that the series in (3) is dominated, term by term, by the series

$$(16)\quad \sum_{n=0}^{\infty} B_{k,m}\,|\, a_{k+2m,0}\,| \cdot |\, (z/2)^{\alpha+k+2m}\,|\, e^{|z/2|^2 E_{k+2m}}.$$

Denote by U_m the mth term of (16), and consider the limit of the ratio U_{m+1}/U_m as m becomes infinite. From (16) and (10) we have

$$(17)\quad U_{m+1}/U_m = [B_{k,m+1}\,|\, a_{k+2m+2,0}\,|/B_{k,m}\,|\, a_{k+2m,0}\,|]\,|\, z/2\,|^2 \exp[\,|\, z/2\,|^2 \{E_{k+2m+2} - E_{k+2m}\}]$$

$$\leq \left[\frac{\alpha_{k,m+1}\,|\, c_{k,0}\,| + \alpha_{k+2,m}\,|\, c_{k,1}\,| + \cdots + \alpha_{k+2m-2,2}\,|\, c_{k,m-1}\,|}{\alpha_{k,m}\,|\, c_{k,0}\,| + \alpha_{k+2,m-1}\,|\, c_{k,1}\,| + \cdots + \alpha_{k+2m-2,1}\,|\, c_{k,m-1}\,|} + \frac{\alpha_{k+2m,1}}{|\, a_{k+2m,0}\,|}\right] |\, z/2\,|^2 \exp[\,|\, z/2\,|^2 \{E_{k+2m+2} - E_{k+2m}\}].$$

Consider the limit $\underset{m\to\infty}{L} f(m,s)$ where $f(m,s) = \alpha_{k+2s,\, m+1-s}/\alpha_{k+2s,\, m-s}$, in which s may take non-negative integral values subject to the restriction that it be not greater than m. From the definition of $\alpha_{r,s}$ (Art. 4) we see that

$$f(m,s) = \frac{[|\alpha|+k+2s+n(m-s)+1][|\alpha|+k+2s+n(m-s)+2]\cdots[|\alpha|+k+2s+n(m-s)+n]}{(m-s+1)[|\alpha|+k+s+m+1][\mu_1+\theta_1(k+2s)+m-s+1]\times[\mu_2+\theta_2(k+2s)+m-s+1]\cdots[\mu_n+\theta_n(k+2s)+m-s+1]}.$$

---

[*] We note that if $k = |\, u_i\,|/\theta_i$ for any i of the set $i = 1, 2, \cdots, n$, the quantities $P_{k,m}(a')$ are undefined. Hence we exclude all such values of k.

[†] *Mathematischen Annalen*, Vol. 52, (1899) p. 231; see Watson, *loc. cit.*, p. 44.

We observe that the factors of the numerator may be paired with the last n factors of the denominator so that each of the resulting n ratios is bounded as to m and s. Let M denote the product of these n bounds. The limit $\underset{m\to\infty}{L} f(m, s)$ is then seen to be zero uniformly as to s when s is in the range $0 \leqq s \leqq m$.

Then, for every positive N, there exists an integer m' such that for all values of $m > m'$ and for all values of s in the range $0 \leqq s \leqq m$, we have

$$\alpha_{k+2s,\, m+1-s}\, N < \alpha_{k+2s, m-s}. \tag{18}$$

Then for all values of $m > m'$ we have

$$\frac{U_{m+1}}{U_m} \leq \frac{1}{N}$$

$$\left[\frac{N\alpha_{k,m+1}\,|c_{k,0}| + N\alpha_{k+2,m}\,|c_{k,1}| + \cdots + N\alpha_{k+2m-2,2}\,|c_{k,m-1}|}{\alpha_{k,m}\,|c_{k,0}| + \alpha_{k+2,m-1}\,|c_{k,1}| + \cdots + \alpha_{k+2m-2,1}\,|c_{k,m-1}|} + \frac{N\alpha_{k+2m,1}}{|\,a_{k+2m,0}\,|}\right] R_m(z)$$

$$\leq \frac{1}{N}$$

$$\left[\frac{\alpha_{k,m}\,|c_{k,0}| + \alpha_{k+2,m-1}\,|c_{k,1}| + \cdots + \alpha_{k+2m-2,1}\,|c_{k,m-1}|}{\alpha_{k,m}\,|c_{k,0}| + \alpha_{k+2,m-1}\,|c_{k,1}| + \cdots + \alpha_{k+2m-2,1}\,|c_{k,m-1}|} + \frac{\alpha_{k+2m,1}\,N}{|\,a_{k+2m,0}\,|}\right] R_m(z)$$

$$= \frac{1}{N}\left[1 + \frac{N\,\alpha_{k+2m,1}}{|\,a_{k-2m,0}\,|}\right] R_m(z),$$

where $R_m(z) = |\, z/2\,|^2 \exp\left[\,|\, z/2\,|^2 \{E_{k+2m+2} - E_{k+2m}\}\right]$,

and, since $\underset{m\to\infty}{L} \alpha_{k+2m,1}/|\, a_{k+2m,0}\,| = 0$, and $\underset{m\to\infty}{L} E_m = 0$, we see that

$$\underset{m\to\infty}{L} U_{m+1}/U_m \leqq |\, z/2\,|^2/N < 1 \qquad \text{if} \qquad |\, z/2\,|^2 < N.$$

Hence the series (16) and consequently the series in (3) converges absolutely for all finite values of z.

The uniformity of convergence of the series (7) throughout any closed region is readily established. For, let R' be any constant, and, let $R > R'$. Then, if in (16) each term be divided by $(z/2)^a$ and z be replaced by R, we have a convergent series of positive constants by which the series in (7) is dominated term by term provided that $|\, z\,| \leqq R'$. Hence the uniformity of convergence throughout the given region follows from the test of Weierstrass.

Since the power series expansions of the functions $(z/2)^{-a}\mathfrak{F}_{k+2m}(z)$ all converge absolutely for all finite values of z, and (7) converges uniformly in the circle $|\, z\,| = R'$, we may sum by columns the repeated series obtained by replacing each of the functions $(z/2)^{-a}\mathfrak{F}_{k+2m}(z)$ in (7) by its power series expansion,* and this sum is $(z/2)^k$, as is seen from the relations (4). Hence the series in (3) represents the function $(z/2)^{a+k}$.

---

* See Osgood, *Lehrbuch der Funktionentheorie*, Vol. I, 4th Ed., p. 343.

III. EXPANSION OF $f(z)$ IN TERMS OF THE FUNCTIONS $F_m(z-a)$ AND OF THE FUNCTIONS $B_m(z-a)$.

6. *Expansion of $z^a/(t-z)$ in terms of $\mathfrak{F}_m(z)$.* In the series

$$z^a/(t-z)=\sum_{k=0}^{\infty} z^{a+k}/t^{k+1}, \tag{19}$$

which is valid when $|t|>|z|$, substitute the series expansions for z^{a+k} obtained from (3). This gives

$$z^a/(t-z)=\sum_{k=0}^{\infty} 2^{a+k}/t^{k+1} \sum_{m=0}^{\infty} c_{k,m}\,\mathfrak{F}_{k+2m}(z). \tag{20}$$

Rearrange this repeated series, assuming temporarily that this rearrangement is permissible, into the simple series

$$z^a/(t-z)=\sum_{m=0}^{\infty} B_m(t)\,\mathfrak{F}_m(z), \tag{21}$$

where

$$B_m(t)=\sum_{r=0}^{\leqq m/2} c_{m-2r,r}\,2^{a+m-2r}/t^{m+1-2r},$$

the sum terminating with $m/2$ if m is even, and with $(m-1)/2$ if m is odd.

Since the absolute convergence of a repeated series implies the absolute convergence of the corresponding double series,[*] the above rearrangement to obtain (21) is permissible provided the repeated series in (20) is absolutely convergent. To establish this fact we first observe that the repeated series in (20) is dominated, term by term, by the series

$$\sum_{k=0}^{\infty} |z^{a+k}|/|t|^{k+1} \sum_{m=0}^{\infty} P_{k,m}(a')\,|z/2|^{2m}\,e^{|z/2|^2\,E_{k+2m}}. \tag{22}$$

If we make use of the relations (12) we are able, by arguments entirely analogous to those employed in Art. 5, to prove that

$$\sum_{m=0}^{\infty} P_{k,m}(a')\,|z/2|^{2m}\,e^{|z/2|^2\,E_{k+2m}} \tag{23}$$

is convergent for every finite value of z and k. Also, by the aid of (13) we may prove in a similar manner that

$$\sum_{m=0}^{\infty} P_{k,m}(\alpha')\,|z/2|^{2m}\,e^{|z/2|^2\,E_{k+2m}} \tag{24}$$

is convergent for every finite value of z and for every $k>k_1$ where

---

[*] Bromwich, "Theory of Infinite Series," p. 78.

$k_1 > |\mu_i|/\theta_i$ for $i = 1, 2, \cdots, n$. (This restriction is placed upon k to insure that all the quantities $P_{k,m}(\alpha')$ shall exist.) For fixed $k \geqq k_1$ we see from (14) that the series (23) is dominated, term by term, by the series (24). Denote by $S_k(z)$ the sum of (23), and by $\sigma_k(z)$ the sum of (24), when $k > k_1$. Now, if R is any constant and $|z| \leqq R$, then $\sigma_k(z)$ is bounded both as to z and k. For, in the series (24) each term is a monotone decreasing function of k when $k \geqq k_1$. Hence, if z and k are in the given regions, $\sigma_{k_1}(R) \geqq \sigma_k(z)$. Clearly, if $0 \leqq k < k_1$, $S_k(z)$ has a bound, since $S_k(z)$ exists for each value of k, and the number of S's is finite. Let this bound be M_1, and let M be the greater of the two quantities $\sigma_{k_1}(R)$ and M_1. Then we have

$$(25)\qquad \sum_{k=0}^{\infty} |2^{a+k}|/|t|^{k+1} \sum_{m=0}^{\infty} |c_{k,m}| \cdot |\mathfrak{F}_{k+2m}(z)| \leqq M \sum_{k=0}^{\infty} |z^{a+k}|/|t|^{k+1},$$

which converges absolutely when $|z| < |t|$. Whence the rearrangement in obtaining (21) is justified.

The uniformity of convergence of the series $\sum_{m=0}^{\infty} B_m(t) z^{-a} \mathfrak{F}_m(z)$ with regard to both z and t throughout the regions $|t| \geqq R$, $|z| \leqq R'$, where $R > R'$, is readily established. For, when these inequalities are satisfied this series is dominated, term by term, by a series of positive quantities independent of t and z, and the uniformity of convergence follows from the test of Weierstrass. We may state this result as

THEOREM I. *Let z and t be two complex variables such that $|z| \leqq R'$ and $|t| \geqq R$, where $R > R'$; then the expansion*

$$(26)\qquad 1/(t - z) = \sum_{m=0}^{\infty} B_m(t)\, F_m(z),$$

where $F_m(z) = z^{-a}\mathfrak{F}_m(z)$, is valid. If t be a variable point on or outside the circle $|z| = R'$, and z a variable point on or within the circle $|z| = R'$, then the expansion converges uniformly with regard to t and z.

7. *Expansion of $z^a f(z)$ in terms of the functions $\mathfrak{F}_m(z)$.* Now, by the aid of Cauchy's integral formula and the expansion (26) we are able to obtain an expansion of a function which is analytic in the neighborhood of the origin in terms of the functions $F_m(z)$. For we have

$$(27)\qquad f(z) = (1/2\pi i)\int_C \frac{f(t)\,dt}{(t-z)}$$

$$= (1/2\pi i)\int_C \{\sum_{m=0}^{\infty} B_m(t)\,F_m(z)\}\,f(t)\,dt$$

$$= \sum_{m=0}^{\infty} a_m F_m(z),$$

where

$$a_m = (1/2\pi i)\int_C B_m(t)\,f(t)\,dt,$$

the path of integration being the circle $|z| = R$, within and upon which $f(z)$ is single-valued and analytic. This expansion converges uniformly with regard to z when $|z| \leq R'$ and $R > R'$. Hence the expansion

$$(28)\qquad z^a f(z) = \sum_{m=0}^{\infty} a_m \mathfrak{F}_m(z),$$

is valid in the same region. We may state this result as

THEOREM II. *Let $f(z)$ be a function of z which is analytic and single-valued within and upon the circle C defined by the relation $|z| = R$; then the expansion*

$$(29)\qquad z^a f(z) = \sum_{m=0}^{\infty} a_m \mathfrak{F}_m(z),$$

where

$$a_m = (1/2\pi i)\int_C B_m(t)\,f(t)\,dt,$$

is valid when z is withn the circle $|z| = R$. If $R > R'$ the expansion obtained from (29) by dividing each term by z^a converges uniformly with regard to z when $|z| \leq R'$.

If we let $g(z) = 1/(t-z)$ and form from (21) $g(0), g'(0), g''(0), \cdots$, where the differentiation is with respect to z, it is clear that the polynominals $B_m(t)$ are uniquely determined.[*] And in a similar manner the expansion (27) is shown to be unique.

8. *Expansion of $f(z)$ in terms of $B_m(z)$.* Again if z and t be two complex variables which satisfy the relations $|z| \geq r$, $|t| \leq r'$, and $r > r'$;

---

[*] To be exact this proves that all expansions of $1/(t-z)$ in terms of the functions $F_m(z)$ which are differentiable, term by term, are identical. This is sufficient for our purpose. It can be proved, however, that *any* expansion of $1/(t-z)$ in terms of $F_m(z)$ which is convergent has a region of uniform convergence, and therefore is identical with (21).

then, by merely interchanging the roles of t and z in Art. 6 we obtain the expansion

$$(30) \qquad 1/(z-t)=\sum_{m=0}^{\infty} F_m(t)\,B_m(z).$$

If t be a variable point on or within the circle C defined by the relation $|t|=r'$, and z is any point on or outside the circle $|z|=r$, then the expansion (30) converges uniformly with regard to t and z. If $f(z)$ be any function of z which is analytic and single-valued upon and outside the circle C it can be shown, by arguments analogous to those used in Art. 7, to have the expansion

$$(31) \qquad f(z)=\sum_{m=0}^{\infty} a'_m B_m(z), \quad \text{where} \quad a'_m=(1/2\pi i)\int_C F_m(t)\,f(t)\,dt.$$

And this expansion converges uniformly with regard to z provided z is on or outside the circle $|z|=r$.

9. *Expansion of* $f(z)$ *in terms of the functions* $F_m(z-a)$ *and* $B_m(z-a)$. If in the expansion (26) we replace t and z by $t-a$ and $z-a$, respectively, where a is any complex quantity, we obtain the expansion

$$(32) \qquad 1/(t-z)=\sum_{m=0}^{\infty} B_m(t-a)\,F_m(z-a),$$

which is valid when $|t-a|\geqq R$, $|z-a|\leqq R'$, and $R>R'$. This expansion converges uniformly with regard to t and z when these inequalities are satisfied.

In a similar manner we obtain from the expansion (30),

$$(33) \qquad 1/(z-t)=\sum_{m=0}^{\infty} F_m(t-a)\,B_m(z-a),$$

which is valid when $|t-a|\leqq r'$, $|z-a|\geqq r$, and $r>r'$. And likewise this expansion converges uniformly with respect to t and z when these inequalities hold.

If $f(z)$ is analytic and single-valued within and upon the circle C defined by the relation $|z-a|=R$, then the expansion

$$(34) \quad f(z)=\sum_{m=0}^{\infty} a_m F_m(z-a), \quad \text{where} \quad a_m=(1/2\pi i)\int_C B_m(t-a)f(t)\,dt,$$

is valid; and, if z remains in the circle $|z-a|=R'$, where $R>R'$, this expansion converges uniformly with respect to z.

And, if $f(z)$ is analytic and single-valued upon and outside the circle C', defined by the relation $|z-a| = r'$, then the expansion

$$(35)\quad f(z) = \sum_{m=0}^{\infty} a'_m B_m(z-a), \text{ where } a'_m = (1/2\pi i)\int_{C'} F_m(t-a) f(t)\,dt,$$

is valid; and, if z remains on or outside the circle $|z-a| = r$, where $r > r'$, this expansion converges uniformly with respect to z. When $a = 0$ the expansions (34) and (35) reduce to (27) and (31), respectively.

IV. RELATION TO THE TAYLOR AND LAURENT EXPANSIONS.

10. If the Taylor's expansion of $f(z)$ is $f(z) = \sum_{m=0}^{\infty} b_m (z-a)^m$, while the expansion (34) is

$$f(z) = \sum_{m=0}^{\infty} a_m F_m(z-a),$$

it is possible to obtain a relation between the coefficients b_m and a_m in these two expansions. For

$$\begin{aligned}(36)\qquad a_m &= (1/2\pi i)\int_C B_m(t-a)\, f(t)\, dt \\ &= (1/2\pi i)\int_C B_m(t-a)\, \{\sum_{s=0}^{\infty} b_s (t-a)^s\}\, dt \\ &= \sum_{s=0}^{\infty} b_s\, (1/2\pi i)\int_C (t-a)^s \sum_{r=0}^{\leq m/2} c_{m-2r,r} \frac{2^{\alpha+m-2r}}{(t-a)^{m+1-2r}}\, dt \\ &= \sum_{r=0}^{\leq m/2} 2^{\alpha+m-2r}\, c_{m-2r,r}\, b_{m-2r}.\end{aligned}$$

By means of these relations we are able to readily construct the expansion (29) when the Taylor's expansion for the given function $f(z)$ is known.

Next, let $f(z)$ be an analytic and single-valued function of z in the ring-shaped region S, bounded by the two circles C_2 and C_1 defined by the relations $|z-a| = r$ and $|z-a| = R$, respectively, where $R > r$. Let z be a fixed point in the region S. Then we have from Cauchy's integral formula

$$(37)\qquad f(z) = (1/2\pi i)\int_{C_1} \frac{f(t)\,dt}{t-z} + \frac{1}{2\pi i}\int_{C_2} \frac{f(t)\,dt}{t-z},$$

the two paths of integration being taken in a counter-clockwise direction. If we apply the expansions (32) and (33) to the first and second integrals, respectively, we have

$$(38)\qquad f(z) = (1/2\pi i)\int_{C_1} \sum_{m=0}^{\infty} B_m(t-a)F_m(z-a)f(t)\,dt$$
$$+ (1/2\pi i)\int_{C_2} \sum_{m=0}^{\infty} F_m(t-a)\,B_m(z-a)f(t)\,dt$$
$$= \sum_{m=0}^{\infty} a_m F_m(z-a) + \sum_{m=0}^{\infty} a'_m B_m(z-a),$$

where

$$a_m = (1/2\pi i)\int_{C_1} B_m(t-a)f(t)\,dt \quad\text{and}\quad a'_m = (1/2\pi i)\int_{C_2} F_m(t-a)f(t)\,dt,$$

and this expansion is valid when z is in S. This result may be stated as

THEOREM III. *Let $f(z)$ be a function of z, which is analytic and single-valued in the region defined by the inequalities $0 < r \leqq |z-a| \leqq R$, and let C_1 and C_2 denote the circles $|z-a| = R$ and $|z-a| = r$, respectively. Then, the expansion*

$$(39)\qquad f(z) = \sum_{m=0}^{\infty} a_m F_m(z-a) + \sum_{m=0}^{\infty} a'_m B_m(z-a),$$

where

$$a_m = (1/2\pi i)\int_{C_1} B_m(t-a)f(t)\,dt$$

and

$$a'_m = (1/2\pi i)\int_{C_2} F_m(t-a)f(t)\,dt,$$

is valid and converges uniformly with regard to z when $r' \leqq |z-a| \leqq R'$, and $R > R' > r' > r > 0$.

If the Laurent expansion of $f(z)$ in the region $r' \leqq |z-a| \leqq R'$ is

$$(40)\qquad f(z) = \sum_{m=0}^{\infty} b_m(z-a)^m + \sum_{m=1}^{\infty} b'_m(z-a)^{-m},$$

we are able to obtain relations between the coefficients a_m, a'_m and the coefficients b_m, b'_m in the expansion (39). In fact the value of a_m is expressed in terms of b_m by a relation notationally equivalent to the relation (36) already obtained, and from (40) we have

$$(41)\qquad a'_m = (1/2\pi i)\int_C F_m(t-a)f(t)\,dt$$
$$= (1/2\pi i)\int_C F_m(t-a)\,\{\sum_{s=1}^{\infty} b'_s(t-a)^{-s}\}\,dt$$
$$= \sum_{s=1}^{\infty} b'_s\,(1/2\pi i)\int_C (t-a)^{-s}F_m(t-a)\,dt$$
$$= \sum_{r=0}^{\infty} \frac{(-1)^r a_{m,r}}{2^{m+2r}}\, b'_{m+1+2r}.$$

By means of these relations it is possible to construct the Neumann series expansion of a given function when the Taylor or Laurent series for the given function is known.*

V. ORTHOGONALITY PROPERTIES.

11. We shall next obtain the following integral formulae:

$$\int_C F_m(t-a)\,F_r(t-a)\,dt = 0, \text{ for all values of } m \text{ and } r,$$

$$\int_C B_m(t-a)\,B_r(t-a)\,dt = 0, \text{ for all values of } m \text{ and } r,$$

(42)

$$\int_C F_m(t-a)\,B_m(t-a)\,dt = 2\pi ik, \text{ for } c \text{ enclosing the point } z=a,$$

$$= 0, \text{ for } c \text{ not enclosing the point } z=a,$$

$$\int_C F_m(t-a)\,B_r(t-a)\,dt = 0, \text{ for } m \neq r,$$

where the contour C, in each case, is any closed curve not passing through the point $z=a$. Denote by S the region enclosed by the contour C.

The first formula is true because the integrand is analytic throughout S and along its boundary, and hence Cauchy's theorem applies.

The second formula is seen to be true, by observing that the residue of $B_m(t-a)B_r(t-a)$ at $t=a$ is zero, and that it is analytic at all other points in S.

The third formula follows from the fact that the integrand has a pole of the first order at $t=a$ and at this point the residue is unity. We see that the integral is $2\pi i\kappa$ or 0 according as C does or does not enclose the point $z=a$ where κ is the excess of the number of positive over the number of negative circuits of C about the point $z=a$.

To establish the fourth formula we separate the argument into two cases, the one for which $m > r$ and the other for which $m < r$. If $m > r$ the residue of the integrand at $t=a$ is readily seen to be zero. Hence the formula is true for this case. If $m < r$ we may write $m = r - p$ where p is of the set $1, 2, 3, \cdots, r$. If p is odd the residue at $t=a$ is seen to be zero. If p is even denote it by $2s$. Then

---

* Cf. Neumann, *Theorie der Bessel'schen Funktionen* (Leipzig, 1867), pp. 33-35; also Watson, *l. c.*, pp. 523-524.

The residue of $F_m(t-a)B_{m+2s}(t-a)$ at $t=a$ is

$$a_{m,0}\,c_{m,s} - a_{m,1}\,c_{m+2,s-1} + \cdots + (-1)^s\,a_{m,s}\,c_{r,0},$$

which may be written in the form *

$$\frac{\begin{vmatrix} a_{m,0} & a_{m,0} & 0 & 0 & \cdots & 0 \\ a_{m,1} & a_{m,1} & a_{m+2,0} & 0 & \cdots & \cdot \\ a_{m,2} & a_{m,2} & a_{m+2,1} & a_{m+4,0} & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{m,s-1} & a_{m,s-1} & a_{m+2,s-2} & a_{m+4,s-3} & \cdots & a_{m+2s-2,0} \\ a_{m,s} & a_{m,s} & a_{m+2,s-1} & a_{m+4,s-2} & \cdots & a_{m+2s-2,1} \end{vmatrix}}{a_{m,0}\quad a_{m+2,0}\quad\cdots\quad a_{m+2s,0}}$$

Since the first two columns of the determinant are identical the residue at $t=a$ is zero, and the fourth formula is verified. The formulae (42) afford a convenient means for determining the coefficients in some of the expansions which we have developed, provided the expansions are known to exist. For example, if we wish to determine the coefficients in the expansion (27) we assume an expansion of the form

$$f(z) = \sum_{m=0}^{\infty} a_m\,F_m(z),$$

multiply both members by $B_m(z)$, and integrate along the circle $|z|=R$, within and upon which $f(z)$ is assumed analytic and single-valued. Making use of the third and fourth formulae of (42) we obtain for a_m the value given in (27). It is clear that the coefficients in other of our expansions may be similarly determined.

VI. EXPANSIONS FOR WHICH THE SUM OF THE θ'S IS 2.

12. Thus far in our development we have assumed that the sum of the θ's in the functions $\mathfrak{F}_m(z)$ is unity. In fact that assumption was essential to the argument which was employed. For, if the sum of the θ's has *any* value other than unity, and k be not an even integer, we are led to the consideration of a system of inconsistent equations for determining the coefficients

---

* This statement is verified if we expand the determinant by minors with respect to the elements of the first column, and observe that the minor obtained from the determinant by striking out the first $(r+1)$ rows and the first $(r+1)$ columns is given by the relation

$$\Delta'_{m+2r,\,s-r} = c_{m+2r,\,s-r}\,a_{m+2r,\,0}\,a_{m+2r+2,\,0}\cdots a_{m+2s,\,0}.$$

in the formal expansion of $(z/2)^{a+k}$ in terms of the functions $\mathfrak{F}_m(z)$. However, if k be an even integer, and the sum of the θ's is 2, then we may obtain a system of equations which is consistent, and then it is possible to get an expansion for $(z/2)^{a+2r}$ in a series of the functions $\mathfrak{F}_{r+m}(z)$, in which the sum of the θ's is 2, and r is any non-negative integer. This suggests the possibility of the expansion of even analytic functions in terms of $z^{-a}\mathfrak{F}_m(z)$, in which the sum of the θ's is 2. Therefore, we next obtain such an expansion, then a similar expansion for odd functions, and finally by combining these two expansions get an expansion for any function which is analytic and single-valued in the neighborhood of the origin in terms of the functions $z^{-a}\mathfrak{F}_m(z)$ and $z^{-a+1}\mathfrak{F}_m(z)$ when the sum of the θ's is 2. In this article the quantities $a_{m,s}$ and $\mathfrak{F}_m(z)$ have precisely the same form as in the previous articles, but we shall understand that throughout this article we are carrying the assumption that the sum of the θ's is 2. The argument for this case is very similar to that used in the first part of this paper, and, consequently, much of the details may be omitted.

Assume an expansion of the form

$$(z/2)^{a+2r} = \sum_{m=0}^{\infty} C_{r,m}\,\mathfrak{F}_{r+m}(z) \tag{43}$$

and substitute for each of the functions $\mathfrak{F}_{r+m}(z)$ the appropriate series expansion, and equate coefficients of like powers of $z/2$. From the resulting recurrence relations the quantities $C_{r,m}$ may be determined, and are seen to be

$$C_{r,m} = \frac{\begin{vmatrix} a_{r,1} & a_{r+1,0} & 0 & 0 & 0 & \cdots & 0 \\ a_{r,2} & a_{r+1,1} & a_{r+2,0} & 0 & 0 & \cdots & \cdot \\ a_{r,3} & a_{r+1,2} & a_{r+2,1} & a_{r+3,0} & 0 & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{r,m-1} & a_{r+1,m-2} & a_{r+2,m-3} & a_{r+3,m-4} & \cdots & & a_{r+m-1,0} \\ a_{r,m} & a_{r+1,m-1} & a_{r+2,m-2} & a_{r+3,m-3} & \cdots & & a_{r+m-1,1} \end{vmatrix}}{a_{r,0}\; a_{r+1,0}\; \cdots\; a_{r+m,0}}$$

The convergence of the series in (43) and its representation of the function $z/2^{a+2r}$ is proved by arguments analogous to those used in Arts. 2 to 5.

For each of the powers of $z/2$ in the expansion

$$z^a/(t^2 - z^2) = \sum_{r=0}^{\infty} (z^{a+2r}/t^{2r+2}),$$

substitute the corresponding expansion from (43) and rearrange the repeated series into a simple series. The resulting formal expansion,

$$(44) \qquad z^a/(t^2 - z^2) = \sum_{m=0}^{\infty} B_m(t)\, \mathfrak{F}_m(z),$$

where $$B_m(t) = \sum_{s=0}^{m} c_{s,m-s}\, (2^{a+2s}/t^{2s+2}),$$

is shown to be valid by the method used in Art. 6.

Now, let $f(z)$ be an even function of the complex variable z, single-valued and analytic, within and upon the circle C defined by the relation $|z| = R$. Then by the special form of Cauchy's integral formula employed by Neumann,* namely,

$$f(z) = (1/2\pi i) \int_C \frac{f(t)\, t\, dt}{(t^2 - z^2)},$$

and the expansion (44) we have

$$(45) \qquad z^a f(z) = (1/2\pi i) \int_C \frac{z^a f(t)\, t\, dt}{t^2 - z^2} = \sum_{m=0}^{\infty} c_m\, \mathfrak{F}_m(z),$$

where $$c_m = (1/2\pi i) \int_C B_m(t)\, f(t)\, t\, dt.$$

This expansion is valid when z is inside the circle $|z| = R$, and converges uniformly with regard to z when $|z| \leq R'$, where $R > R'$. Since any odd function of z, which is single-valued and analytic at $z = 0$, may be expressed as z multiplied by an even analytic function, we are able to get the expansion for odd analytic functions in a series very similar to (45). For, let $\phi(z) = z f(z)$, where $\phi(z)$ is an odd analytic function of z. Then $f(z)$ is even, and has the expansion

$$f(z) = \sum_{m=0}^{\infty} z^{-a}\, C_m\, \mathfrak{F}_m(z).$$

Consequently,

$$\phi(z) = \sum_{m=0}^{\infty} z^{-a+1}\, C_m\, \mathfrak{F}_m(z), \quad \text{where } C_m = (1/2\pi i) \int_C B_m(t)\, \phi(t)\, dt.$$

We may state these results as

THEOREM IV. *Let $f(z)$ and $\phi(z)$ be an even and an odd function of z, respectively, each analytic and single-valued in the region $|z| = R$, and along its contour C. Then the expansions*

$$(46) \qquad f(z) = \sum_{m=0}^{\infty} z^{-a}\, c_m\, \mathfrak{F}_m(z), \quad \textit{where} \quad c_m = (1/2\pi i) \int_C B_m(t)\, f(t)\, t\, dt,$$

$$(47) \qquad \phi(z) = \sum_{m=0}^{\infty} z^{-a+1}\, C_m \mathfrak{F}_m(z), \ \textit{where}\ C_m = (1/2\pi i) \int_C B_m(t)\, \phi(t)\, dt,$$

---

* *Mathematische Annalen*, Vol. 3, p. 594.

are valid when z is within the circle $|z| = R$, and these expansions both converge uniformly with regard to z when $|z| \leqq R'$, where $R > R'$.

Let $f(z)$ be any function of z which is analytic and single-valued in the region $|z| = R$ and upon its contour C. And, let $f(z) = \phi_1(z) + \phi_2(z)$, where $\phi_1(z)$ is an odd function, and $\phi_2(z)$ is an even function, each of which is analytic and single-valued in the circle $|z| = R$. Then by the expansions (46) and (47) we have at once the expansion

$$f(z) = \sum_{m=0}^{\infty} z^{-a+1} c_m \mathfrak{J}_m(z) + \sum_{m=0}^{\infty} z^{-a} c'_m \mathfrak{J}_m(z), \tag{48}$$

where
$$c_m = (1/2\pi i) \int_C B_m(t)\, \phi_1(t)\, dt,$$

and
$$c'_m = (1/2\pi i) \int_C B_m(t)\, \phi_2(t)\, t\, dt.$$

This expansion is valid when z is within the circle $|z| = R$, and the convergence is uniform with regard to z when $|z| \leqq R'$, where $R > R'$. Furthermore, the uniqueness of the expansions (44) and (45) can be established in a manner analogous to that used in Art. 7.

VII. Special Cases.

13. In order to make explicit the sense in which these expansions are generalizations of known results it seems desirable to call attention to some of the special cases which have been treated. It will then become apparent what generalizations have been made.

When $n = 1$, $\mu_1 = 0$, and $\theta_1 = 1$ the expansion (29) reduces to that obtained by Neumann,[*] namely

$$f(z) = \sum_{m=0}^{\infty} a_m J_m(z),$$

where
$$a_m = (\epsilon_m/2\pi i) \int_C O_m(t)\, f(t)\, dt, \qquad \epsilon_m = 2 \text{ if } m \geqq 1, \quad \epsilon_0 = 1,$$

and
$$\begin{cases} O_m(t) = (1/4) \displaystyle\sum_{r=0}^{\leq m/2} \frac{m(m-r-1)!}{r!\,(t/2)^{m+1-2r}} & \text{for } m \geqq 1, \\ O_0(t) = 1/t. \end{cases}$$

For, since the expansion (21) is unique it follows that $\epsilon_m O_m(t) = B_m(t)$ when $\mu_1 = 0$ and $\theta_1 = 1$, n now having the value unity.

---

[*] *Theorie der Bessel'schen Funktionen*, pp. 33-35; Watson, *loc. cit.*, p. 523.

When $n = 2$, $\theta_1 = \theta_2 = 1$, and $\mu_1 = \mu_2 = 0$ we see from the uniqueness of the expansion (44) that the expansion of even functions given in (46) reduces to that obtained by Neumann[*] for even functions in terms of squares of Bessel Functions, namely,

$$f(z) = \sum_{m=0}^{\infty} a'_m \left[J_m(z)\right]^2, \quad \text{where} \quad a'_m = (\epsilon_m/2\pi i)\int_C \Omega_m(t)\, f(t)\, t\, dt,$$

and
$$\begin{cases} \Omega_m(t) = (1/4) \displaystyle\sum_{s=0}^{m} \frac{m(m+s-1)!\,(s!)^2}{(m-s)!(2s)!(t/2)^{2s+2}} & \text{for } m \geq 1, \\ \Omega_0(t) = 1/t^2. \end{cases}$$

And we have $\epsilon_m \Omega_m(t) = B_m(t)$, where $\theta_1 = \theta_2 = 1$ and $\mu_1 = \mu_2 = 0$, n now having the value 2.

When $n = 1$, $\mu_1 = \nu$, and $\theta_1 = 1$ the expansion (29) reduces to the following expansion, due to Gegenbauer,[†]

$$z^{\nu} f(z) = \sum_{m=0}^{\infty} a_m J_{\nu+m}(z), \text{where } a_m = (1/2\pi i)\int_C A_{m,\nu}(t)\, f(t)\, dt,$$
and
$$A_{m,\nu}(t) = \frac{2^{\nu+m}(\nu+m)}{t^{m+1}} \sum_{s=0}^{\leq m/2} \frac{\Gamma(\nu+m-s)}{s!}(t/2)^{2s}.$$

From the uniqueness of the expansion (21) we have $A_{m,\nu}(t) = B_m(t)$ when $\mu_1 = \nu$ and $\theta_1 = 1$, n now having the value 1.

When $n = 2$, $\mu_1 = \mu$, $\mu_2 = \nu$ and $\theta_1 = \theta_2 = 1/2$ we obtain from (29) another expansion, due also to Gegenbauer,[‡]

$$z^{\mu+\nu} f(z) = \sum_{m=0}^{\infty} a_m J_{\mu+(m/2)}(z)\, J_{\nu+(m/2)}(z),$$

$$a_m = (1/2\pi i)\int_C B_{m,\mu,\nu}(t)\, f(t)\, dt,$$

and
$$B_{m,\mu,\nu}(t) = \frac{2^{\mu+\nu+m}(\mu+\nu+m)}{t^{m+1}}$$

$$\sum_{s=0}^{\leq m/2} \frac{\Gamma[\mu+(m/2)-s+1]\,\Gamma[\nu+(m/2)-s+1]\,\Gamma(\mu+\nu+m-s)}{s!(\mu+\nu+m-2s+1)}(t/2)^{2s}.$$

And, likewise, $B_{m,\mu,\nu}(t) = B_m(t)$ when $\mu_1 = \mu$, $\mu_2 = \nu$ and $\theta_1 = \theta_2 = 1/2$, n now having the value 2.

---

[*] *Mathematische Annalen*, Vol. 3 (1871), p. 599; Watson, *loc. cit.*, p. 525.

[†] *Wiener Sitzungsberichte*, 74, (2), (1877), pp. 124-130; Watson, *loc. cit.*, p. 525.

[‡] *Wiener Sitzungsberichte*, 75, (2), (1877), pp. 218-222; Watson, *loc. cit.*, p. 525.

VIII. EXPANSIONS OF ANALYTIC FUNCTIONS OF SEVERAL VARIABLES.

14. Much of the foregoing expansion theory which has been developed for functions of a single variable is capable of extension to functions of several variables. We shall next indicate briefly the argument for one of the expansions which can be obtained.

Since $\mathfrak{F}_m(z)$ is a function of the quantities μ_i and θ_i, we introduce the following notation:

$$\mathfrak{F}_m{}^{(j)}(z) = J_{\mu_{j1}+\theta_{j1}m}(z)\, J_{\mu_{j2}+\theta_{j2}m}(z) \cdots J_{\mu_{jn_j}+\theta_{jn_j}m}(z),$$

$$B_m{}^{(j)}(t) = \sum_{r=0}^{\leq m/2} c_{m-2r,r}\, \frac{2^{a_j+m-2r}}{t^{m+1-2r}},$$

$$\alpha_j = \mu_{j1} + \mu_{j2} + \cdots + \mu_{jn_j},$$

with the restriction that $\mu_{ji} + \theta_{ji}r$ be not a negative integer for any i or j occurring in the expansion, and for $r = 0, 1, 2, \cdots$.

For simplicity in argument the expansion is developed for the case of two variables. The extension is obvious.

Let z_1, t_1, and z_2, t_2 be two pairs of complex quantities in the z_1 and z_2 planes, respectively, such that

$$(49) \quad \begin{cases} |z_1| \leq r_1, & |t_1| \geq R_1, & r_1 < R_1, \\ |z_2| \leq r_2, & |t_2| \geq R_2, & r_2 < R_2; \end{cases}$$

then, by Theorem I the expansions

$$(50) \qquad \begin{cases} 1/(t_1 - z_1) = \sum_{m_1=0}^{\infty} B_{m_1}{}^{(1)}(t_1)\, F_{m_1}{}^{(1)}(z_1), \\ 1/(t_2 - z_2) = \sum_{m_2=0}^{\infty} B_{m_2}{}^{(2)}(t_2)\, F_{m_2}{}^{(2)}(z_2), \end{cases}$$

where
$$F_{m_i}{}^{(r)}(z_i) = z^{-a_r}\mathfrak{F}_{m_i}{}^{(r)}(z_i),$$

are valid. If t_1 and t_2 are variable points on or outside the circles C_1 and C_2, respectively, defined by the relations $|z_1| = R_1$, $|z_2| = R_2$, and if z_1 and z_2 are any variable points within or upon the circles $|z_1| = r$, and $|z_2| = r_2$, respectively, then the expansions (50) converge uniformly with regard to t_1, z_1 and t_2, z_2, respectively. Since both expansions converge uniformly when conditions (49) are fulfilled and are dominated by convergent series of constants, then under the same conditions the double series in the second member of the equation

$$(51) \qquad 1/(t_1 - z_1) \cdot 1/(t_2 - z_2) = \sum_{m_1, m_2 = 0}^{\infty} B_{m_1}^{(1)}(t_1)\, B_{m_2}^{(2)}(t_2)\, F_{m_1}^{(1)}(z_1)\, F_{m_2}^{(2)}(z_2)$$

converges uniformly.

Now let $f(z_1, z_2)$ be an arbitrary function of the two independent complex variables z_1 and z_2, which is analytic and single-valued in z_1 and z_2 separately when z_1 and z_2 are, respectively, in the regions bounded by the contours C_1 and C_2; then, by Cauchy's generalized integral formula, we have

$$\begin{aligned} f(z_1, z_2) &= (1/2\pi i)^2 \int_{C_1} \int_{C_2} \frac{f(t_1, t_2)\, dt_1\, dt_2}{(t_1 - z_1)(t_2 - z_2)} \\ &= (1/2\pi i)^2 \int_{C_1} \int_{C_2} \{ \sum_{m_1, m_2 = 0}^{\infty} B_{m_1}^{(1)}(t_1)\, B_{m_2}^{(2)}(t_2) \\ &\qquad F_{m_1}^{(1)}(z_1)\, F_{m_2}^{(2)}(z_2) \}\, f(t_1, t_2)\, dt_1\, dt_2 \\ &= \sum_{m_1, m_2 = 0}^{\infty} A_{m_1, m_2} F_{m_1}^{(1)}(z_1)\, F_{m_2}^{(2)}(z_2), \end{aligned}$$

where $A_{m_1, m_2} = 1/(2\pi i)^2 \int_{C_1} \int_{C_2} B_{m_1}^{(1)}(t_1) B_{m_2}^{(2)}(t_2)\, f(t_1, t_2)\, dt_1\, dt_2.$

This expansion is valid and converges uniformly with regard to z_1 and z_2 when z_1 and z_2 are, respectively, in the circles $|z_1| = r_1$ and $|z_2| = r_2$. This result may be stated for ν variables as

THEOREM V. *Let $f(z_1, z_2, \cdots, z_\nu)$ be a function of the complex quantities $z_1, z_2, \cdots, z_\nu$, which is analytic and single-valued when $z_1, z_2, \cdots, z_\nu$ are, respectively, in the regions $|z_1| \leqq R_1$, $|z_2| \leqq R_2$, $\cdots$, $|z_\nu| \leqq R_\nu$, whose contours are, respectively, C_1, C_2, $\cdots$, C_ν, then the expansion*

$$f(z_1, z_2, \cdots, z_\nu) = \sum_{m_1, m_2, \ldots, m_\nu = 0}^{\infty} A_{m_1, m_2, \ldots m_\nu} F_{m_1}^{(1)}(z_1)\, F_{m_2}^{(2)}(z_2) \cdots F_{m_\nu}^{(\nu)}(z_\nu$$

where

$$A_{m_1, m_2, \ldots m_\nu} = 1/(2\pi i)^\nu \int_{C_1} \int_{C_2} \cdots \int_{C_\nu} B_{m_1}^{(1)}(t_1)\, B_{m_2}^{(2)}(t_2) \cdots B_{m_\nu}^{(\nu)}(t_\nu) f(t_1, t_2, \cdots, t_\nu)\, dt_1\, dt_2 \cdots dt_\nu,$$

is valid and converges uniformly when $z_1, z_2, \cdots, z_\nu$ are, respectively, in the circles $|z_1| = r_1$, $|z_2| = r_2$, $\cdots$, $|z_\nu| = r_\nu$, where $r_i < R_i$, $i = 1, 2, \cdots, \nu$.

It is now obvious that expansions of functions of several variables, analogous to the expansions (31), (34), (35) and (39) exist.

UNIVERSITY OF ILLINOIS,
URBANA, ILLINOIS.

Canonical Forms for Ordinary Homogeneous Linear Differential Equations of the Second Order with Periodic Coefficients.

BY ELIJAH SWIFT.

Even such a simple matter as the actual reduction to canonical form of the ordinary homogeneous linear differential equation of the second order,

$$L: \qquad y'' + p(x)y' + q(x)y = 0,$$

(p and q continuous in the interval (a, b), i. e. $a \leq x \leq b$) by transformations of the form

$$y = \lambda(x)\bar{y}, \qquad x = \phi(\bar{x}),$$

(where we assume that throughout (a, b) λ and ϕ are continuous together with their first two derivatives and $\lambda(x) \neq 0$, $\phi'(\bar{x}) \neq 0$) does not appear to have been carried out in the literature. However, in his lectures at Harvard University Professor G. D. Birkhoff has effected this very simply and for purposes of orientation, at his suggestion, I have given this material in the second section of this paper.

But there is another less simple case of great importance in the applications, namely, that in which $p(x)$ and $q(x)$ are periodic with period 2π. It is the problem of reducing a general equation of this type to a canonical form, preserving the periodic character of the coefficients, the solution of which is given in the following paper. Incidentally we classify such equations and investigate the general character of their solutions.

1. *Introductory.* In the following we consider the differential equation L: $y'' + p(x)y' + q(x)y = 0$, where the functions p and q are real and continuous in a closed interval (a, b), $a \leq x \leq b$, which may in particular include the whole of the X-axis. We consider only real values of x and real solutions of L. The following facts are well known:[*]

a. There exists one and only one function, $y(x)$, which is continuous together with its first two derivatives and satisfies the differential equation identically throughout (a, b) and which is such that $y(x_0) = \alpha$, $y'(x_0) = \alpha'$, where x_0 is any arbitrarily chosen point of (a, b) and α and α' are arbitrarily chosen constants. From this follows that if a solution and its derivative vanish at the same point, then the solution vanishes identically throughout (a, b).

---

[*] See e. g. Bocher, Leçons sur les Méthodes de Sturm, Chap. I for a, b, d; p. 46 for c.

b. Two solutions, y_1, y_2, are said to be *linearly dependent* or simply *dependent,* if a relation of the form $C_1y_1 + C_2y_2 = 0$ (C_1, C_2, constants not both zero) holds identically throughout (a, b): linearly independent, if no such relation exists. If y_1, y_2 are two independent solutions, then any other solution may be written in the form $C_1y_1 + C_2y_2$ (C_1, C_2, suitably chosen constants).

c. If y_1, y_2 are two independent solutions, they cannot vanish at the same point and the zeros of y_1 and the zeros of y_2 occur alternately.

d. The "wronskian" of two solutions y_1, y_2, is defined as the determinant $y_1y'_2 - y'_1y_2$. Its value is

$$W(y_1, y_2) = K \exp\left[-\int p\,dx\right],$$

where K is a constant whose value depends upon which pair of solutions is chosen. If W vanishes at one point of (a, b) it vanishes identically throughout (a, b). Its vanishing is a necessary and sufficient condition that y_1 and y_2 be dependent.

With suitable restrictions the only transformations of the form

$$x = \phi(\bar{x}, \bar{y}), \quad y = \psi(\bar{x}, \bar{y}),$$

changing the original variables, x, y, to the new set, $\bar{x}, \bar{y}$, which carry *any* equation, L, into one of the same form are

$$A \qquad y = \lambda(x)\bar{y},$$

changing the dependent variable, and

$$B \qquad x = \phi(\bar{x}),$$

changing the independent variable.* I shall refer to these by the letters A and B respectively. We assume that both $\lambda(x)$ and $\phi(\bar{x})$ are continuous and possess continuous first and second derivatives throughout the interval under consideration,—(a, b) for λ: $(\bar{a}, \bar{b})$, the interval corresponding to (a, b) by the transformation B, for ϕ.

---

* Although a proof of this fact is comparatively simple, I have been unable to find it in the standard texts. It may be noted that there exist *special* transformations carrying *special* equations of the form L into equations of the same form. Such a one e. g. is $y = \lambda(x)\bar{y} + y_1$ where y_1 is a solution of the original equation.

Applied formally A transforms the differential equation L into

$$(1) \qquad \bar{y}'' + (p + 2\lambda'/\lambda)\bar{y}' + [(\lambda'' + p\lambda' + q\lambda)/\lambda]\bar{y} = 0.$$

If the coefficients of $\bar{y}'$ and $\bar{y}$ are to be continuous throughout (a, b), it is necessary and sufficient that $\lambda(x)$ be not zero in (a, b)[*] and this is also necessary and sufficient that the inverse of A shall exist throughout (a, b).

Similarly B formally transforms L into

$$(2) \qquad y'' + [p(\phi)\phi' - \phi''/\phi']y' + [q(\phi)/\phi'^2]y = 0,$$

where $\phi = \phi(\bar{x})$ and the differentiations are performed with respect to $\bar{x}$. Here a necessary and sufficient condition for the continuity of the coefficients and the existence of a single valued inverse of the transformation is that $\phi'(\bar{x})$ shall not vanish for values of $\bar{x}$ in $(\bar{a}, \bar{b})$.

Transformations satisfying these conditions,—(λ, ϕ, continuous, possessing first and second derivatives, λ, ϕ' nowhere zero)—we term *proper*. It is clearly necessary to consider only *proper* transformations if our work is to be more than formal and if the coefficients of the transformed equation are to be continuous.

The proper A transformations form a group, since 1) the succession of two such is equivalent to a single one, 2) the identical transformation is one of the group, 3) the inverse of any proper A transformation exists and is a proper A. Similarly the proper B transformations form a group. Furthermore if a finite number of proper A and of proper B transformations be carried out in any order, they will be equivalent to a single proper A and a single proper B.

In the greater part of this paper we shall be concerned with equations whose coefficients are periodic with period 2π and shall consider only those (proper) transformations which leave the coefficients periodic (with the same period). For an A transformation this means that the coefficient of $\bar{y}'$, $p(x) + 2\lambda'/\lambda$, [see (1)] must be periodic and hence λ'/λ also. Consequently λ must be of the form $e^{cx}P(x)$, where c is a constant and $P(x)$ a periodic function which cannot vanish for any value of x in (a, b). Clearly if λ is of this form both coefficients of the transformed equation will be periodic.

---

[*] If λ is a function of x, continuous throughout (a, b) and satisfying there the equation $\lambda'/\lambda = f(x)$, where $f(x)$ is continuous, λ cannot vanish. For a continuous solution of the differential equation $\lambda' - f(x)\lambda = 0$ either vanishes identically throughout (a, b) or not at all.

If the *same* B transformation is to carry *any* equation with periodic coefficients into one whose coefficients are periodic, it must in particular do this if p, q, are constants. Referring to the form of the transformed equation [see (2)] we see from the form of the coefficient of y that this is possible only if $\phi'(\bar{x}+2\pi)=\phi'(\bar{x})$,[*] or $\phi(\bar{x}+2\pi)=\phi(\bar{x})+C$, C a constant. For equations in which q has a minimum period of 2π, C must be a multiple of 2π [†] and we shall take it equal to 2π, since taking it equal to $2k\pi$ is equivalent to following a transformation where $C=2\pi$ by $\bar{x}=k\bar{\bar{x}}$. Conversely if ϕ is such that $\phi(\bar{x}+2\pi)=\phi(\bar{x})+2\pi$, the coefficients of the transformed equation will be periodic.

For lack of a better term we shall call transformations of the above types "*periodic*":—that is, a *periodic A transformation* is one for which

$$\lambda(x)=e^{cx}\cdot P(x), \quad P \text{ periodic and nowhere zero;}$$

a *periodic B transformation* is one for which

$$\phi(\bar{x}+2\pi)=\phi(\bar{x})+2\pi, \quad \phi' \text{ nowhere zero,}$$

which is clearly equivalent to

$$\phi(\bar{x})=\bar{x}+P(\bar{x}), \quad P \text{ periodic.}$$

In § 2 we shall reduce the general equation to a canonical form using general A and B transformations: in the following paragraphs we shall reduce the equation with periodic coefficients to one of a number of canonical forms using only periodic transformations.

2. *Reduction of the general Differential Equation to a Canonical Form.*[‡] Suppose that we transform a given differential equation, L, by arbitrary A and B transformations. Since we have two functions, λ and ϕ, at our disposal and only two coefficients, p and q, appear in the original equation, it seems reasonable to assume that we can so determine λ and ϕ that the transformed equation will have coefficients which do not depend upon p and q, but which are the same for all differential equations or for all of a large class.

This has been effected formally somewhat as follows: Let y_1 and y_2 be any two independent solutions of the given equation, L. Employ the transformations

---

[*] $\phi'(x+2\pi) = -\phi'(x)$ implies the vanishing of ϕ' contrary to hypothesis.

[†] $C=0$ implies the vanishing of ϕ'.

[‡] The results contained in this section are due to Professor G. D. Birkhoff: see Introduction.

$$y = y_2(x)\bar{y}. \tag{3}$$

$$\bar{x} = y_1(x)/y_2(x). \tag{4}$$

Since any solution of L may be written in the form $C_1y_1 + C_2y_2$, any solution of the transformed equation will have the form $C_1\bar{x} + C_2$, so that the transformed equation is simply $d^2\bar{y}/d\bar{x}^2 = 0$. Since, however, *we are studying the equation L in connection with an interval* (a, b), we must take care that the transformations employed are proper throughout this interval if our work is to be more than merely formal. But here the transformations (3) and (4) both cease to be proper if $y_2(x)$ vanishes. But if any solution vanishes twice in (a, b) every solution vanishes at least once (see § 1, c) and we can find no y_2 such that these transformations are proper throughout (a, b). In other words, while the above reduction is valid for any differential equation and an interval sufficiently short, it is not in general more than a purely formal reduction for a general equation and a general interval.

We may, however, reduce the equation L to a canonical form employing only transformations proper throughout (a, b). Let y_2 be a solution of L such that $y_2(a) = 0$ (see § 1, a). If $y_2(b)$ is not 0, choose the solution y_1 such that $y_1(b) = 0$: if $y_2(b) = 0$, let y_1 be any independent solution. We use the transformations

$$\arctan\left[y_2(x)/y_1(x)\right] = \bar{x}, \tag{5}$$

$$y_2/(y_1^2 + y_2^2)^{1/2} = \sin\bar{x}, \tag{6}$$

$$y = (y_1^2 + y_2^2)^{1/2}\cdot\bar{y}, \tag{7}$$

where of course (5) and (6) are essentially the same. With suitable conventions there is no discontinuity in the left-hand member of (5) at points where $y_1 = 0$: in particular we may use the equivalent equation

$$\operatorname{arccot}(y_1/y_2) = \bar{x}.$$

Thus $d\bar{x}/dx$ and $d^2\bar{x}/dx^2$ exist and are continuous even at such points. Since y_1 and y_2 cannot vanish together, it is clear that the coefficient of $\bar{y}$ in (7) cannot vanish at any point of (a, b). Moreover this coefficient possesses continuous first and second derivatives, since y_1 and y_2 do. Consequently (7) is a proper A transformation.

From (5) $d\bar{x}/dx$ is $W(y_1, y_2)/(y_1^2 + y_2^2)$, which exists and is different from 0 throughout (a, b) [see § 1, d]. Since we may take any constant multiple of y_2, in particular $-y_2$, in place of y_2, we may assume this positive. The theorems on implicit functions show that we may solve equation (5), for x in terms of $\bar{x}$, that $dx/d\bar{x}$ will exist and be different from 0 and that $d^2x/d\bar{x}^2$ will exist and be continuous throughout (a, b). Thus (5) defines a proper B transformation.

From equations (5), (6), (7), we see that y_1 and y_2 are carried over by the above transformations into $\cos\bar{x}$ and $\sin\bar{x}$ respectively, so that the transformed equation is

$$d^2\bar{y}/d\bar{x}^2 + \bar{y} = 0.$$

To find the transformed interval, let x change continuously in (a, b) starting with the value a. We may take the value of $\bar{x}$ corresponding to a as 0. Then as x increases from a, $\bar{x}$ will increase from 0. When x has increased from a to the next following zero of y_2, (5) or (6) shows that $\bar{x}$ has increased from 0 to π, when x increases from this zero to the next, $\bar{x}$ increases from π to $2\pi, \cdots$. If y_2 has $k+1$ zeros in (a, b) (including a) and $y_2(b)$ is not zero, $\bar{x}$ will increase from 0 to a value between $k\pi$ and $(k+1)\pi$ when x increases from a to b. But since $y_1(b) = 0$, (5) tells us that this value is exactly $(k+1/2)\pi$. If y_2 has $k+1$ zeros in (a, b) including a and b, this value is clearly $k\pi$.

We may state our results as follows: *Any linear differential equation of the second order, whose coefficients p and q are continuous functions of the independent variable, x, in an interval (a, b), $a \leq x \leq b$, may be transformed by proper transformations of the types $y = \lambda(x)\cdot\bar{y}$, $x = \phi(\bar{x})$, into the equation $d^2\bar{y}/dx^2 + \bar{y} = 0$, while the interval (a, b) is carried over into the interval*

$$0 \leq \bar{x} \leq (k+1/2)\pi \text{ or } 0 \leq \bar{x} \leq k\pi,$$

the first for the case that that solution of the original equation which vanishes at a has exactly k other zeros in (a, b) but does not vanish at b: the second for the case that that solution which vanishes at a has exactly k other zeros in (a, b) of which b is one.

Suppose now we have a second equation, L', and a closed interval, (a', b'), in which its coefficients are continuous. Then a necessary and sufficient condition that L' be transformable into L by proper transformations, while the interval $(a', b',)$ is transformed into (a, b), is that that solution of L' which vanishes at a' has as many zeros in (a', b') as that solution of L which vanishes at a, has in (a, b) and that these solutions either both vanish at the other end of the respective intervals or both fail to vanish there.

That this condition is necessary is clear from the fact that the transformations considered do not change the number of zeros of a solution in a given interval nor its vanishing at the end of the interval: that it is sufficient follows from the fact that if satisfied, when we transform L and L' into the canonical equation, the canonical intervals will be the same. Since every transformation considered has an inverse, we can transform L' into L directly and (a', b') will go over into (a, b).

Thus if we consider the group of proper A and B transformations and a differential equation together with an interval in which its coefficients are continuous, essentially the only invariants are the number of zeros in the interval of a solution vanishing at one end and the existence or non-existence of a solution vanishing at both ends of the interval.

3. *Preliminary Study of the Differential Equation with Periodic Coefficients.*[*] In what follows we shall deal exclusively with differential equations, L, whose coefficients, $p(x)$, $q(x)$, are continuous for all values of the real variable, x, and periodic with period 2π. For convenience I shall use the term periodic to mean periodic with period 2π.

If $y_1(x)$ and $y_2(x)$ are any two independent solutions of such an equation, $y_1(x+2\pi)$ and $y_2(x+2\pi)$ will also be solutions on account of the periodic character of the coefficients, and hence expressible in the form (see § 1, b)

$$(8)\qquad \begin{aligned} y_1(x+2\pi) &= C_1 y_1(x) + C_2 y_2(x), \\ y_2(x+2\pi) &= C'_1 y_1(x) + C'_2 y_2(x), \qquad (C_1, C_2, C'_1, C'_2, \text{ constants}). \end{aligned}$$

Since we are dealing with real solutions, the C's will all be real. The value of their determinant may be found directly from the differential equation, L: it is †

$$(9)\qquad \Delta(C) = C_1 C'_2 - C_2 C'_1 = \exp\left[-\int_0^{2\pi} p(x)\,dx\right]$$

and is accordingly positive.

Although the value of $\Delta(C)$ [and, as we shall see below, $C_1 + C'_2$] is definitely determined when the differential equation is given, the values of the C's themselves depend upon the particular solutions, y_1, y_2, chosen. If, for instance, we had chosen αy_1 (α any constant) as one of the pair above in place of y_1, we should have found αC_2 instead of C_2, C_1 would have remained the same, etc. Below in tabular form we give the new values of the C's corresponding to different pairs of solutions. α and β are arbitrary constants, not zero.

---

[*] For most of the results of this paragraph see e. g. Goursat-Hedrick, *Mathematical Analysis,* Volume 2, Part 2, pp. 146 ff.

† Goursat-Hedrick, *loc. cit.*, p. 147.

Replacing	we find the new values of the C's to be of C_1	of C_2	of C'_1	of C'_2
y_1 by αy_1	C_1	αC_2	C'_1/α	C'_2
y_2 by βy_2	C_1	C_2/β	$\beta C'_1$	C'_2
y_1 by $y_1 + y_2$	$C_1 + C'_1$	$C'_2 + C_2 - C_1 - C'_1$	C'_1	$C'_2 - C'_1$

No one of these substitutions changes the value of $C_1 + C'_2$ or of $\Delta(C)$ and hence it may easily be shown that each of these quantities has the same value no matter which pair of solutions we choose in place of y_1, y_2.[*] The value of $\Delta(C)$ was given above: the value of $C_1 + C'_2$ cannot in general be found without integrating the differential equation.

In general the differential equation will have no periodic solution, but it may have one or more real solutions, $Y(x)$, not identically zero, which have the property [†] that

$$(10)\qquad Y(x + 2\pi) = \rho Y(x),\ \rho \text{ a constant.}$$

I shall call such a solution a *principal solution*. We may find all principal solutions as follows. Any solution, in particular Y, may be written in the form $Ay_1 + By_2$, (A, B constants not both zero). Consequently, using equations (8)

$$(11)\qquad \begin{aligned} Y(x + 2\pi) &= Ay_1(x + 2\pi) + By_2(x + 2\pi) \\ &= (AC_1 + BC'_1)y_1(x) + (AC_2 + BC'_2)y_2(x). \end{aligned}$$

But if Y is a principal solution,

$$(12)\qquad Y(x + 2\pi) = \rho Y(x) = \rho A y_1(x) + \rho B y_2(x).$$

Comparing (11) and (12) and noting that y_1 and y_2 are independent (see § 1, b), we have

$$(13)\qquad A(C_1 - \rho) + BC'_1 = 0, \qquad AC_2 + B(C'_2 - \rho) = 0.$$

Since A and B are not both zero, the determinant of their coefficients in equations (13) must vanish, that is

$$(14)\qquad (C_1 - \rho)(C'_2 - \rho) - C'_1C_2 = 0,$$

---

[*] This may be proved directly: see e. g. Forsyth, *Theory of Differential Equations*, Part III, Vol. 4, pp. 405-6.

[†] Of course any constant multiple of Y has this same property. When we speak of two (distinct) principal solutions, we understand neither is a constant multiple of the other.

which we call the *characteristic equation.* It may be written in the form

$$(14') \qquad \rho^2 - (C_1 + C'_2)\rho + C_1C'_2 - C_2C'_1 = 0,$$

and hence is independent of the particular pair of solutions, y_1, y_2, chosen. We also note that the constant term is positive. See (9).

Conversely if ρ is a root of this characteristic equation, we can find A and B, not both zero, satisfying equations (13) and $Ay_1(x) + By_2(x)$ will be a principal solution. The value of ρ uniquely determines the ratio A/B and the corresponding principal solution (except for a constant factor) unless equations (13) are identically satisfied, that is unless $C_2 = 0$, $C'_1 = 0$, $C_1 = C'_2 = \rho$, in which case, as appears from equations (8), y_1 and y_2 are principal solutions (with the same multiplier, ρ) and every solution is a principal solution.

Since the characteristic equation is a quadratic in ρ with constant term positive, the following cases are possible:

I. Roots real and distinct, both positive or both negative.

II. Roots real and equal.
   (a) Every solution a principal solution.
   (b) One and only one principal solution.

III. Roots conjugate complex.

In the following paragraphs we classify differential equations by the roots of their characteristic equations as above and reduce those of each type to simple canonical forms employing only periodic transformations of types A and B.

4. *Roots of Characteristic Equation real and distinct.* Let the roots be ρ_1 and ρ_2 where we call ρ_2 that root for which $|\rho_2| \geq |\rho_1|$. Corresponding to these are two principal solutions, Y_1, Y_2 such that

$$Y_i(x + 2\pi) = \rho_i Y_i(x), \qquad (i = 1, 2).$$

Consider the expression

$$Y_i(x) \cdot \exp\left[-(x/2\pi) \log |\rho_i|\right].$$

When x is replaced by $x + 2\pi$, this preserves the same value (ρ_i positive) or changes sign (ρ_i negative). It is then periodic or semi-periodic* and we may write

---

* I use the latter term to denote a function, $f(x)$, such that $f(x + 2\pi) = -f(x)$.

$$Y_1(x) = \exp\left[(x/2\pi)\log|\rho_1|\right]\cdot P_1(x), \quad (15)$$
$$Y_2(x) = \exp\left[(x/2\pi)\log|\rho_2|\right]\cdot P_2(x),$$

where P_1 and P_2 are either both periodic $(\rho_1, \rho_2 > 0)$ or both semi-periodic $(\rho_1, \rho_2 < 0)$ and both possess continuous first and second derivatives, since Y_1 and Y_2 do.

If the roots of the characteristic equation are equal but there are two independent principal solutions, we have essentially the above case and accordingly shall not rule out the possibility $\rho_1 = \rho_2$.

Y_1 and P_1 have the same zeros. If P_1 vanishes at x_1, it also vanishes at $x_1 + 2n\pi$, $(n = \pm 1, 2, \cdots)$, and since the zeros of Y_1 and Y_2 occur alternately, P_1 and P_2 vanish the same number of times in any interval of length 2π (including exactly one end point).

CASE (*i*). Y_1, Y_2 nowhere zero. If P_1 does not vanish in the interval $0 \leq x < 2\pi$, it can vanish nowhere on the X-axis and neither can P_2. Y_2/Y_1 is everywhere continuous and its derivative

$$(Y_1Y'_2 - Y_2Y'_1)/Y_1^2 = W(Y_1, Y_2)/Y_1^2$$

is nowhere zero, since Y_1 and Y_2 are independent solutions (see § 1, d). Incidentally we note that here we cannot have $\rho_1 = \rho_2$. For then Y_2/Y_1 would be a continuous periodic function (from (15)) whose derivative exists, is continuous and does not vanish for any value of x. But this is impossible, for a continuous periodic function cannot be monotonic.

We may state this result thus: *An ordinary linear differential equation of the second order with coefficients continuous in the interval* $0 \leq x \leq 2\pi$ *cannot have as independent solutions two functions of the form* $e^{cx}P_1(x)$, $e^{cx}P_2(x)$, *where* c *is a constant and* P_1 *and* P_2 *are continuous and periodic with period* 2π, *unless* P_1 (*and therefore* P_2 *also*) *vanishes at least once in the above interval.*

We now consider the transformation defined by the equation

$$\log (Y_2/Y_1)^{*} = (x/2\pi)\log(\rho_2/\rho_1) + \log\left[P_2(x)/P_1(x)\right] \quad (16)$$
$$= (\bar{x}/2\pi)\log(\rho_2/\rho_1)\cdot$$

or, for convenience setting

$$2\alpha = (1/2\pi)\log(\rho_2/\rho_1). \quad (17)$$

$$x + (1/2\alpha)\log(P_2/P_1) = \bar{x}. \quad (16')$$

The left-hand member of equation (16) possesses a continuous non-vanishing

---

[*] To avoid imaginaries we choose Y_1, Y_2 so that Y_2/Y_1 is positive.

derivative, namely $W(Y_1, Y_2)/Y_1Y_2$, and also a continuous second derivative (since Y_1 and Y_2 possess continuous second derivatives). From the theorems on implicit functions it follows that (16) defines x as a function of $\bar{x}$, $x = \phi(\bar{x})$, so that ϕ' and ϕ'' exist and are continuous and that ϕ' is nowhere zero. Furthermore, on account of the periodicity of P_1 and P_2, when x increases by 2π, so will $\bar{x}$ and conversely. Thus equation (16) [or (16′)] defines a proper periodic B transformation.

Using equations (15), (16) and (17) we may write

$$(18)\qquad \begin{cases} Y_1 = \exp(-\alpha\bar{x}) \exp[(x/2\pi)\log(\rho_1\rho_2)^{1/2}] \cdot [P_1(x)P_2(x)]^{1/2}, \\ Y_2 = \exp(2\alpha\bar{x}) \cdot Y_1. \end{cases}$$

Consider now the A transformation

$$(19)\qquad \begin{aligned} &y = \lambda(x) \cdot \bar{y}, \text{ where} \\ &\lambda(x) = \exp[(x/2\pi)\log(\rho_1\rho_2)^{1/2}] \cdot [P_1(x)P_2(x)]^{1/2}. \end{aligned}$$

λ possesses continuous first and second derivatives and is itself nowhere zero. It is also of the form given on p. 394 for a periodic transformation. It is therefore a proper, periodic A transformation.

From (18) we see that this transformation carries the solutions Y_1, Y_2 of the original equation over into the canonical forms $\exp(-\alpha\bar{x})$ and $\exp(+\alpha\bar{x})$, and the equation becomes

$$(20)\qquad d^2\bar{y}/d\bar{x}^2 - \alpha^2\bar{y} = 0.$$

Thus any equation may, in this case, be reduced to the canonical form (20) by proper periodic transformations; the only invariant being essentially α, which is determined by the ratio of the roots of the characteristic equation.

CASE (*ii*). Y_2 vanishes for some value of x. Suppose that $Y_2(x)$ vanishes k times in the interval $0 \leq x < 2\pi$ where k is even if P_2 is periodic and odd if P_2 is semi-periodic. We assume $P_2(0) = 0$: if it is not, the transformation $x = \bar{x} + a$ will carry a zero, a, into the origin. We consider the B transformation defined by the equation

$$(21)\qquad Y_2/Y_1 = \tan(k/2)\bar{x} \cdot \exp(2\alpha\bar{x}) \cdot \exp[(-2\alpha^2/k^2)\cos k\bar{x}],$$

where α has the meaning assigned above by (17). In particular if $\rho_1 = \rho_2$, $\alpha = 0$.

The last (exponential) factor on the right of (21) is inserted to insure that the derivative of that member of the equation shall always be positive,—which is necessary for the validity of the transformation (see below). In fact the derivative of $\exp(2\alpha x) \cdot \tan(k/2)x$ is

$$[\exp(2\alpha x)/2\cos^2(k/2)x](2\alpha\sin kx + k),$$

which will vanish for some value of x if $k \leq 2\alpha$, while the derivative of the right-hand member of (21) is $(4\alpha^2 \sin^2 k\bar{x} + 4\alpha k \sin kx + 2k^2)$ multiplied by a non-vanishing factor and can never vanish. The final factor in (21) is not uniquely determined: in particular instead of $2\alpha^2$ we might use $b\alpha^2$, $b > 1$. Of course if $k > 2\alpha$ we may omit this factor and have simpler canonical forms: those forms which I give, however, are valid for all values of k and α.

We now show that (21) defines a proper periodic transformation. For convenience denote the zeros of Y_2 by $0, a_1, a_2, \cdots, a_k (= 2\pi)$ and those of Y_1 by $b_1, b_2, \cdots, b_k$. These occur in the order $0, b_1, a_1, b_2, a_2, \cdots, a_{k-1}, b_k, a_k = 2\pi$ (§ 1, c). The derivative of the left-hand member of (21) is $W(Y_1, Y_2)/Y_1^2$ and consequently has everywhere the same sign (which we may take as positive), except at the points b_i, where it fails to exist. The derivative of the right-hand member is always positive except at the points $[(2i-1)/k]\pi$, $(i = 1, 2, \cdots, k)$ where it fails to exist.

Since $Y_2(0) = 0$, the values $x = 0$ and $\bar{x} = 0$ satisfy (21) and we shall let these correspond to each other. Now consider the intervals

$$0 \leq x \leq b_1, \qquad 0 \leq \bar{x} \leq \pi/k.$$

When x increases from 0 to b_1, the left-hand member of (21) increases monotonically from 0 to ∞; and when $\bar{x}$ increases from 0 to π/k, the right-hand member likewise increases monotonically from 0 to ∞, so that (21) defines a continuous one-to-one correspondence between the points of the above intervals. Moreover in these intervals, except at the points $x = b_1$, $\bar{x} = \pi/k$, each member of (21) possesses continuous first and second derivatives and each first derivative is positive. It follows that $dx/d\bar{x}$ and $d^2x/d\bar{x}^2$ exist and are continuous, except perhaps at the upper ends of the intervals considered. But if we equate the inverse of one member of (21) to the inverse of the other, we see that here also the correspondence is continuous and the derivatives exist and are continuous.

We next consider the intervals $b_1 \leq x \leq b_2$, $\pi/k \leq \bar{x} \leq 3\pi/k$, for which similar considerations hold, each member of (21) increasing from $-\infty$ to $+\infty$ when x and $\bar{x}$ increase from the lower to the upper end of the respective intervals. Proceding in this manner we find that for the intervals $0 \leq x \leq 2\pi$, $0 \leqq \bar{x} \leqq 2\pi$ equation (21) defines x as a function of $\bar{x}$, $x = \phi(\bar{x})$, where ϕ is continuous and possesses continuous first and second derivatives and ϕ' is nowhere zero.

If in (21) we replace x by $x + 2\pi$, the left hand member is multiplied by ρ_2/ρ_1: similarly replacing $\bar{x}$ by $\bar{x} + 2\pi$ the right hand member is multiplied by $e^{4\pi a}$ or ρ_2/ρ_1 (see (17)), so that if (21) is true for a pair of values, x, x, it is also true for the pair, $x + 2\pi$, $\bar{x} + 2\pi$. It follows at once that we may obtain the correspondence between x and $\bar{x}$ for the interval $(2\pi, 4\pi)$ by taking corresponding values of x and $\bar{x}$ for the interval $(0, 2\pi)$ and increasing each by 2π: for the interval $(4\pi, 6\pi)$ from the interval $(2\pi, 4\pi)$ in the same manner etc.

Thus equation (21) defines x as a function of $\bar{x}$, $x = \phi(\bar{x})$, for all values of these variables, where ϕ' and ϕ'' exist and are continuous and ϕ' is nowhere zero, and $\phi(\bar{x} + 2\pi) = \phi(\bar{x}) + 2\pi$. Thus the transformation is a proper periodic transformation of type B.

To investigate the form into which the solutions of the original differential equation are carried by this transformation, we write

$$(22) \quad Y_1(\bar{x}) = \exp(-\alpha x) \cdot \cos(k/2)\bar{x} \cdot \exp[(2\alpha^2/k^2)\cos^2(k/2)\bar{x}] \\ \times \exp[(\bar{x}/2\pi)\log(\rho_1\rho_2)^{1/2}] \cdot F(\bar{x}),$$

where $F(\bar{x})$ is a function whose properties are deduced below. From (21) follows

$$(22') \quad Y_2(\bar{x}) = \exp(+\alpha\bar{x}) \cdot \sin(k/2)\bar{x} \cdot \exp[(2\alpha^2/k^2)\sin^2(k/2)\bar{x}] \\ \times \exp[(\bar{x}/2\pi)\log(\rho_1\rho_2)^{1/2}] \cdot F(\bar{x}).$$

From the manner of setting up the correspondence between x and $\bar{x}$, it is clear that the zeros of $Y_1(\bar{x})$ are the same as those of $\cos(k/2)\bar{x}$ and the zeros of $Y_2(\bar{x})$ are the same as those of $\sin(k/2)\bar{x}$. From (22) it appears that $F(x)$ is continuous and possesses continuous first and second derivatives everywhere except possibly at the zeros of $\cos(k/2)\bar{x}$. But (22′) shows that it is continuous and possesses continuous first and second derivatives at these points also. Furthermore F can vanish only at a common zero of Y_1 and Y_2 and is therefore nowhere zero. If in either (22) or (22′) we replace $\bar{x}$ by $\bar{x} + 2\pi$ and remember that Y_i is a principal solution (see p. 599), we verify that F is periodic.

The A transformation

$$(23) \qquad y = \lambda(\bar{x})\bar{y}, \lambda = \exp[(\bar{x}/2\pi)\log(\rho_1\rho_2)]^{1/2} \cdot F(\bar{x})$$

is proper and periodic. It carries our solutions into the canonical forms

$$\exp(-\alpha\bar{x}) \cdot \cos(k/2)\bar{x} \cdot \exp[(2\alpha^2/k^2)\cos^2(k/2)\bar{x}],$$
$$\exp(+\alpha\bar{x}) \cdot \sin(k/2)\bar{x} \cdot \exp[(2\alpha^2/k^2)\sin^2(k/2)\bar{x}].$$

The canonical form of the equation may be found from the solutions with some labor. It is

$$(24)\quad 4k^2[k^2+2\alpha k\sin k\bar{x}+2\alpha^2\sin^2 k\bar{x}]\bar{y}''-8k^3[\alpha k\cos k\bar{x}+\alpha^2\sin 2k\bar{x}]\bar{y}'$$
$$+[(k^5-8\alpha^2k^4)-(2\alpha k^5+32\alpha^3k^3)\sin k\bar{x}$$
$$-(36\alpha^4k^2+6\alpha^2k^4)\sin^2 k\bar{x}-24\alpha^5k\sin^3 k\bar{x}-8\alpha^6\sin^4 k\bar{x}]\bar{y}=0.$$

For the special case where every solution is a principal solution, $\rho_1=\rho_2$, $\alpha=0$ and the canonical forms for the solutions and the equation become

$$\sin(k/2)\bar{x},\ \cos(k/2)\bar{x} \text{ and } \bar{y}''-(k^2/4)\bar{y}=0.$$

It thus appears that if the roots of the characteristic equation are real and distinct, the only invariants (under proper periodic transformations) are essentially (a) the ratio of these roots, (b) the number of zeros of a principal solution in the interval $0\leq x<2\pi$.

5. *Roots of Characteristic Equation equal.* If the roots of the characteristic equation are equal, there will be a principal solution, Y_1, such that $Y_1(x+2\pi)=\rho Y_1(x)$, where ρ is the root. If there is an independent solution, $Y_2(x)$, such that $Y_2(x+2\pi)=\rho Y_2(x)$, every solution will have this property. This case has been implicitly covered in the preceding paragraph (namely for $\alpha=0$) and we shall assume that it does not occur.*

Consider Y_1 and any independent solution, y_2. Equations (8) become for these

$$(25)\qquad Y_1(x+2\pi)=\rho Y_1(x),\qquad y_2(x+2\pi)=C'_1Y_1(x)+C'_2y_2(x).$$

From the form of the characteristic equation, the roots being equal, $C'_2=\rho$. C'_1 cannot be zero on account of our assumption above. By taking a suitable multiple of y_2, we can make $C'_1=2\pi\rho$ (see table, p. 598). Calling this multiple Y_2 we have

$$(26)\qquad Y_2(x+2\pi)=2\pi\rho Y_1(x)+\rho Y_2(x).$$

We readily verify from (25), (26) that $Y_1\cdot e^{-\beta x}$ and $[Y_2-xY_1]e^{-\beta x}$, where $\beta=(1/2\pi)\log|\rho|$ are periodic (semi-periodic if ρ is negative) so that we may write

$$(27)\qquad Y_1=e^{\beta x}P_1(x),\qquad Y_2=xY_1+e^{\beta x}P_2(x),$$

---

* This case and that mentioned below appear as distinct if we use elementary divisors in classifying the determinant, which, equated to zero, gives the characteristic equation. See Forsyth, *loc. cit.*

where P_1 and P_2 are periodic (semi-periodic) and possess continuous first and second derivatives, since Y_1, Y_2 do.

Although y_2 was any solution and Y_1 is not definitely determined because of a possible arbitrary constant factor, the wronskian of $Y_1, Y_2, W(Y_1, Y_2)$, has a definite sign. For choosing AY_1 in place of Y_1, Y_2 becomes AY_2 by (26) and $W(AY_1, AY_2) = A^2W(Y_1, Y_2)$ and has the same sign. Again any solution, Y^*, may be written in the form $C_1Y_1 + C_2Y_2$, where, if

$Y^* = xY_1 + e^{\beta x}P$, C_2 must be 1. But $W(Y_1, C_1Y_1 + Y_2) = W(Y_1, Y_2)$.

As in the preceding paragraph we consider two cases in reducing the differential equation to a canonical form: one in which Y_1 does not vanish for any value of x, the other in which Y_1 has at least one zero.

CASE (i). $P_1(x) \neq 0$. If P_1 does not vanish for any value of x, the transformations

$$y = e^{\beta x}P_1(x)\bar{y}, \tag{28}$$

$$Y_2/Y_1 = x + (P_2/P_1) = \bar{x}, \tag{29}$$

may be easily shown to be proper and periodic. They carry the solutions into the canonical forms 1 and x and the differential equation into the canonical form $d^2\bar{y}/d\bar{x}^2 = 0$. It thus appears that for this case there are no invariants.

CASE (ii). $P_1(x)$ vanishes. Now let Y_1 vanish k times in the interval $0 \leq x < 2\pi$, where, as before, if ρ is positive, k will be even and P_1 periodic: if ρ is negative, k will be odd and P_1 semi-periodic. If $Y_1(0)$ is not 0, we can choose Y_2 so that $Y_2(0) = 0$: if $Y_1(0) = 0$, a linear transformation, $x = \bar{x} + a$, will reduce this to the above.[*]

We employ the transformation defined by the equation

$$Y_2/Y_1 = x + (P_2/P_1) = \bar{x} + \mu \tan (k/2)\bar{x} \tag{31}$$

where $\mu = 1$, $W(Y_1, Y_2) > 0$; $\mu < 0$ and $-\mu > 2/k$, $W(Y_1, Y_2) < 0$.[†] Essentially the same reasoning as used for the transformation defined by (21), with a slight change if $W < 0$, shows that (31) defines a proper periodic B transformation and employing the same methods as on pages 603 and 604, we see that (31) together with a proper periodic A transformation will reduce our solutions to the canonical forms

---

[*] The transformation $Y_2/Y_1 = \bar{x} - \mu \cot (k/2)\bar{x}$ may be used in this case to reduce the equation to a simple canonical form. Its effect is to interchange $\sin (k/2)\bar{x}$ and $\cos (k/2)\bar{x}$ in the canonical forms of the solutions, and change the sign of μ.

[†] If $W < 0$, the derivative of the left hand member of (31) is negative and if x and $\bar{x}$ are to increase together, the derivative of the right hand member must be negative also. The single value, $\mu = -3$, will clearly satisfy both inequalities since $k \geq 1$.

$$(32) \qquad \cos(k/2)\bar{x} \quad \text{and} \quad \bar{x}\cos(k/2)\bar{x} + \mu\sin(k/2)\bar{x}.$$

The canonical form of the equation is

$$(33) \quad 4\bar{y}''[\cos k\bar{x} + 1 + \mu k] + 4k\bar{y}'\sin k\bar{x} + \bar{y}[3k^2 + \mu k^3 - k^2\cos k\bar{x}] = 0.$$

If the roots of the characteristic equation are equal, the only invariants are essentially (a) k, the number of zeros of the principal solution in the interval $0 \leq x < 2\pi$, (b) the sign of the wronskian, $W(Y_1, Y_2)$, where Y_1 is the principal solution and Y_2 is written in the form $Y_2 = xY_1 + e^{\beta x}P_1(x)$.

6. *Roots of Characteristic Equation Complex.* Let the absolute value of each of the roots be ρ and their angles be ϕ, $2\pi - \phi$, where we may take ϕ as lying between 0 and 2π, leaving undecided for the moment whether we shall call ϕ the larger or the smaller angle. If ρ is not 1, the proper periodic transformation

$$y = \exp[(x/2\pi)\log\rho]\cdot\bar{y}$$

will reduce the equation to one for which $\rho = 1$. We assume this carried out so that the roots are of the form $\cos\phi \pm i\sin\phi$.

For any pair of independent solutions, y_2, y_1, as we saw in § 3, the characteristic equation has the form

$$(14) \qquad (C_1 - \rho)(C'_2 - \rho) - C'_1C_2 = 0.$$

Noting that C'_1C_2 is not 0 (since the C's are real and the roots of (14) imaginary) we see, referring to the table of § 3, that we can find two independent solutions, Y_1, Y_2, readily determined from y_1, y_2 and the constants, $C_1, \cdots, C'_2$, which have the property that the (new) constants, $C_1, \cdots, C'_2$, associated with Y_1, Y_2 by equations (8) are such that $C_1 = C'_2$, $C'_1 = \pm C_2$, and C'_1 has the same sign as $\sin\phi$. Since the roots of (14) are $\cos\phi \pm i\sin\phi$, we must then have

$$C_1 = C'_2 = \cos\phi, \quad C'_1 = -C_2 = \sin\phi.$$

We have thus found a pair of solutions, Y_1, Y_2, such that (substituting the values for the C's in (8))

$$(34) \qquad \begin{cases} Y_1(x + 2\pi) = \cos\phi\, Y_1(x) - \sin\phi\, Y_2(x), \\ Y_2(x + 2\pi) = \sin\phi\, Y_1(x) + \cos\phi\, Y_2(x). \end{cases}$$

I shall call this pair, *written in this order*, a pair of principal solutions.

Equations (34) may be written in the form

$$Y_2(x+2\pi) = \cos(2\pi - \phi)\, Y_2(x) - \sin(2\pi - \phi)\, Y_2(x),$$
$$Y_1(x+2\pi) = \sin(2\pi - \phi)\, Y_2(x) + \cos(2\pi - \phi)\, Y_1(x),$$

which are of the same form as (34) except that $2\pi - \phi$ appears in place of ϕ and Y_1 and Y_2 are interchanged. It follows that if Y_1, Y_2 are a pair of principal solutions for one of the two possible determinations of the angle, ϕ, Y_2, Y_1 will be such a pair for the other. Since $W(Y_1, Y_2) = -W(Y_2, Y_1)$, we can choose for ϕ that one of the two possible values, ϕ, $2\pi - \phi$, for which the wronskian of a pair of principal solutions is positive. We shall assume this done at the outset, so that $W(Y_1, Y_2) > 0$. That this choice does not depend upon the particular pair of principal solutions chosen will appear below.

Consider the functions, $P_1(x)$, $P_2(x)$, defined by the equations

$$(35) \qquad \begin{cases} P_1(x) = \cos[(x/2\pi)\phi]\, Y_1(x) + \sin[(x/2\pi)\phi]\, Y_2(x), \\ P_2(x) = -\sin[x/2\pi)\phi]\, Y_1(x) + \cos[(x/2\pi)\phi]\, Y_2(x). \end{cases}$$

They are clearly continuous and possess continuous first and second derivatives: if we substitute $x + 2\pi$ for x and reduce, using equations (34), we see that they are periodic. We may solve equations (35) and express Y_1, Y_2 in terms of P_1, P_2, thus

$$(36) \qquad \begin{cases} Y_1(x) = \cos[(x/2\pi)\phi]\, P_1(x) - \sin[(x/2\pi)\phi]\, P_2(x), \\ Y_2(x) = \sin[(x/2\pi)\phi]\, P_1(x) + \cos[(x/2\pi)\phi]\, P_2(x). \end{cases}$$

Since Y_1, Y_2 are independent solutions, they cannot vanish together and hence neither can P_1, P_2. Thus any pair of principal solutions may be written in the form (36) and conversely we may readily verify that any pair of solutions written in this form will satisfy equations (34) and be a pair of principal solutions.

Equations (36) show that we may obtain Y_1, Y_2 from P_1, P_2 by rotating the P_1, P_2 plane about the origin through the angle $(x/2\pi)\phi$. It is clear that if we rotate this whole figure through any constant angle, we shall get a new set of functions, Y_3, Y_4, P_3, P_4 in the same relation to each other as Y_1, Y_2, P_1, P_2 and should expect Y_3, Y_4 to constitute a pair of principal solutions. In fact we readily verify that Y_3, Y_4 will be a pair of principal solutions *if and only if* they are expressible in the form

$$(37) \qquad Y_3 = A_1 Y_1 - A_2 Y_2, \quad Y_4 = A_2 Y_1 + A_1 Y_2,$$
$$A_1, A_2 \text{ constants not both } 0,$$

corresponding to a rotation of Y_1, Y_2 through the angle $\arctan A_2/A_1$ (and multiplication by a constant factor). From (37) follows

$$W(Y_3, Y_4) = (A_1^2 + A_2^2)\, W(Y_1, Y_2),$$

so that the sign of the wronskian of any pair of principal solutions will be the same as the sign of the wronskian of Y_1, Y_2, thus showing that the determination of ϕ adopted above did not depend upon the particular pair of principal solutions chosen.

From (36) we have

$$\arctan\,(Y_2/Y_1) = (\phi/2\pi)x + \arctan\,(P_2/P_1).\text{*} \tag{38}$$

Since P_1 and P_2 are continuous and do not vanish together, $\arctan\,(P_2/P_1)$ is continuous. When x increases continuously from 0 to 2π, $\arctan\,(P_2/P_1)$ will, on account of the periodicity of P_1 and P_2, change by an integral multiple of 2π, say $2k\pi$. Hence $\arctan\,(Y_2/Y_1)$, which is also everywhere continuous, changes by $\phi + 2k\pi$. Since the derivative of $\arctan\,(Y_2/Y_1) = W(Y_1, Y_2)/(Y_1^2 + Y_2^2)$ is positive, $\arctan\,(Y_2/Y_1)$ must actually increase and k is 0 or positive.

By a suitable choice of A_1, A_2 in (37), we can find a pair of principal solutions, uniquely determined except for a constant factor, such that the second vanishes for $x = 0$. We shall assume this to be the pair Y_1, Y_2.

We employ the transformation defined by the equations

$$\arctan\,(Y_2/Y_1) = (\phi/2\pi)x + \arctan\,(P_2/P_1) = (\phi/2\pi)\bar{x} + k\bar{x}, \tag{39}$$

or the equivalent

$$Y_2/Y_1 = \tan\,[(\phi/2\pi) + k]\bar{x}. \tag{39'}$$

Using reasoning similar to that previously employed (§ 4), we see that this defines a proper periodic B transformation.

Rewriting (39′) in the equivalent forms

$$\begin{aligned} Y_2/(Y_1^2 + Y_2^2)^{1/2} &= \sin\,[(\phi/2\pi) + k]\bar{x}, \\ Y_1/(Y_1^2 + Y_2^2)^{1/2} &= \cos\,[\phi/2\pi) + k]\bar{x}, \end{aligned} \tag{40}$$

we note that as $\bar{x}$ increases from 0 to 2π, $[(\phi/2\pi) + k]\bar{x}$ increases from 0 to $2k\pi + \phi$ and will consequently take on at least $2k + 1$ values for which Y_2 vanishes, namely $0, \pi, 2\pi, \cdots, 2k\pi$. If $0 < \phi < \pi$, there will be no other value for which Y_2 vanishes, but if $\pi < \phi < 2\pi$, there will be one other such value, namely $(2k + 1)\pi$. Since the intervals $0 \leq x \leq 2\pi$ and $0 \leq \bar{x} \leq 2\pi$ correspond in a one-to-one manner and continuously by virtue of (39), Y_2 has the same number of zeros in the interval $(0, 2\pi)$ whether we regard x or $\bar{x}$ as the independent variable. Furthermore any solution which vanishes at 0 is a constant multiple of Y_2. It follows that if a solution (not identically zero) which vanishes for $x = 0$ has n zeros (including 0) in the interval $(0, 2\pi)$, then if n is odd, $2k + 1 = n$ and ϕ lies between 0 and π, while if n is even, $2k + 2 = n$ and ϕ lies between π and 2π. Thus this number of zeros determines the value of k and also the half plane in which ϕ lies.

---

* Assuming all inverse tangents suitably defined at points where the denominator vanishes.

From (36) we have $Y_1{}^2 + Y_2{}^2 = P_1{}^2 + P_2{}^2$. Hence the A transformation

$$y = (Y_1{}^2 + Y_2{}^2)^{1/2}\,\bar{y} \tag{41}$$

is proper and periodic. Together with (39) it carries the solutions, Y_1, Y_2, into the canonical forms [note (40)]

$$\cos\,[(\phi/2\pi) + k]\bar{x} \quad \text{and} \sin\,[(\phi/2\pi) + k]\bar{x}. \tag{42}$$

The corresponding canonical form for the equation is

$$\bar{y}'' + [(\phi/2\pi) + k]^2\bar{y} = 0. \tag{43}$$

Thus if the roots of the characteristic equation are complex, the only invariants are (a) the angle (but not the absolute value) of the roots, (b) the number of zeros in the interval $(0, 2\pi)$ of a solution which vanishes at 0.

7. *Impossibility of further Reduction.* The proper periodic A transformation, $y = e^{cx}P(x)\cdot\bar{y}$, c a real constant, P nowhere zero, divides the roots of the characteristic equation by $e^{2\pi c}$, but does not change the character of the roots nor their ratio nor their angle, if they are complex. Clearly also it does not change the zeros of a solution and it carries a principal solution into a principal solution.

Again the proper periodic B transformation, $x = \phi(\bar{x})$, ϕ' nowhere zero, since $x + 2\pi = \phi(\bar{x} + 2\pi)$, leaves unchanged the characteristic equation and the number of zeros of any solution in an interval of length 2π. It also transforms a principal solution into a principal solution.

Examining the canonical forms and invariants as listed in the Summary, § 8, it appears at once from the above that the classes I (i), I (ii), II a, II b (i), II b(ii), III are actually distinct, in that no equation of one class may be transformed into one of another class by proper periodic A and B transformation and that the numbers α, k and ϕ are true invariants.

In II b (ii) we assumed Y_1 a principal solution and wrote Y_2 in the form $Y_2 = xY_1 + e^{\beta x}P_2(x)$. Noting that for the proper periodic B transformation $\phi(\bar{x}) = x + P(\bar{x})$, P periodic (see end of § 1), we see that such a transformation will not change the form of Y_2. Clearly a proper periodic A transformation will not do so either. And neither of these will change the sign of $W(Y_1, Y_2)$, provided that for the B transformation $\phi' > 0$. Hence here also there is no possibility of further reduction.

To sum up: Under proper periodic transformations of types A and B where for type B $\phi' < 0$ there is no possibility of reducing the canonical forms found [*] to a smaller number and the quantities listed as invariants are actually such. Two equations of different classes or of the same class with different invariants cannot be transformed the one into the other: two equations of the same class with the same invariants may be transformed the one into the other.

8. *Further Restriction of the Transformations employed.* In the two cases treated below we used a transformation $x = \bar{x} + a$ to bring a zero of a principal solution to the origin or from the origin to some other point. If we restrict ourselves to those transformations which carry the interval $(0, 2\pi)$ into the same interval, it is still possible to reduce the equation to simple canonical forms, but we shall be obliged to distinguish further cases.

If the roots of the characteristic equation are equal and the principal solution vanishes at 0, we may use the transformation and the canonical forms suggested in the footnote on p. 605. If we employ only transformations carrying $(0, \pi)$ into $(0, 2\pi)$, it is clear that $Y_1(0) = 0$ and $Y_1(0) \neq 0$ are distinct cases, since our proper periodic transformations carry a principal solution into a principal solution.

If the roots of the characteristic equation are real and unequal and a principal solution has at least one zero, we must now distinguish four cases. Remembering that we assumed $\rho_2 \geqq \rho_1$ and consequently $\alpha \geqq 0$, thus definitely fixing ρ_2 and its associated principal solution, Y_2, we have:

(A) If $Y_2(0) = 0$, our work above holds and we have the canonical forms given.

(B) If $Y_1(0) = 0$, we may interchange Y_1 and Y_2 in (A). This is the same as changing the sign of α in our forms.

(C) Let a and b respectively be the smallest zeros of Y_2 and Y_1 in the interval $(0, 2\pi)$. Then if $0 < a < b$ we may in equation (21) on the right change $\bar{x}$ to $\bar{x} - \pi/2k$ and let the values $x = a$ and $\bar{x} = \pi/2k$ correspond. The transformation thus defined carries the interval $(a, a + 2\pi)$ into $(\pi/2k, 2\pi + \pi/2k)$ and by suitably choosing Y_2 on the left,—we may of course replace Y_2 by any constant multiple of itself,—the values $x = 0$ and $\bar{x} = 0$ will correspond. We will thus obtain the canonical forms of (A) with $\bar{x}$ replaced by $\bar{x} - \pi/2k$.

(D) Finally if $0 < b < a$, we may interchange Y_1 and Y_2 in (C) thus changing the sign of α.

---

[*] We may of course choose other forms.

9. *Summary.* We conclude this paper by giving a summary of the different classes of equations considered together with the invariants[*] and the canonical forms for the solutions and equations of each class. Any linear differential equation of the second order whose coefficients are continuous in $(0, 2\pi)$ and periodic with period 2π may be reduced to one of these forms by transformations of the types

$$(A) \quad y = \lambda(x)\bar{y}, \qquad (B) \quad x = \phi(\bar{x}),$$

satisfying the conditions for being "proper" and periodic (§ 1) and carrying $(0, 2\pi)$ into $(0, 2\pi)$. A necessary and sufficient condition that two equations be reducible the one to the other by transformations of the types given above is that they belong to the same class and possess the same invariants.

I. Roots of characteristic equation real and distinct, ρ_1, ρ_2. Assume $\rho_2 > \rho_1$. Let $2\alpha = (1/2\pi) \log (\rho_2/\rho_1)$.

(i) Neither principal solution vanishes $0 \leq x < 2\pi$.

Invariant: ρ_2/ρ_1.

Canonical form of solutions: $e^{+\alpha x}$, $e^{-\alpha x}$,

Canonical form of equation: $y'' - \alpha^2 y = 0$.

(ii) A principal solution vanishes k times in the interval $0 \leq x < 2\pi$.

Invariants: ρ_2/ρ_1. The integer k.

The vanishing or not vanishing of a principal solution at 0, classified as follows:

(A) $Y_2(0) = 0$:

Canonical form of solutions:

$$\exp(+\alpha x) \sin (k/2)\bar{x} \cdot \exp [(2\alpha^2/k^2) \sin^2 (k/2)\bar{x}]$$
$$\exp(-\alpha\bar{x}) \cos (k/2)x \cdot \exp [(2\alpha^2/k^2) \cos^2 (k/2)\bar{x}].$$

Canonical form of equation: Equation (24).

(B) $Y_1(0) = 0$.

Canonical forms same as for (A) with $-\alpha$ in place of α.

(C) Let a and b be the smallest zeros of Y_2 and Y_2 respectively in the interval $(0, 2\pi)$ and $0 < a < b$.

Canonical forms same as for (A) with x replaced by $x - \pi/2k$.

(D) $0 < b < a$.

Canonical forms same as for (C) with $-\alpha$ in place of α.

II. Roots of characteristic equation real and equal.

a. Every solution a principal solution. Any solution vanishes exactly k times, $k \neq 0$, in the interval $0 \leq x < 2\pi$.

---

[*] The quantities named are essentially the only invariants. Of course any combinations or functions of them would also be invariants.

Invariant: k.

Canonical form of solutions: $\sin(k/2)x, \cos(k/2)x$.

Canonical form of equation: $y'' + (k^2/4)y = 0$.

b. There exists one and only one principal solution, Y_1.

(i) Y_1 does not vanish, $0 \leq x < 2\pi$.

Invariants: None.

Canonical form of solutions: $1, x$.

Canonical form of equation: $y'' = 0$.

(ii) Y_1 vanishes k times in the interval $0 \leq x < 2\pi$.

Invariants: k. Sign of $W(Y_1, Y_2)$, where Y_2 is any other solution written in the form $Y_2 = xY_1 + e^{\beta x}P(x)$, see § 5.

Vanishing or not vanishing of Y_1 at 0.

(A) $Y_1(0) \neq 0$.

Canonical form of solutions:

$$\cos(k/2)x, x\cos(k/2)x + \mu\sin(k/2)x,$$

where $\mu = 1$, if $W(Y_1, Y_2) > 0$;

$\mu < 0$ and $-\mu > 2/k$, (e. g. $\mu = -3$) if $W(Y_1, Y_2) < 0$.

Canonical form of equation:

$$4y''[1 + \mu k + \cos kx] + 4ky' \sin kx + [3k^2 + \mu k^3 - k^2 \cos kx]y = 0.$$

(B) $Y_1(0) = 0$. Canonical form of solutions:

$$\sin(k/2)x, x\sin(k/2)x - \mu\cos(k/2)x.$$

Canonical form of equation:

$$4y''[1 + \mu k - \cos kx] - 4ky' \sin kx + y[3k^2 + \mu k^3 + k^2 \cos kx] = 0.$$

III. Roots of characteristic equation complex, $\rho(\cos\phi \pm i\sin\phi)$.

The solution vanishing at 0 vanishes n times in the interval $0 \leq x < 2\pi$.

If n is even, let $2k + 2 = n$ and take ϕ so that $\pi < \phi < 2\pi$.

If n is odd, let $2k + 1 = n$ and take ϕ so that $0 < \phi < \pi$.

Invariants: ϕ, determined as above. n (or k determined from it).

Canonical form of solutions:

$$\cos[(\phi/2\pi) + k]x, \quad \sin[(\phi/2\pi) + k]x.$$

Canonical form of equation:

$$y'' + [(\phi/2\pi) + k]^2 y = 0.$$

UNIVERSITY OF VERMONT,
BURLINGTON, VERMONT.

Periodic Orbits of Three Finite Masses about the Equilateral Triangle Points.

BY H. E. BUCHANAN.

Introduction. Since the announcement of the straight line and equilateral triangle solutions of the three body problem by Lagrange, many periodic orbits near these positions have been found. Most of these have been limited to the case in which one of the bodies is infinitesimal. The equilateral triangle configuration has received much less attention than the straight line cases probably because of the greater difficulty of the Algebra encountered. A special case of the problem of this paper was discussed by Professor Thomas Buck in his dissertation,* one of the bodies being infinitesimal. Professor Daniel Buchanan † has discussed the asymptotic orbits for an infinitesimal body near the equilateral triangle positions. The paper of Professor Buck has been valuable to the author as a check, and his results appear here as a special case.

The Equation of Variation. Let the units of mass and distance be chosen so that the gravitational constant and the distances between the masses are equal to unity. When the axes are rotating uniformly in the ξ, η plane, the positions in the triangle solutions are given by equations 57, page 310, Moulton's *Celestial Mechanics.* The square of the angular velocity is $\omega^2 = m_1 + m_2 + m_3$. We make the transformation—

$$(1) \qquad \xi_i = \xi_i^{(0)} + x_i', \quad \eta_i = \eta_i^{(0)} + y_i', \quad \zeta_i = z_i', \quad (i = 1, 2, 3).$$

Then the differential equations which define the variations become

$$\frac{d^2x_i'}{dt^2} - 2\omega \frac{dy_i'}{dt} = \omega^2(\xi_i^{(0)} + x_i') - \sum_1^3 m_j \frac{(x_i' + \xi_i^{(0)} - x_j' - \xi_j^{(0)})}{r^3_{ij}},$$

$$\frac{d^2y_i'}{dt^2} + 2\omega \frac{dx_i'}{dt} = \omega^2(\eta_i^{(0)} + y_i') - \sum_1^3 m_j \frac{(y_i' + \eta_i^{(0)} - y_j' - \eta_j^{(0)})}{r^3_{ij}},$$

$$\frac{d^2z_i'}{dt^2} = -\sum_1^3 m_j \frac{(z_i' - z_j')}{r^3_{ij}}, \qquad (i = 1, 2, 3, j \neq i).$$

---

[*] *Periodic Orbits,* Publication No. 161, Carnegie Institution.

[†] *Transactions of the Cambridge Philosophical Society,* Vol. 22 (1919).

Expansion of the Right Members. From this point on we omit the superscripts on $\xi_i^{(0)}$ and $\eta_i^{(0)}$ and wherever these symbols occur it will be understood that they refer to the equilateral triangle positions.

We expand the right members as power series in x_i', y_i' and z_i'. The x-axis is taken parallel to the line joining m_1 and m_2. Then $\xi_i - \xi_j$ can never vanish. Hence the regions of convergence will be determined by *

$$\frac{(y_i' + \eta_i - y_j' - \eta_j)^2 + (z_i' - z_j')^2}{(x_i' + \xi_i - x_j' - \xi_j)^2} < 1,$$

$$\left| \frac{x_i' - x_j'}{\xi_i - \xi_j} \right| < 1, \qquad (i = 1, 2, 3,\ j \neq i.)$$

Inside this region we may expand the right members as power series in x_i', y_i' and z_i' in the form

$$\begin{gathered} X_{1i} + X_{2i} + \cdots, \\ Y_{1i} + Y_{2i} + \cdots, \\ Z_{1i} + Z_{2i} + \cdots, \end{gathered}$$

where each $X_{ji}\, Y_{ji}\, Z_{ji}$ is a homogeneous function of degree j in $x_i'\, y_i'\, z_i'$. Further X_{ji} and Y_{ji} are functions of $z_i'^2$ and Z_{ji} is z_i' times a function of $z_i'^2$.

The Parameters ϵ and δ. Applying the transformation

$$x_i' = \epsilon' x_i, \quad y_i' = \epsilon' y_i, \quad z_i' = \epsilon' z_i, \quad t = (1 + \delta)\tau, \text{ we find}$$

$$(2) \qquad \begin{aligned} (d^2x_i/d\tau^2) - 2\omega(1+\delta)(dy_i/d\tau) &= (1+\delta)^2\{X_{1i} + \epsilon' X_{2i} + \cdots\} \\ (d^2y_i/d\tau^2) + 2\omega(1+\delta)(dx_i/d\tau) &= (1+\delta)^2\{Y_{1i} + \epsilon' Y_{2i} + \cdots\} \\ (d^2z_i/d\tau^2) &= (1+\delta)^2\{Z_{1i} + \epsilon' Z_{2i} + \cdots\}, \end{aligned}$$

$$(i = 1, 2, 3).$$

These equations are valid for the physical problem as long as the convergence of the right members is assured. We generalize the problem by replacing ϵ' by ϵ where ϵ may take all values in the vicinity of zero. We shall have a solution of the physical problem only when $\epsilon = \epsilon'$. The Method will be to find all periodic solutions of these when $\epsilon = \delta = 0$ and discuss the analytic continuation of these when ϵ increases to ϵ'.

---

* Cf. H. E. Buchanan, "Periodic Orbits," *American Journal of Mathematics*, Vol. 45, No. 2 (1923), where a similar problem is treated.

The Center of Gravity Integrals. Since the origin is at the center of mass, we have

$$m_1x_1 + m_2x_2 + m_3x_3 = 0,$$
$$m_1y_1 + m_2y_2 + m_3y_3 = 0,$$
$$m_1z_1 + m_2z_2 + m_3z_3 = 0.$$

By means of these we eliminate the x_2, y_2, z_2 equations so that hereafter i has only the values 1 and 3.

It is evident that

$$\partial x_2/\partial x_1 = \partial y_2/\partial y_1 = \partial z_2/\partial z_1 = - m_1/m_2,$$
$$\partial x_2/\partial x_3 = \partial y_2/\partial y_3 = \partial z_2/\partial z_3 = - m_3/m_2.$$

The Linear Terms of the Differential Equations. We choose our axes so that the x-axis is parallel to the line joining m_1 and m_2 and m_1 is in the negative direction from m_2. It follows that

$$\xi_2 - \xi_1 = 1,\ \ \xi_3 - \xi_1 = \tfrac{1}{2},\ \xi_3 - \xi_2 = -\tfrac{1}{2},$$
$$\eta_2 - \eta_1 = 0,\ \eta_3 - \eta_1 = \eta_3 - \eta_2 = 3^{1/2}/2.$$

In equation (2) put $\epsilon = \delta = 0$. There results

$$(3)\quad \begin{cases} (d^2x_i/d\tau^2) - 2\omega(dy_i/d\tau) = A_{1i}x_1 + A_{3i}x_3 + B_{1i}y_1 + B_{3i}y_3, \\ (d^2y_i/d\tau^2) + 2\omega(dx_i/d\tau) = C_{1i}x_1 + C_{3i}x_3 + D_{1i}y_1 + D_{3i}y_3, \end{cases}$$

$$(4)\qquad d^2z_i/d\tau^2 = E_{1i}z_1 + E_{3i}z_3, \qquad (i = 1, 3),$$

where

$$A_{11} = 3\omega^2 - (9m_3/4),\ A_{31} = 9m_3/4,\ A_{13} = D_{13} = 0,$$
$$A_{33} = 3\omega^2/4,\ B_{11} = C_{11} = (3\cdot 3^{1/2}/4)m_3,\ B_{31} = C_{31} = -(3\cdot 3^{1/2}/4)m_3,$$
$$B_{13} = C_{13} = -(3\cdot 3^{1/2}/2)m_1,\ B_{33} = C_{33} = (3\cdot 3^{1/2}/2)m_1 - (3\cdot 3^{1/2}/4)\omega^2,$$
$$D_{11} = 9m_3/4,\ D_{31} = -(9m_3/4),\ D_{33} = 9\omega^2/4),\ E_{11} = E_{33} = -\omega^2,$$
$$E_{31} = E_{13} = 0.$$

Equations (3) and (4) are mutually independent. The solutions of (4) are

$$z_1 = M_{11}\cos\omega\tau + M_{12}\sin\omega\tau,$$
$$z_3 = M_{31}\cos\omega\tau + M_{33}\sin\omega\tau.$$

We consider the solutions of the xy-equations in subsequent paragraphs.

The Characteristic Equation. Let us substitute in (3)

$$x_i = K_i e^{\lambda\tau}, \qquad y_i = L_i e^{\lambda\tau}$$

there results a set of equations linear and homogeneous in K_i and L_i. In order that there be any solution different from $K_i = L_i = 0$ we must have

$$\begin{array}{llll}
-3\omega^2+\dfrac{9m_3}{4}, & -\dfrac{9m_3}{4}, & -2\lambda\omega-\dfrac{3\sqrt{3}\,m_3}{4}, & \dfrac{3\sqrt{3}\,m_3}{4} \\
0, & \lambda^2-\dfrac{3\omega^2}{4}, & \dfrac{3\sqrt{3}\,m_1}{2}, & -2\lambda\omega-\dfrac{3\sqrt{3}\,m_1}{2}+\dfrac{3\sqrt{3}\,\omega^2}{4} \\
\omega-\dfrac{3\sqrt{3}\,m_3}{4}, & \dfrac{3\sqrt{3}\,m_3}{4}, & \lambda^2-\dfrac{9m_3}{4}, & \dfrac{9m_3}{4} \\
\dfrac{3\sqrt{3}m_1}{2}, & 2\lambda\omega-\dfrac{3\sqrt{3}\,m_1}{2}+\dfrac{3\sqrt{3}\,\omega^2}{4}, & 0, & \lambda^2-\dfrac{9\omega^2}{4}
\end{array} = 0.$$

The Roots $+\iota\omega$ *and* $-\iota\omega$.—Lagrange[*] has proved that there are elliptic orbits in which the bodies all remain at the vertices of an equilateral triangle. These ought to appear here as oscillations near the circular triangle solutions. Therefore we expect equation (5) to have $\lambda = \pm\iota\omega$ as roots. Substituting $\lambda = +\iota\omega$ we find

$$\left|\begin{array}{llll}
-4\omega^2+9m_3/4, & -9m_3/4, & -2\iota\omega^2-(3\cdot 3^{1/2}/4)m_3, & (3\cdot 3^{1/2}/4)m_3 \\
0, & -7\omega^2/4, & (3\cdot 3^{1/2}/2)m_1, & -2\iota\omega^2-(3\cdot 3^{1/2}/2)m_1+(3\cdot 3^{1/2}/4)\omega^2 \\
2\iota\omega^2-(3\cdot 3^{1/2}/4)m_3, & (3\cdot 3^{1/2}/4)m_3, & -\omega^2-9m_3/4, & 9m_3/4 \\
(3\cdot 3^{1/2}/2)m_1, & 2\iota\omega^2-(3\cdot 3^{1/2}/2)m_1+(3\cdot 3^{1/2}/4)\omega^2, & 0, & (-13\omega^2/4).
\end{array}\right|$$

Adding the second column plus 2ι times the sum of the third and fourth to the first column and then adding -2ι times the second row to the first row and expanding we find that this determinant vanishes. Hence $\lambda = \iota\omega$ is a root. A similar computation shows that $\lambda = -\iota\omega$ is also a root.

The Roots 0, 0. In a paper on Periodic Orbits near the Straight line solutions the author[†] found $\lambda = 0$ to be a double root of the characteristic equation. The physical situation being similar here it is natural to expect that $\lambda = 0$ will be a double root. Direct substitution of 0 for λ gives, after considerable computation,

---

[*] Tisserand, *Méchanique Céleste,* Vol. 1, Chap. 8; Moulton, *Celestial Mechanics,* p. 217.

[†] *American Journal of Mathematics,* April, 1923.

$$(27\omega^4/16)\begin{vmatrix} -4, & -9m_3/4, & 0, & (3\cdot 3^{1/2}/4)m_3 \\ 0, & -3m_1/2, & 0, & (-3\cdot 3^{1/2}/2)m_1 \\ 0, & (3\cdot 3^{1/2}/4)m_3, & 0, & 9m_3/4 \\ 1-3^{1/2}, & (-3/2)m_1+3\omega^2/4, & -1, & (-3\cdot 3^{1/2}\omega^2/4). \end{vmatrix}$$

This is obviously equal to

$$(-27/4)\omega^4 \begin{vmatrix} -3m_1/2, & (-3\cdot 3^{1/2}/2)m_1 \\ (3\cdot 3^{1/2}/4)m_3, & 9m_3/4, \end{vmatrix} = 0.$$

Hence $\lambda = 0$ is a root.

To show that $\lambda = 0$ is a double root we differentiate (5) with respect to λ and then set $\lambda = 0$. Discarding the common factor 2ω we have the sum of four third order determinants. One of these vanishes identically and the sum of the others reduces to

$$(81\cdot 3^{1/2}/8)m_1m_3\omega^2 + (81\cdot 3^{1/2}/16)m_3\omega^2(m_3-\omega^2) \\ -(81\cdot 3^{1/2}/16)m_3\omega^2(-\omega^2+m_3+2m_1)$$

which vanishes. Hence $\lambda = 0$ is a double root.

The Conditions on the Masses. From the above discussion it follows that $\lambda^4 + \omega^2\lambda^2$ must be a factor of equation (5). Expanding and dividing by $\lambda^4 + \omega^2\lambda^2$ there is found

$$\lambda^4 + \omega^2\lambda^2 + (27/4)(m_1m_2 + m_1m_3 + m_2m_3) = 0$$

whence

$$\lambda^2 = \{-\omega^2 \pm [\omega^4 - 27(m_1m_2 + m_1m_3 + m_2m_3)]^{1/2}\}/2. \tag{6}$$

There are three cases which arise. They are expressed by the following conditions on the masses:

I. $\omega^4 - 27(m_1m_2 + m_1m_3 + m_2m_3) > 0,$

II. $\omega^4 - 27(m_1m_2 + m_1m_3 + m_2m_3) = 0,$

III. $\omega^4 - 27(m_1m_2 + m_1m_3 + m_2m_3) < 0.$

In case I the characteristic exponents are

$$\iota\omega,\ -\iota\omega,\ 0,\ 0,\ \iota\nu,\ -\iota\nu,\ \iota\rho,\ -\iota\rho$$

where $\nu = \{\omega^2 - [\omega^4 - 27(m_1m_2 + m_1m_3 + m_2m_3)]^{1/2}\}^{1/2}/2^{1/2}$,

and $\rho = \{\omega^2 + [\omega^4 - 27(m_1m_2 + m_1m_3 + m_2m_3)]^{1/2}\}^{1/2}/2^{1/2}$.

It follows that

$$\rho > \nu.$$

In case II the characteristic exponents are

$$\iota\omega,\ -\iota\omega,\ 0,\ 0,\ \iota\omega/2^{1/2},\ \iota\omega/2^{1/2},\ -\iota\omega/2^{1/2},\ -\iota\omega/2^{1/2}.$$

In case III the exponents are

$$\iota\omega,\ -\iota\omega,\ 0,\ 0,\ a - \iota\beta,\ -a + \iota\beta,\ a + \iota\beta,\ -a - \iota\beta.$$

The Solutions of the xy Equations, Case I. With the characteristic exponents given above, the solutions of equations (3) are

$$(7)\quad \left\{\begin{aligned}
x_1 &= K_{11}e^{\iota\omega\tau} + K_{12}e^{-\iota\omega\tau} + K_{13} + K_{14}\tau + K_{15}e^{\iota\nu\tau} + K_{16}e^{-\iota\nu\tau} + K_{17}e^{\iota\rho\tau} + K_{18}{}^{-\iota\rho}\\
x_3 &= K_{31}e^{\iota\omega\tau} + K_{32}e^{-\iota\omega\tau} + K_{33} + K_{34}\tau + K_{35}e^{\iota\nu\tau} + K_{36}e^{-\iota\nu\tau} + K_{37}e^{\iota\rho\tau} + K_{38}e^{-\iota}\\
y_1 &= L_{11}e^{\iota\omega\tau} + L_{12}e^{-\iota\omega\tau} + L_{13} + L_{14}\tau + L_{15}e^{\iota\nu\tau} + L_{16}e^{-\iota\nu\tau} + L_{17}e^{\iota\rho\tau} + L_{18}e^{-\iota\rho\tau}\\
y_3 &= L_{31}e^{\iota\omega\tau} + L_{32}e^{-\iota\omega\tau} + L_{33} + L_{34}\tau + L_{35}e^{\iota\nu\tau} + L_{36}{}^{-\iota\nu\tau} + L_{37}e^{\iota\rho\tau} + L_{38}e^{-\iota\rho\tau}.
\end{aligned}\right.$$

There must be only eight arbitrary constants. Hence we choose K_{ij}, $j = 1 \cdots 8$ arbitrarily. Then K_{3j} and L_{ij}, $j = 1, 2, 5, 6, 7, 8$ are determined in terms of K_{1j} by any three of the equations,

$$(8)\quad \left\{\begin{aligned}
&[\lambda^2 - \omega^2 + (9m_3/4)]K_{1j} - (9m_3/4)K_{3j} - [2\omega\lambda + (3\cdot 3^{1/2}/4)m_3]\,L_{1j}\\
&\qquad\qquad + (3\cdot 3^{1/2}/4)m_3L_{3j} = 0,\\
&[\lambda^2 - (3\omega^2/4)]K_{3j} + (3\cdot 3^{1/2}/2)m_1L_{1j} - [2\omega\lambda - (3\cdot 3^{1/2}/2)m_1\\
&\qquad\qquad + (3\cdot 3^{1/2}/4)\omega^2]L_{3j} = 0,\\
&[2\lambda\omega - (3\cdot 3^{1/2}/4)m_3]K_{1j} + (3\cdot 3^{1/2}/4)m_3K_{3j} + [\lambda^2 - (9m_3/4)]\,L_{1j}\\
&\qquad\qquad + (9m_3/4)L_{3j} = 0,\\
&(3\cdot 3^{1/2}/2)m_1K_{1j} + [2\lambda\omega - (3\cdot 3^{1/2}/2)m_1 + (3\cdot 3^{1/2}/4)\omega^2]\,K_{3j}\\
&\qquad\qquad + [\lambda^2 - (9\omega^2/4)]L_{3j} = 0.
\end{aligned}\right.$$

To determine K_{33}, K_{34}, L_{13}, L_{14}, L_{33} and L_{34} we substitute in equations (3)

$$x_1 = K_{13} + K_{14}\tau,\qquad y_1 = L_{13} + L_{14}\tau,$$
$$x_3 = K_{33} + K_{34}\tau,\qquad y_3 = L_{33} + L_{34}\tau.$$

There results, since the equations must be identities in τ,

$$(9)\qquad \left\{\begin{aligned} A_{11}K_{14}+A_{31}K_{34}+B_{11}L_{14}+B_{31}L_{34}&=0,\\ A_{33}K_{34}+B_{13}L_{14}+B_{33}L_{34}&=0,\\ B_{11}K_{14}+B_{31}K_{34}+D_{11}L_{14}+D_{31}L_{34}&=0,\\ B_{13}K_{14}+B_{33}K_{34}\qquad\qquad\quad +D_{33}L_{34}&=0,\end{aligned}\right.$$

$$(9')\qquad \left\{\begin{aligned} B_{31}K_{33}+A_{31}L_{13}-A_{31}L_{33}&=2\omega K_{14}-B_{11}K_{13},\\ A_{31}K_{33}+B_{11}L_{13}+B_{31}L_{33}&=-2\omega L_{14}-A_{11}K_{13},\\ A_{33}K_{33}+B_{13}L_{13}+B_{33}L_{33}&=-2\omega L_{34},\\ B_{33}K_{33}\qquad\qquad +3A_{33}L_{33}&=2\omega K_{34}-B_{13}K_{13}.\end{aligned}\right.$$

Solving the first, second and fourth equations of (9) for K_{34}, L_{14}, and L_{34} in terms of K_{14} there results

$$(10)\qquad K_{34}=-\frac{m_1+m_2}{m_3}K_{14},\qquad L_{14}=-\frac{2m_2+m_3}{3^{1/2}\,m_3}K_{14},\qquad L_{34}=\frac{m_1-m_2}{3^{1/2}\,m_3}K_{14}.$$

These values satisfy the third equation identically. Substituting these in equations (9′) and solving the last three equations for K_{33}, L_{13} and L_{33} we have

$$(11)\qquad \left\{\begin{aligned} K_{33}&=-\frac{m_1+m_2}{m_3}K_{13}+\frac{4m_2\omega}{3(3)^{1/2}\,m_3^{\,2}}K_{14},\\ L_{13}&=-\frac{2m_2+m_3}{(3)^{1/2}\,m_3}K_{13}+\frac{8}{9}\,\frac{(m_3^{\,2}+m_2^{\,2}+m_2m_3}{m_3^{\,2}\omega}K_{14},\\ L_{33}&=\frac{m_1-m_2}{(3)^{1/2}\,m_3}K_{13}-\frac{4}{9}\,\frac{(2m_3m_1+m_2\omega^2-2m_2^{\,2})}{m_3^{\,2}\omega}K_{14}.\end{aligned}\right.$$

These values satisfy the first equation of (9′) identically.

The last three equations of (8) yield the following for K_{3j}, L_{1j} and L_{3j} in terms of K_{1j} $j=1, 2, 5, 6, 7, 8.$

$$(12)\qquad \begin{aligned} K_{3j}&=(27m_1/8D)\,[(m_2-m_1)\lambda^2-2\cdot 3^{1/2}\omega(m_1+m_2)\lambda]K_{1j},\\ L_{3j}&=(3\cdot 3^{1/2}m_1/2D)\{-\lambda^4+\lambda^2\,[-(13\omega^2/4)+(9m_3/4)]\\ &\qquad\qquad +(3\cdot 3^{1/2}/2)\omega(m_1-m_2)\lambda\}K_{1j},\\ L_{1j}&=(K_{1j}/D)\,[-2\omega\lambda^3(\lambda^2+\omega^2)+(3\cdot 3^{1/2}/8)m_3\lambda^2\\ &\qquad\qquad (2\lambda^2+2\omega^2+9m_1)+(27\cdot 3^{1/2}/4)m_1m_3\omega\lambda],\end{aligned}$$

where

$$D = \lambda^6 + \lambda^4[\omega^2 - (9m_3/4)] + (27/8)\lambda^2[2m_1m_2 + m_1m_3 - (2/3)m_3\omega^2] + (27 \cdot 3^{1/2}/4)\omega m_1 m_3 \lambda.$$

The Generating Solutions. The generating solutions must be periodic. Hence we have the following sub-cases of case I to consider.

Case I. (a) ρ, ω and ν incommensurable,

$$(1) \quad x_i = K_{i5}e^{\iota\nu\tau} + K_{i6}e^{-\iota\nu\tau}, \quad y_i = L_{i5}e^{\iota\nu\tau} + L_{i6}e^{-\iota\nu\tau}, \quad z_1 = 0,$$

$$(2) \quad x_i = K_{i7}e^{\iota\rho\tau} + K_{i8}e^{-\iota\rho\tau}, \quad y_i = L_{i7}e^{\iota\rho\tau} + L_{i8}e^{-\iota\rho\tau}, \quad z_i = 0,$$

$$(3) \quad x_i = K_{i1}e^{\iota\omega\tau} + K_{i2}e^{-\iota\omega\tau}, \quad y_i = L_{i1}e^{\iota\omega\tau} + L_{i2}e^{-\iota\omega\tau},$$
$$z_i = M_{i1}\cos\omega\tau + M_{i2}\sin\omega\tau.$$

(b) ρ and ν commensurable with each other but not with ω.

$$(4) \quad \left.\begin{aligned} x_i &= K_{i5}e^{\iota\nu\tau} + K_{i6}e^{-\iota\nu\tau} + K_{i7}e^{\iota\rho\tau} + K_{i8}e^{-\iota\rho\tau}, \\ y_i &= L_{i5}e^{\iota\nu\tau} + L_{i6}e^{-\iota\nu\tau} + L_{i7}e^{\iota\rho\tau} + L_{i8}e^{-\iota\rho\tau}, \end{aligned}\right\}, \quad z_i = 0.$$

(c) ρ and ω commensurable but ν incommensurable with them.

$$(5) \quad \left.\begin{aligned} x_i &= K_{i1}e^{\iota\omega\tau} + K_{i2}e^{-\iota\omega\tau} + K_{i7}e^{\iota\rho\tau} + K_{i8}e^{-\iota\rho\tau}, \\ y_i &= L_{i1}e^{\iota\omega\tau} + L_{i2}e^{-\iota\omega\tau} + L_{i7}e^{\iota\rho\tau} + L_{i8}e^{-\iota\rho\tau}, \end{aligned}\right\},$$
$$z_i = M_{i1}\cos\omega\tau + M_{i2}\sin\omega\tau.$$

(d) ν and ω commensurable but ρ incommensurable with them.

$$(6) \quad \left.\begin{aligned} x_i &= K_{i1}e^{\iota\omega\tau} + K_{i2}e^{-\iota\omega\tau} + K_{i5}e^{\iota\nu\tau} + K_{i6}e^{-\iota\nu\tau} \\ y_i &= L_{i1}e^{\iota\omega\tau} + L_{i2}e^{-\iota\omega\tau} + L_{i5}e^{\iota\nu\tau} + L_{i6}e^{-\iota\nu\tau} \end{aligned}\right\},$$
$$z_i = M_{i1}\cos\omega\tau + M_{i2}\sin\omega\tau.$$

(e) ρ, ω and ν all commensurable.

$$(7) \quad x_i = K_{i1}e^{\iota\omega\tau} + K_{i2}e^{-\iota\omega\tau} + K_{i5}e^{\iota\nu\tau} + K_{i6}e^{-\iota\nu\tau} + K_{i7}e^{\iota\rho\tau} + K_{i8}e^{-\iota\rho\tau},$$
$$y_i = L_{i1}e^{\iota\omega\tau} + L_{i2}e^{-\iota\omega\tau} + L_{i5}e^{\iota\nu\tau} + L_{i6}e^{-\iota\nu\tau} + L_{i7}e^{\iota\rho\tau} + L_{i8}e^{-\iota\rho\tau},$$
$$z_i = M_{i1}\cos\omega\tau + M_{i2}\sin\omega\tau.$$

Transformation to the Normal Form. It will be convenient to make a transformation of variables of the form

$$x_1 = \sum_{j=1}^{8} a_{1j}u_j, \quad dx_1/dt = \sum_{j=1}^{8} a_{2j}u_j, \quad x_3 = \sum_{j=1}^{8} a_{3j}u_j,$$

$$dx_3/dt = \sum_{j=1}^{8} a_{4j}u_j, \quad y_1 = \sum_{j=1}^{8} a_{5j}u_j, \quad dy_1/dt = \sum_{j=1}^{8} a_{6j}u_j,$$

$$y_3 = \sum_{j=1}^{8} a_{7j}u_j, \quad dy_3/dt = \sum_{j=1}^{8} a_{8j}u_j.$$

It is desired to determine the a_{ij} so that the differential equations (2) take the form

$$(13) \qquad \begin{aligned} u_1' &= (1+\delta)\omega\iota u_1 + (1+\delta)\epsilon P_1(u_j, z_i), \\ u_2' &= -(1+\delta)\omega\iota u_2 + (1+\delta)\epsilon P_2(u_j, z_i), \\ u_3' &= (1+\delta)u_4 + (1+\delta)\epsilon P_3(u_j, z_i), \\ u_4' &= \qquad\qquad\quad (1+\delta)\epsilon P_4(u_j, z_i), \\ u_5' &= (1+\delta)\nu\iota u_5 + (1+\delta)\epsilon P_5(u_j, z_i), \\ u_6' &= -(1+\delta)\nu\iota u_6 + (1+\delta)\epsilon P_6(u_j, z_i), \\ u_7' &= (1+\delta)\rho\iota u_7 + (1+\delta)\epsilon P_7(u_j, z_i), \\ u_8' &= -(1+\delta)\rho\iota u_8 + (1+\delta)\epsilon P_8(u_j, z_i), \end{aligned}$$

where the P_j are power series in u_j beginning with terms of the second degree. To determine the a_{ij} integrate equations (13) after all the terms of the right members have been dropped except the linear ones. The result is

$$\begin{aligned} &u_1 = K_{11}e^{(1+\delta)\omega\iota\tau}, \quad u_2 = K_{12}e^{-(1+\delta)\omega\iota\tau}, \quad u_3 = K_{13} + (1+\delta)K_{14}\tau, \\ &u_4 = K_{14}, \quad u_5 = K_{15}e^{(1+\delta)\nu\iota\tau}, \quad u_6 = K_{16}e^{-(1+\delta)\nu\iota\tau}, \\ &u_7 = K_{17}e^{(1+\delta)\rho\iota\tau}, \quad u_8 = K_{18}e^{-(1+\delta)\rho\iota\tau}. \end{aligned}$$

Adding these eight equations we get the value of x_1 obtained from the linear terms of equation (2). The values of x_1', x_3, x_3', y_i and y_i' are obtained from x_1 by multiplying the K_{ij} by certain constants given in equations (12). We may choose all the K_{ij} equal to unity. This gives the following form for the transformation:

$$(14)\quad\left\{\begin{aligned}
x_1 &= u_1 + u_2 + u_3 + u_4 + u_5 + u_6 + u_7 + u_8,\\
x_1' &= \omega\iota(u_1 - u_2) + u_4 + \nu\iota(u_5 - u_6) + \rho\iota(u_7 - u_8),\\
x_3 &= a_{31}u_1 + a_{32}u_2 + a_{33}u_3 + a_{34}u_4 + a_{35}u_5 + a_{36}u_6 + a_{37}u_7 + a_{38}u_8,\\
x_3' &= \omega\iota(a_{31}u_1 - a_{32}u_2) + a_{34}u_4 + \nu\iota(a_{35}u_5 - a_{36}u_6) + \rho\iota(a_{37}u_7 - a_{38}u_8)\\
y_1 &= a_{51}u_1 + a_{52}u_2 + a_{53}u_3 + a_{54}u_4 + a_{55}u_5 + a_{56}u_6 + a_{57}u_7 + a_{58}u_8,\\
y_1' &= \omega\iota(a_{51}u_1 - a_{52}u_2) + a_{54}u_4 + \nu\iota(a_{55}u_5 - a_{56}u_6) + \rho\iota(a_{57}u_7 - a_{58}u_8)\\
y_3 &= a_{71}u_1 + a_{72}u_2 + a_{73}u_3 + a_{74}u_4 + a_{75}u_5 + a_{76}u_6 + a_{77}u_7 + a_{78}u_8,\\
y_3' &= \omega\iota(a_{71}u_1 - a_{72}u_2) + a_{74}u_4 + \nu\iota(a_{75}u_5 - a_{76}u_6) + \rho\iota(a_{77}u_7 - a_{78}u_8)
\end{aligned}\right.$$

Orbits with Period $2\pi/\nu$. We choose to investigate Case I, (a), (1). In the normal variables the generating solution is

$$u_5 = K_5 e^{\nu\iota\tau}, \quad u_6 = K_6 e^{-\iota\nu\tau}, \quad u_j = 0, \quad (j = 1, 2, 3, 4, 7, 8),$$
$$z_i = 0.$$

We shall prove that these orbits do not exist outside the xy-plane. Let the initial conditions be

$$u_5 = k_5 + a_5,\ u_6 = k_6 + a_6,\ u_j = a_j,\ j = 1, 2, 3, 4, 7, 8,$$
$$z_i = g_i,\ z'_i = h_i,\ i = 1, 3 \text{ at } \tau = 0.$$

It is evident that if $g_i = h_i = 0$, $i = 1, 3$, the orbits, if they exist at all, must lie wholly in the xy-plane.

The conditions that the solutions be periodic are

$$(15)\qquad u_j(2\pi/\nu) - u_j(0) = 0,\ z_i(2\pi/\nu) - z_i(0) = 0,$$
$$z_i'(2\pi/\nu) - z_i'(0) = 0.$$

We integrate equations (13) as power series in the initial constants, ϵ and δ. The linear terms of the z_i equations are independent of the u_i equations. Integrating the linear terms of the z_i equations and imposing the initial conditions,

$$z_{10} = g_1 \cos\omega\tau + (h_1/\omega)\sin\omega\tau,\quad z'_{10} = -\omega g_1 \sin\omega\tau + h_1\cos\omega\tau,$$
$$z_{30} = g_3 \cos\omega\tau + (h_3/\omega)\sin\omega\tau,\quad z'_{30} = -\omega g_3 \sin\omega\tau + h_3\cos\omega\tau.$$

Then the periodicity conditions for the z_i equations are:

$$0 = g_1[\cos(2\pi\omega/\nu) - 1] + (h_1/\omega)\sin(2\pi\omega/\nu) + \epsilon Q_1(a_j, g_1, h_1, \epsilon, \delta),$$

$$0 = -g_1\omega\sin(2\pi\omega/\nu) + h_1[\cos(2\pi\omega/\nu) - 1] + \epsilon Q_1'(a_j, g_i, h_i, \epsilon, \delta),$$

$$0 = g_3[\cos(2\pi\omega/\nu) - 1] + (h_3/\omega)\sin(2\pi\omega/\nu) + \epsilon Q_3(a_j, g_i, h_i, \epsilon, \delta),$$

$$0 = -g_3\sin(2\pi\omega/\nu) + h_3[\cos(2\pi\omega/\nu) - 1] + \epsilon Q_3'(a_j, g_i, h_i, \epsilon, \delta).$$

In the first three of these equations the determinant of the linear terms in g_1, h_1, h_3 does not vanish. There is a term independent of g_1, h_1, h_3. Hence we may solve uniquely for g_1, h_1, and h_3 in terms of a_j, g_3, ϵ, δ. Substitute these solutions in the last equation. Since all the terms of the right members of the z_i equations of equation (2) carry either z_1 or z_3 as a factor, it follows that after the values of g_1, h_1, h_3 have been substituted, the fourth equation will carry g_3 as a factor. Removing this factor leaves a term independent of a_i, ϵ and δ. Therefore solution for g_3 in terms of a_i, ϵ, δ vanishing with these parameters is impossible and the orbits do not exist outside the xy-plane.

Suppression of the u_4 and u_6 Equations. The periodicity equations now become

$$u_j[(2\pi/\nu)] - u_j(0) = 0, \qquad (j = 1, \cdots, 8).$$

These are not independent. They are related by the integral of areas and the energy integral.

$$F_1 = \sum_{i=1}^{3} m_i(\bar{x}_i\bar{y}_i' - \bar{y}_i\bar{x}_i') - c_1 = 0,$$

$$F_4 = \tfrac{1}{2}\sum_{i=1}^{3}[(\bar{x}_i')^2 + (\bar{y}_i')^2 + (\bar{z}_i')^2] - h = 0.$$

These variables are related to the u_i equations by the following transformations:

$$\bar{x}_i = \xi_i\cos\omega t - \eta_i\sin\omega t,$$

$$\bar{y}_i = \xi_i\sin\omega t + \eta_i\cos\omega t,$$

$$\xi_i = \xi_i^{(0)} + \epsilon x_i, \quad \eta_i = \eta_i^{(0)} + \epsilon y_i, \quad \bar{z}_i = \epsilon z_i,$$

$$t = (1 + \delta)\tau, \text{ and equations (14).}$$

Now let

$$u_j = a_j + s_j, \quad (j = 1, 2, 3, 4, 7, 8), \quad u_5 = (k_5 + a_5)e^{(1+\delta)\nu\iota\tau} + s_5,$$

$$u_6 = (k_6 + a_6)e^{-(1+\delta)\nu\iota\tau} + s_6.$$

---

[*] Moulton, *Periodic Orbits*, Chap. I, page 1.

It follows that the $s_j = 0$ for $\tau = 0$. Since c_1 and h can be expressed in terms of the initial constants, we may consider F_1 and F_4 as functions of s_j and a_j vanishing with s_j whatever values a_j may have. Therefore they may be solved for s_4 and s_6 as power series in s_j, $(j = 1, 2, 3, 5, 7, 8)$, provided the Jacobian

$$J = \begin{vmatrix} \partial F_1/\partial s_4, & \partial F_1/\partial s_6 \\ \partial F_4/\partial s_4, & \partial F_4/\partial s_6 \end{vmatrix} \neq 0,$$

for zero values of the arguments.

The elements of J are found by direct computation to be

$$\partial F_1/\partial s_6 = -2m_1\omega - m_3\omega K_{35} + m_3\omega\, 3^{1/2}\, L_{35},$$

$$\partial F_1/\partial s_4 = -2m_1\omega - m_3\omega K_{34} + m_3\omega\, 3^{1/2}\, L_{34},$$

$$\begin{aligned}\partial F_4/\partial s_4 = 2\omega^2\{\xi_1 - (m_1/m_2)\xi_2 + [\xi_3 - (m_3/m_2)\xi_2]K_{34} \\ + [\eta_1 - (m_1/m_2)\eta_2]L_{14} + [\eta_3 - (m_3/m_2)\eta_2]L_{34}\}.\end{aligned}$$

The values of K_{34} and L_{34} are given in equations (10). When these are substituted $\partial F_1/\partial s_4$ vanishes, which makes it unnecessary to compute $\partial F_4/\partial s_6$. Supplying the values of K_{34}, L_{34} and L_{14} in $\partial F_4/\partial s_4$ we find

$$\frac{\partial F_4}{\partial s_4} = 2\omega^2 \left[\frac{3m_3 - \omega^2}{m_3}\ \xi_1 - \frac{m_1 - m_2}{m_3}\ (3)^{1/2}\eta_1 + \tfrac{1}{2}\ \frac{m_1 + m_2}{m_2}\right].$$

Now if we choose $m_3 \leqq m_2 \leqq m_1$, then $3m_3 - \omega^2$ is negative or zero, $m_1 - m_2$ is positive or zero. Since ξ_1 and η_1 are negative it follows that one factor of J does not vanish.

Substituting the values of K_{35} and L_{35} from (12) in $\partial F_1/\partial s_6$ we find

$$\frac{\partial F_1}{\partial s_6} = -\frac{16\lambda^6 + 16\omega^2\lambda^4 + 81(m_1m_2 + m_1m_3 + m_2m_3)\lambda^2}{8D} \tag{16}$$

When λ^2 is taken from (6) this reduces to

$$\frac{\partial F_1}{\partial s_6} = \frac{27\{-\omega^2 + [\omega^4 - 27(m_1m_2 + m_1m_3 + m_2m_3)]^{1/2}\}(m_1m_2 + m_1m_3 + m_3m_3}{16D}$$

which cannot vanish. Therefore we may suppress the u_4 and u_6 equations.

Existence of Orbits with Period $2\pi/\nu$. With the initial conditions $u_j = a_j$ $(j = 1, 2, 3, 4, 7, 8)$, $U_5 = K_5 + a_5$, $U_6 = K_6 + a_6$, at $\tau = 0$, we may integrate equations (13) as power series in the initial constants, ϵ and δ in the form:

$$
\begin{aligned}
u_1 &= a_1 e^{(1+\delta)\omega\iota\tau} + \epsilon P_1(a_j, \epsilon, \delta),\\
u_2 &= a_2 e^{-(1+\delta)\omega\iota\tau} + \epsilon P_2(a_j, \epsilon, \delta),\\
u_3 &= a_3 + a_4\tau + \epsilon P_3(a_j, \epsilon, \delta),\\
u_4 &= \qquad a_4 + \epsilon P_4(a_j, \epsilon, \delta),\\
u_5 &= (K_5 + a_5) e^{(1+\delta)\nu\iota\tau} + \epsilon P_5(a_j, \epsilon, \delta),\\
u_6 &= (K_6 + a_6) e^{-(1+\delta)\nu\iota\tau} + \epsilon P_6(a_j, \epsilon, \delta),\\
u_7 &= a_7 e^{(1+\delta)\rho\iota\tau} + \epsilon P_7(a_j, \epsilon, \delta),\\
u_8 &= a_8 e^{-(1+\delta)\rho\iota\tau} + \epsilon P_8(a_j, \epsilon, \delta).
\end{aligned}
$$

The periodicity conditions are now

$$u_j(2\pi/\nu) - u_j(0), \quad (j = 1, 2, 3, 5, 7, 8),$$

since the u_4 and u_6 equations have been suppressed. There are six equations and ten parameters. Therefore we may choose a_4, a_5, a_6 and ϵ arbitrarily and solve for a_1, a_2, a_3, a_7, a_8 and δ in terms of the arbitrarily chosen ones. The solutions will vanish with these parameters. This solution is uniquely possible since the determinant of the linear terms is

$$K_5\{e^{\omega\iota(2\pi/\nu)} - 1\}\{e^{-\omega\iota(2\pi/\nu)} - 1\}\{e^{\rho\iota(2\pi/\nu)} - 1\}\{e^{-\rho\iota(2\pi/\nu)} - 1\}\ (2\pi/\nu) \neq 0$$

and since $a_1 = a_2 = a_3 = a_7 = a_8 = \delta = 0$ is not a solution. This proves the existence of an unique set of periodic orbits in the xy-plane with period $2\pi/\nu$. We could have used the period $2K\pi/\nu$ just as well and found an unique set of that period. Hence all these orbits are re-entrant after one revolution.

Orbits of Period $2\pi/\rho$. Following the method of the last two paragraphs we may establish the existence of plane orbits of period $-2\pi/\rho$. The steps are:

1. The initial conditions are $u_j = a_j$, $j = 1 \cdots 6$, $u_7 = k_7 + a_7$, $u_8 = k_8 + a_8$ at $\tau = 0$.

2. The orbits exist only in the xy-plane. The proof is exactly the same as the preceding case.

3. The u_4 and u_8 equations of the periodicity conditions may be suppressed. The only difference being in $\partial F_1/\partial s_8$ which may be found from $\partial F_1/\partial s_6$, equation (16), by replacing λ^2 by $-\rho^2$ from equation (6). There results

$$\partial F_1/\partial s_8 = 27M/D[\omega^2 + (\omega^4 - 27M)^{1/2}], \text{ where } M = m_1m_2 + m_2m_3 + m_3m_1.$$

Hence the u_4 and u_8 equations may be suppressed.

4. The remaining periodicity equations may be solved and an unique set of orbits of period $2\pi/\rho$ re-entrant after one revolution is established.

Conclusion. Thus we have established the existence of two types of periodic orbits, both of which lie in the plane of motion of the finite bodies. Since the linear terms of the z-equations give only the period $2\pi/\omega$ one expects that there will exist no periodic orbits in space except those of period $2\pi/\omega$. The orbits discussed exist if the condition on the masses

$$\omega^4 - 27(m_1m_2 + m_1m_3 + m_2m_3) > 0$$

is satisfied. The discussion of the construction of the solutions and the further geometrical properties of the orbits whose existence is proved as well as the other sub-cases is reserved for a later paper.

TULANE UNIVERSITY,
JUNE, 1927.

f Plane Involutions of Order $t > 2$.

By F. R. Sharpe.

A plane involution of order t, I_t, is defined as the
of a plane (x), by the methods of algebraic projective
t points in such a way that a general point occurs in
astelnuovo* has proved that any I_t of this kind is
can therefore be put into $(1, 1)$ correspondence with
) by a transformation which can be expressed by equa-

$$z_i = f_i(x_1,\ x_2,\ x_3) \qquad\qquad (i = 1,\ 2,\ 3,)$$

nogeneous polynomials in x_1, x_2, x_3 having no common
tional determinant which does not vanish identically.
hich can be changed to the same form by a Cremona
(x) or (z) plane or of both planes, are considered to
the functions $f_i(x)$ in (1) are reduced to a minimum
transformation is said to be reduced to an irreducible
, to the net of lines in (z)

$$a_1z_1 + a_2z_2 + a_3z_3 = 0$$

ive net of curves

$$a_1f_1 + a_2f_2 + a_3f_3 = 0.$$

termine a point P on (z), the corresponding curves of
iable points, the group of I_t corresponding to P. The
said to map the plane (x) on a t-fold plane (z). The
tions of order t is to determine the irreducible normal
(3) with t variable intersections. To define a group
two equations are necessary. From the solutions by
e analogous problem of determining a line of a $(1, n)$
$S_3(y)$, (1) being the analogue of the planes through a point,
wo other ways of defining groups of points in the plane, (a)

---

ovo, *Mathematische Annalen,* Vol. 44 (1894).
mer, *Berlin Abhandlungen* (1866).

by the intersections of two projective nets of curves,[*] (b) by the intersection of the curves of a pencil with the algebraically corresponding curves of a linear system. The type (a) is analogous to the bisecants of a cubic curve and (b) to the lines meeting a fixed line l and a fixed curve C_{2n-2} meeting l in $2n-3$ points. If the linear system in (b) is of dimension $r > 2$[†] then I_t belongs to a family of I_t, n being arbitrary. In the case of both (a) and (b) the corresponding I_t's can be reduced to the normal form (1) but are more easily discovered when in the form (a) or (b). It is now clear that the problem of plane involutions is closely connected with the theory of linear systems of plane curves. It will be shown that if the pencil of curves in (b) is of genus p then a finite number of irreducible normal forms of I_t exist for each $p \leq (t-1)(t-2)/2$. The actual determination of forms of I_t requires a knowledge of the linear systems of curves of least
$p \leq t(t-1)/2$. A fairly complete determination is made
less complete one for $t = 4$. The mapping of the plane (x) o
$r > 2$ is shown to give valuable information concerning possi

2. *Properties of linear systems of curves*[‡] (G. Castelnı
a system of curves of dimension r and order d

$$\sum_{i=1}^{r+1} a_i f_i(x) = 0 \tag{4}$$

determined by the basic points, a k-fold basis point bein
$k(k+1)/2$ conditions so that

$$r - [d(d+3)]/2 + \sum k(k+1)/2 \geq 0 = s, \text{ the} \tag{5}$$

The genus $\quad \pi = (d-1)(d-2)/2 - \sum k(k-1)/2$

and the grade (number of variable intersections) $n = d^2 - \sum$

$$r = n - \pi + 1 + s. \tag{6}$$

The virtual dimension $\quad {}_v r = n - \pi + 1.$
For a regular system $\quad {}_v r = r \quad \text{and} \quad s = 0.$
It is known that $\quad s \leq \pi - r + 1.$

---

[*] A. M. Howe, T. R. Hollcroft, *American Journal of Mathemati*

[†] If $r = 2$ the involution is reducible to a single type not c
meter n.

[‡] G. Castelnuovo, *Torino Memorie*, Ser. 2, Vol. 42 (1891); G.
Matematico, Ser. 2, Vol. 15 (1887), pp. 277-312; Ser. 2, Vol. 16 (1

Given two linear systems $| C_1 |$, $| C_2 |$, their sum $| C_1 | + | C_2 |$ is the system of order $d_1 + d_2$ with a $k_1 + k_2$ fold point at a point which is k_1, k_2 fold for $| C_1 |$, $| C_2 |$, respectively. If a curve of $| C_1 |$ meets a curve of $| C_2 |$ in t variable points, $[C_1, C_2] = t$, then $| C_1 | + | C_2 |$ is of grade $n_1 + n_2 + 2t$, of genus $\pi_1 + \pi_2 + t - 1$ and virtual dimension ${}_v r_1 + {}_v r_2 + t$.

3. *Properties of linear series of points on a fixed curve C of genus p and order m*[*] (L. Berzolari). The system of adjoint curves of order $m - 3$ having a $k - 1$ fold point at a k-fold point of C is of dimension $p - 1$ and cuts on C groups of $2p - 2$ variable points the canonical G_{2p-2}^{p-1}. A G_n^r included in the G_{2p-2}^{p-1} is special and $n \geq r(r + p + 1)/(1 + r)$ (A. Clebsch[†] with $r > n - p$. For a non-special G_n^r, $r = n - p$. The residual G_{2p-2-n} is of dimension $r + p - 1 - n$ by the Riemann-Roch theorem. The curves of a linear system of grade n and dimension r cut on a curve of the system a G_n^{r-1}. If $n > 2p - 2$ then $r = n - p + 1$, if $n \leq 2p - 2$ the G_n^{r-1} can be cut by the adjoint curves of order $m - 3$ and $r > n - p + 1$ (C. Segre)[‡] subject to $n \geq (r + 1)(r + p)/r$ for curves of general moduli.

4. *Types of plane involutions I_t.* Type (b). For a given pencil of curves $| C |$ of genus p, $s = p$. Hence if we consider the t variable intersections of a curve of C with the curves of a second system $| C | + | \bar{C} |$ there are two cases, (1) C of grade p with $[C, \bar{C}] = t - p$, (2) $| C |$ of grade 0 with $[C, \bar{C}] = t$. If the system $| C | + | \bar{C} |$ represents

$$(a_1 C_1 + a_2 C_2)\bar{C} + a_3 C_3 + \cdots a_{r+1} C_{r+1} = 0 \tag{7}$$

the involution I_t is determined by

$$\begin{gathered} z_2 C_1 - z_1 C_2 = 0 \\ (b_1 z_3 + c_1) C_1 \bar{C} + (b_2 z_3 + c_2) C_2 \bar{C} + (b_3 z_3 + c_3) C_3 \\ + \cdots + (b_{r+1} z_3 + c_{r+1}) C_{r+1} = 0 \end{gathered} \tag{8}$$

where b_i and c_i are of arbitrary orders $n - 1$, n in z_1, z_2. Solving for $z_1 : z_2 : z_3$, we find for I_t in the normal form (1)

$$z_1 = C_1 u, \quad z = C_2 u, \quad z_3 = v, \tag{9}$$

where u and v are linear in $C_1 \bar{C}$, $C_2 \bar{C}$, $C_3, \cdots, C_{r+1}$ with coefficients of arbitrary orders $n-1$, n respectively in C_1, C_2. The form (9) therefore determines for $r > 2$ a family of I_t.

---

[*] L. Berzolari, *Pascal Repertorium*, Vol. 2, p. 312 (1910).

[†] A. Clebsch, *Geometric*, Vol. 3, p. 37.

[‡] C. Segre, *Rendiconti di Circolo Matematico Palermo*, Vol. I.

Type (a). If two projective nets of curves belong to linear systems of grade n_1, n_2, genus p_1, p_2, virtual dimensions r_1, r_2, then $| C_1 | + | C_2 |$ is of grade $n_1 + n_2 + 2t$, genus $p_1 + p_2 + t - 1$ and virtual dimension $r_1 + r_2 + t$. The net defining the involution must therefore have $n_1 + n_2 + t$ additional simple basis points and has therefore superabundance

$$(n_1 + n + t) - 2 - (r_1 + r_2 + t) = p_1 + p_2.$$

The case of a net of lines and a net of general curves of order t always exists and the net defining the involution is of genus $t(t-1)/2$ and superabundance $(t-1)(t-2)/2$.

Type (1). In order that a given linear system of curves of grade N, genus p, dimension $r > 2$ determine an I_t, it must be possible to determine $N - t$ additional simple basis points so that through them passes a net of curves of the system. For a superabundant net, the curves of the net cut on a curve of the net a G'_t which must be included in the canonical G^{p-1}_{2p-2}. For curves of general moduli therefore $t \geq (p + 2)/2$ that is

$$p \leq 2t - 2.$$

5. *The mapping of an I_t on $S_3(y)$.* For Type (b) we may take for $r > 2$

(10) $\qquad y_1 = C_1\bar{C}, \quad y_2 = C_2\bar{C}, \quad y_3 = C_3, \quad y_4 = C_4$ where $C_3 = 0$,

$C_4 = 0$ are curves of the system $| C | + | \bar{C} |$. If $| C | + | \bar{C} |$ is of grade N then (10) represents a surface $F_N{}^3$, of order N, having $l \equiv y_1 = y_2 = 0$ for $N - t$-fold line, $\bar{C}$ being the image of l. The plane sections of $F_N{}^3$ through l are C_s the images of the curves of the pencil $a_1C_1 + a_2C_2 = 0$. Hence

(11) $$p \leq (t-1)(t-2)/2.$$

The surface $F_N{}^3$ may have other multiple curves or points necessary for its rationality. The adjoint surfaces of order $N - 3$ meet F in l and the other multiple curves or points and in variable curves. The image of the variable curves is the system of adjoint curves of order $d - 3$ of the system $| C | + | \bar{C} |$ and the residue of $(N-3)(| C | + | \bar{C} |)$ consists of the image curves of l and the other multiple curves and points of $F_N{}^3$. The involution I_t on $F_N{}^3$ is cut by the lines of the rational congruence defined by the image of the net

(12) $$c_1C_1u + c_2C_2u + c_3v = 0.$$

The involution may also be mapped in $S_r(y)$ on an $F_N{}^r$ by the equations

(13) $$y_1 = C_1\bar{C},\quad y_2 = C_2\bar{C},\quad y_3 = C_3,\ \cdots,\ y_{r+1} = C_{r+1}.$$

The pencil of $S_{r-1}(y) a_1y_2 + a_2y_2 = 0$ passes through a fixed C_{N-t} in the S_{r-2}, $y_1 = y_2 = 0$ and meets $F_N{}^r$ in residual curves C_t. Two V_{r-1}'s of the net corresponding to $c_1C_1u + c_2C_2u + c_3v = 0$ meet in a group of I_t on $F_N{}^r$. The representation on $F_N{}^3$ is a special case found by projecting from $S(_ry)$ into $S_3(y)$. For Type (1) we may take

(14) $$y_1 = C_1\bar{C},\quad y_2 = C_2\bar{C},\quad y_3 = C_3\bar{C},\quad y_4 = C_4,$$

where $\bar{C}$ may be 1 of $C_1 = 0$, $C_2 = 0$, $C_3 = 0$ belong to a linear system of dimension > 2 with possibly fewer simple basis points. In the case $\bar{C} = 1$, $F_N{}^3$ has a multiple $N - t$-fold curve. In the case $\bar{C} \neq 1$, $C_4 = 0$ belongs to a linear system including the net $a_1C_1 + a_2C_2 + a_3C_3 = 0$ with a residual $\bar{C}$. In this case $F_N{}^3$ has an $N - t$-fold point $(0, 0, 0, 1)$ whose image is a fundamental curve of $| C | + | \bar{C} |$.

Type (a) may be reduced to the form (1) and treated similarly. If the two nets belong to the same linear system of dimension 3 we may take $y_2 = C_2$, $(i = 1 \text{ to } 4)$. In this case $N = t$ and the involution I_t on $F_t{}^3$ is cut by the bisecants of a cubic curve.

6. *The known* I_2 (M. Noether).[*] For subsequent use the known types of I_2 are derived by the methods developed in the preceding articles.

Type (b). $p = 0$, $a_1x_1 + a_2x_2 = 0$ [†]
$$(a_1x_1 + a_2x_2)x_3 + a_3x_3{}^2 + a_4x_1{}^2 + a_5x_1x_2 + a_6x_2{}^2 = 0.$$

The involution defined by

$$z_1 = x_1u,\quad z_2 = x_2u,\quad z_3 = v$$

is easily seen to be defined by the net of curves

$$C_n : A^{n-2}\ (4n - 6)B$$

the point A being $(0, 0, 1)$ and the $4n - 6$ simple basis points B given by $u = v = 0$.

Type (b). $p = 1$, $a_1C_1 + a_2C_2 = 0$ cubics $C_3 : 8AB$
$$(a_1C_1 + a_2C_2)C_1 + a_3C_6 : 8A^2 = 0.$$

---

[*] M. Noether, *Erlangen Medico Berichte* (1878).

[†] M. Noether, *Mathematische Annalen*, Vol. 3 (1871), p. 161.

The involution is defined by the net of curves $C_6 : 8A^2 2B$ the points B bei given by $C_1 = 0$, $C_6 : 8A^2 = 0$.

Type (a). $a_1x_1 + a_2x_2 + a_3x_3 = 0$.
$a_1C_1 + a_2C_2 + a_3C_3 = 0$ (conics).

The I_2 is therefore defined by the net of cubics $C_3 : 7A$. There is a spec case of the first Type (b) for $n = 2$

$$(a_1x_1 + a_2x_2)x_1 + a_3x_3^2 - a_3x_2^2 = 0.$$

The points A and the two points B are collinear. In the case of I_2 there an involutorial transformation of the (x) plane which interchanges the points of a group. These transformations in the order in which the 4 ty are given, are known as the Jonquières, Bertini, Geiser, and the harmo homology.*

7. *Families of* I_3. Type (b), $p = 0$, gives as in the preceding case

$$C_n : A^{n-3}\,(6n - 12)B.$$

Type (b). $p = 1$ with the pencil $C_3 : 8AB$.†

We proceed to determine the different linear systems (7) which inclu $C_3 : 8AB$ with a fixed component $\bar{C}$ and meet $C_3 : 8AB$ in three varia points. If B is a basis point of (7), $\bar{C}$ must meet $C_3 : 8AB$ in three varia points. Hence $\bar{C}$ is reducible to a line and (7) is $C_4 : 9A$. If B is no basis point of (7) either (7) is $C_3 : 6A$ so that $\bar{C} = 1$ or else $\bar{C}$ meets C_3 : in 2 variable points. From the discussion of I_2 it is seen that $\bar{C}$ is C_1 : $C_3 : 7A$ or $C_6 : 8A^2$. For the pencil $C_9 : 8A^3B^3$ we readily find for ($C_9 : 8A^3B^2$, $C_{12} : 8A^4B^3$, or $C_{15} : 8A^5B^4$.

The corresponding families can readily be formed by remembering tl from (9) $c_1C_1u + c_2C_2u + c_3v$ is linear in the curves of (1) with coefficie of order n in the curves of (2). Hence we find the following:

---

* K. Doehlmann, *Geometrische Transformationen*, Vol. 2, p. 170.

† G. H. Halphen, *Bulletin de la Société Mathématique*, Vol. 10 (1882), pp. 162-1

Families of Involutions of Order 3.

	Pencil	*Linear System*	*Family*
)	$C_3 : 6A3B$	$C_3 : 6A$	$C_{3n} : 6A^n 3B^{n-1}\ (6n-6)C$
)	$C_3 : 9A$	$C_4 : 9A$	$C_{3n+1} : 9A^n\ (6n-2)B$
)	$C_3 : A7BC$	$C_4 : A^2 7B$	$C_{3n+1} : A^{n+1} 7B^n C^{n-1}\ (6n-4)D$
)	$C_3 : 7ABC$	$C_6 : 7A^2B$	$C_{3n} : 7A^n B^{n-1} C^{n-2}\ (6n-8)D$
)	$C_3 : 8AB$	$C_9 : 8A^3$	$C_{3n} : 8A^n B^{n-3}\ (6n-12)C$
)	$C_9 : 8A^3B^3$	$C_9 : 8A^3B^2$	$C_{3n} : 8A^n B^{n-1}\ (2n-4)C$
		$C_{12} : 8A^4B^3$	
		$C_{15} : 8A^5B^4$	

each case the smallest possible n gives the linear system from which the mily was derived and corresponds to the net of plane sections of F_N^3 through point on l, the simple basis points being the images of this point. For the milies, the images of points on C_{2n} and F_N^3 but not on l are simple basis ints. The corresponding F_N^3's are

(1) $F_3 : r = 3$, l not on F_3
(2) $F_7 : r = 5 : l^4, l_1^3, l_2^3$ meeting l
(3) $F_5 : r = 4 : l^2, l_1^3$, meeting l
(4) $F_7 : r = 5 : l^4, l_1^3, l_2^3$ concurrent
(5) $F_9 : r = 6 : l^6, l_1^3, l_2^3$ concurrent.

8. *Application to* I_4. With the pencil $C_3 : 8AB$ and $t = 4$ we first educe from I_3, by freeing one point, the linear systems

$$C_3 : 5A,\quad C_4 : 8A,\quad C_4 : A^2 6B,\quad C_6 : 7A^2.$$

B is to be a basis point of (1) then $C_3 : 8AB$ must meet $\bar{C}$ in 4 variable oints. Hence $\bar{C} = C_2 : 2A$. If B is not a basis point of (1) $C_3 : 8AB$ ust meet $\bar{C}$ in 3 variable points, that is, from I_3

$$\bar{C} = C_3 : 6A,\quad C_4 : A^2 7B,\quad C_6 : 7A^2B, \text{ or } C_9 : 8A^3.$$

ence including the pencils $C_6 : 8A^2B^2$ and $C_{12} : 8A^4B^4$ we have the following:

11

Families of Involutions of Order 4, I_4

Pencil	*Linear System*	*Family*
(1) $C_3 : 5A4B$	$C_3 : 5A$	$C_{3n} : 5A^n4B^{n-1}\,(8n-8)C$
(2) $C_3 : 8AB$	$C_4 : 8A$	$C_{3n+1} : 8A^nB^{n-1}\,(8n-4)C$
(3) $C_3 : A6B2C$	$C_4 : A^26B$	$C_{3n+1} : A^{n+1}6B^n2C^{n-1}\,(8n-6)D$
(4) $C_3 : 7A2B$	$C_6 : 7A^2$	$C_{3n} : 7A^n2B^{n-2}\,(8n-12)C$
(5) $C_3 : 2A7B$	$C_5 : 2A^27B$	$C_{3n+2} : 2A^{n+1}7B^n\,(8n-2)C$
(6) $C_3 : 6A2BC$	$C_6 : 6A^22B$	$C_{3n} : 6A^n2B^{n-1}C^{n-2}\,(8n-10)D$
(7) $C_3 : A7BC$	$C_7 : A^37B^2$	$C_{3n+1} : A^{n+1}7B^nC^{n-2}\,(8n-8)D$
(8) $C_3 : 7ABC$	$C_9 : 7A^3B^2$	$C_{3n} : 7A^nB^{n-1}C^{n-3}\,(8n-14)D$
(9) $C_3 : 8AB$	$C_{12} : 8A^4$	$C_{3n} : 8A^nB^{n-4}\,(8n-20)C$
(10) $C_6 : 8A^2B^2$	$C_9 : 8A^3B$	$C_{3n} : 8A^nB^{n-1}\,(2n-5)C$
(11) $C_{12} : 8A^4B^4$	$C_{12} : 8A^4B^3$	
	$C_{15} : 8A^5B^4$	
	$C_{18} : 8A^6B^5$	
	$C_{21} : 8A^7B^6$	

The corresponding surfaces $F_N{}^3$ for (6) of I_3 and (10), (11) of I_4 have complex singular points which will be discussed elsewhere. It can be seen from the method here developed that if we know the linear systems for I_t genus p we can deduce the linear systems for the families of I_{t+1} genus p.

9. *Single types of I_3 not belonging to a family.*

(1) $C_4 : 13A$.* From either

$$z_1x_1 + z_2x_2 + z_3x_3 = 0, \quad z_1C_1 + z_2C_2 + z_3C_3 = 0 \text{ (cubic curves)}$$

or $C_4 : 11A2B$, the two points B being properly chosen. The transformation $y_i = C_i$, $(i = 1 \text{ to } 4)$, $C_i = 0$ being a $C_4 : 11A$, maps the plane on an $F_5{}^3$ with a double space cubic. The I_3 corresponds to cutting $F_5{}^3$ by the lines through a point on the double curve.

(2) $C_7 : A^39B^2C$ † the points $A9B$ being on a cubic, the image of the singular point of the $F_4{}^3$ determined by the transformation, $y_i = C_3 : A9B$, C_i $(i = 1, 2, 3)$, where $C_i = 0$ is a $C_4 : A^29B$, $y_4 = C_7 : A^39B^2$. The I_3 corresponds to cutting $F_4{}^3$ by lines through a point on the surface.

---

[*] F. S. Macaulay, *Proceedings of the London Mathematical Society*, Vol. 26 (1895), p. 514.

[†] M. Noether, *Mathematische Annalen*, Vol. 33 (1889), p. 546.

(3) $C_7 : A^3 8B^2 2CD2E$ * the points $A8B2C$ being on a cubic the image the singular point of the F_5^3 determined by the transformation $y_i =$ $: A8B2C \cdot C_i$, where $C_i = 0$ is a $C_4 : A^2 8BD$, $y_4 = C_7 : A^3 8B^2 2CD$. e I_3 corresponds to cutting F_5^3 by lines through a point on the double ic of the surface.

(4) $C_9 : 7A^3 3B^2 C2D$,* the points $7A3B$ being on a cubic the image of singular point of the F_5^3 determined by the transformation $y_i =$ $: 7A3B \cdot C_i$, where $C_i = 0$ is a $C_6 : 7A^2 3BC$; $y_4 = C_9 : 7A^3 3B^2 C$. The orresponds to cutting F_5^3 by lines through a point on the double conic.

10. *Conclusion.* The families of involutions I_4 determined from pen- of genus 2 and 3 are being investigated by Franklin G. Williams of Cor- University. When they are completed a knowledge of the linear systems

also being determined would enable

not included in families. I wish to

ork of D. Montesano * on Quintic

line or no double curve, but in al

that the surfaces are rational. Many

CORRECTIONS, VOLUME 49.

H. M. GEHMAN, *Irreducible Continuous Curves.*

Page 191, line 16, change “

Page 196, line 2, change “ *cor*

CORRECTI